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OSCILLATION CRITERIA FOR SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS INVOLVING INTEGRAL AVERAGES

JAMES S. W. WONG

ABSTRACT Consider the second order nonlinear differential equation

$$y'' + a(t)f(y) = 0$$

where $a(t) \in C[0,\infty)$, $f(y) \in C^1(-\infty,\infty)$, $f'(y) \geq 0$ and $yf(y) > 0$ for $y \neq 0$ Furthermore, $f(y)$ also satisfies either a superlinear or a sublinear condition, which covers the prototype nonlinear function $f(y) = |y|^\gamma \operatorname{sgn} y$ with $\gamma > 1$ and $0 < \gamma < 1$ known as the Emden-Fowler case The coefficient $a(t)$ is allowed to be negative for arbitrarily large values of t Oscillation criteria involving integral averages of $a(t)$ due to Wintner, Hartman, and recently Butler, Erbe and Mingarelli for the linear equation are shown to remain valid for the general equation, subject to certain nonlinear conditions on $f(y)$ In particular, these results are therefore valid for the Emden-Fowler equation

§**1.** Consider the second order nonlinear differential equation

$$y'' + a(t)f(y) = 0, \quad t \in [0,\infty), \tag{1}$$

where $a(t) \in C[0,\infty)$ and $f(y) \in C^1(-\infty,\infty)$, $f'(y) \geq 0$ for all y, and satisfies $yf(y) > 0$ if $y \neq 0$. The prototype of equation (1) is the so-called Emden-Fowler equation

$$y'' + a(t)|y|^\gamma \operatorname{sgn} y = 0, \quad \gamma > 0. \tag{2}$$

Here we are interested in the oscillation of solutions of (1) when $f(y)$ satisfies, in addition, the following sublinear condition:

$$0 < \int_0^\varepsilon \frac{dy}{f(y)}, \int_{-\varepsilon}^0 \frac{dy}{f(y)} < \infty, \text{ for all } \varepsilon > 0, \tag{F$_1$}$$

which corresponds to the special case $f(y) = |y|^\gamma \operatorname{sgn} y$ when $0 < \gamma < 1$, and also the following superlinear condition:

$$0 < \int_\varepsilon^\infty \frac{dy}{f(y)}, \int_{-\infty}^{-\varepsilon} \frac{dy}{f(y)} < \infty, \text{ for all } \varepsilon > 0, \tag{F$_2$}$$

which corresponds to the special case $f(y) = |y|^\gamma \operatorname{sgn} y$ when $\gamma > 1$. These assumptions were introduced systematically in [14]. The coefficient $a(t)$ is allowed to be negative for

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arbitrarily large values of t. Under these circumstances, in general not every solution to the second order nonlinear differential equation (1) is continuable throughout the entire half real axis. For this reason, we confine ourselves with those solutions of (1) that exist and can be continued on some interval of the form $[t_0, \infty)$ where $t_0 \geq 0$ may depend on the particular solution. A solution $y(t)$ is said to be *oscillatory* if it has arbitrarily large zeros, *i.e.* for each $t \in [t_0, \infty)$, there exists $t_1 \geq t$ such that $y(t_1) = 0$. Equation (1) is called *oscillatory* if all continuable solutions are oscillatory. We are here concerned with sufficient conditions $a(t)$ so that all solutions of (1) are oscillatory.

In the linear case, *i.e.* equation (2) when $\gamma = 1$, the most important simple oscillation criterion is the well known Fite-Wintner theorem which states that if $a(t)$ satisfies

$$(\mathrm{A}_0) \qquad \lim_{T\to\infty} A(T) = \lim_{T\to\infty} \int_0^T a(t)\,dt = +\infty,$$

then equation (2) is oscillatory when $\gamma = 1$. Fite [5] assumed in addition that $a(t)$ is non-negative, whilst Wintner [11] in fact proved a stronger result which required a weaker condition on $a(t)$ and involved the integral average of $A(t)$, namely,

$$(\mathrm{A}_1) \qquad \lim_{T\to\infty} \frac{1}{T} \int_0^T A(t)\,dt = +\infty.$$

Clearly, (A_0) implies (A_1). Wintner's result was later improved by Hartman [6] who proved that (A_1) can be replaced by two weaker conditions

$$(\mathrm{A}_2) \qquad \liminf_{T\to\infty} \frac{1}{T} \int_0^T A(t)\,dt > -\infty,$$

and that the limit in (A_1) does not exist, *i.e.*

$$(\mathrm{A}_3) \qquad \liminf_{T\to\infty} \frac{1}{T} \int_0^T A(t)\,dt < \limsup_{T\to\infty} \frac{1}{T} \int_0^T A(t)\,dt.$$

Recently in connection with the study of oscillation theory for linear systems, Butler, Erbe and Mingarelli [3] discovered another new oscillation criterion for the linear equation, namely, condtion (A_2) together with the following

$$(\mathrm{A}_4) \qquad \limsup_{T\to\infty} \frac{1}{T} \int_0^T A^2(t)\,dt = +\infty,$$

are sufficient for oscillation of the linear equation. Note that condition (A_4) is implied by a weaker form of (A_1) upon application of Schwarz's inequality to

$$(\mathrm{A}_5) \qquad \limsup_{T\to\infty} \frac{1}{T} \int_0^T A(t)\,dt = +\infty.$$

Hence the result of Butler, Erbe and Mingarelli also extends the Wintner's condition (A_1).

In the sixties, efforts have been made to show that oscillation criteria for the linear equation remain valid for the Emden-Fowler equation (2) and under appropriate assumptions on $f(y)$ for the more general equation (1). The first result is due to Waltman [12] who proved that the Fite-Wintner condition (A_0) is sufficient for oscillation of (2) when $\gamma > 1$. This result was extended to the more general equation (1) by Bhatia [1] and Wong [13]. The extension of Wintner's condition (A_1) involving integral averages to the nonlinear differential equation (2) remained elusive until Butler's classical work [2] in which he proved that Hartman's theorem remains valid for equation (2), $\gamma > 1$. Furthermore, in the sublinear case either condition (A_3) or (A_5) alone is sufficient for oscillation. For the more general equation, Butler [2] requires rather restrictive conditions on the nonlinear function some of which are difficult to verify except in the Emden-Fowler case (see also [8], [9] and [10]). Recently, we showed [19] that the Butler-Erbe-Mingarelli criterion remained valid for sublinear equation (2), *i.e.* for $0 < \gamma < 1$. The superlinear case, however, remains open.

For the more general sublinear equation (1) subject to (F_2), we have modified in [18] a condition first introduced by Coles [4] to prove extensions of Belohorec's oscillation theorem. Denote $F(y) = \int_0^y \frac{dv}{f(v)}$. We require that

$$(F_3) \qquad f'(y)F(y) \geq \frac{1}{c} > 0, \text{ for all } y.$$

Similarly, we introduce the corresponding superlinear condition

$$(F_4) \qquad f'(y)G(y) \geq d > 1, \text{ for all } y,$$

where $G(y) = \int_y^\infty \frac{dv}{f(v)}$. It is natural to ask whether oscillation criteria for the Emden-Fowler equation (2) can be extended to the more general equation (1) subject to conditions (F_3) or (F_4) on the nonlinear function $f(y)$. The purpose of this paper is to show that both Hartman and Butler-Erbe-Mingarelli oscillation criteria remain valid for the more general equation (1). In particular, we prove that Butler-Erbe-Mingarelli theorem remains valid for equation (2) in the superlinear case. Our main results are

THEOREM 1. *Let $f(y)$ satisfy (F_1) and (F_3). Then conditions (A_2) and (A_3) imply that equation (1) is oscillatory.*

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COROLLARY 3 ([19]) *Conditions (A_2) and (A_4) are sufficient for oscillation of (2) in the sublinear case, i e* $0 < \gamma < 1$

COROLLARY 4 *Conditions (A_2) and (A_4) are sufficient for oscillation of (2) in the superlinear case, i e* $\gamma > 1$

§**2.** In this section, we shall deal with equation (1) in the sublinear case, *i e* when the nonlinear function $f(y)$ satisfies (F_1) and (F_3) We shall prove by contradiction Suppose that equation (1) has nonoscillatory solution $y(t)$ which can, in view of (F_1) and sign condition $yf(y) > 0$ when $y \neq 0$, be assumed to be positive for $t \geq t_0$, where t_0 may depend on the solution $y(t)$ Define $z(t) = F\big(y(t)\big)$ We deduce from (1) that $z(t)$ satisfies the second order nonlinear differential equation

$$(3) \qquad z'' + a(t) + f'(y)z'^{\gamma} = 0$$

Since $f'(y) \geq 0$, we note that the integral of $f'\big(y(t)\big)z'^{\gamma}(t)$ over $[t_0, \infty)$ exists, finite or infinite First we consider the case when $\int_{t_0}^{\infty} f'(y)z'^{2} = K_0 < \infty$ We shall show that in this case $y(t)$ satisfies the following asymptotic behavior

$$(4) \qquad \lim_{t\to\infty} \frac{z(t)}{t} = \lim_{t\to\infty} \frac{F\big(y(t)\big)}{t} = 0$$

Using the Schwarz Inequality, we estimate $F\big(y(t)\big)$ as follows

$$(5) \qquad F\big(y(t)\big) - F\big(y(t_1)\big) = \left|\int_{t_1}^{t} \frac{y'(s)}{f\big(y(s)\big)}\, ds\right| \leq \Big(\int_{t_1}^{t} \frac{f'(y)y'^2}{f(y)^2}\, ds\Big)^{\frac{1}{2}} \Big(\int_{t_1}^{t} \frac{1}{f'(y)}\, ds\Big)^{\frac{1}{2}},$$

where $t_1 \geq t_0$ Since $f(y)z'^2 \in L^1[t_0, \infty)$, for each $\varepsilon > 0$, we can choose t_1 sufficiently large so that

$$(6) \qquad \int_{t_1}^{\infty} \frac{f'(y)y'^2}{f(y)^2}\, ds < \frac{\varepsilon}{4}$$

Using (F_3) and (6) in (5) above, we find

$$(7) \qquad F\big(y(t)\big) \leq F\big(y(t_1)\big) + \frac{\sqrt{c\varepsilon}}{2}\Big(\int_{t_1}^{t} F\big(y(s)\big)\, ds\Big)^{\frac{1}{2}}$$

Suppose that $F\big(y(t)\big) \in L^1(t_1, \infty)$, then $F\big(y(t)\big)$ is bounded by (7) hence (4) is satisfied Otherwise, we can choose $t_2 \geq t_1$ so that $F\big(y(t_1)\big) \leq \frac{\sqrt{c\varepsilon}}{2}\big(\int_{t_1}^{t} F(y)\, ds\big)^{\frac{1}{2}}$ for $t \geq t_2$ which together with (7) yields

$$(8) \qquad F\big(y(t)\big) \leq \sqrt{c\varepsilon}\Big(\int_{t_1}^{t} F\big(y(s)\big)\, ds\Big)^{\frac{1}{2}}$$

Upon integrating (8), we have for $t \geq t_2$

$$(9) \qquad \Big(\int_{t_1}^{t} F(y)\, ds\Big)^{\frac{1}{2}} - \Big(\int_{t_1}^{t_2} F(y)\, ds\Big)^{\frac{1}{2}} \leq \frac{\sqrt{c\varepsilon}}{2}(t - t_2) \leq \frac{\sqrt{c\varepsilon}}{2} t$$

Once again we can choose $t_3 \geq t_2$ so that $\int_{t_1}^{t_2} F(y)\,ds < \frac{c\varepsilon}{4} t^2$ for all $t \geq t_3$. This together with (8) and (9) prove that $z(t) = F\big(y(t)\big) \leq c\varepsilon t$. Since $\varepsilon > 0$ is arbitrary, this proves (4).

We are now ready to complete the proof in case $f'(y)z'^2 \in L^1[t_0, \infty)$. Integrating (3) from t_0 to t, one obtains

$$A(t) = z'(t_0) + A(t_0) - z'(t) - \int_{t_0}^{t} f'(y)z'^2, \tag{10}$$

from which it follows

$$A^2(t) \leq 3\Big\{C_0^2 + z'^2(t) + \Big(\int_{t_0}^{t} f'(y)z'^2\Big)^2\Big\}, \tag{11}$$

where $C_0 = z'(t_0) + A(t_0)$, or simply

$$A^2(t) \leq 3\big(C_0^2 + K_0^2 + z'^2(t)\big). \tag{12}$$

Next we estimate integral of z'^2 as follows

$$\begin{aligned} \frac{1}{T}\int_{t_0}^{T} z'^2 &\leq \frac{1}{T} \operatorname*{Max}_{t_0 \leq t \leq T} \frac{1}{f'\big(y(t)\big)} \int_{t_0}^{T} f'(y)z'^2 \\ &\leq \frac{K_0 c}{T} \operatorname*{Max}_{t_0 \leq t \leq T} F\big(y(t)\big) \end{aligned} \tag{13}$$

By (4), we can choose $t_4 \geq t_0$ such that $|z(t)| \leq t$, $t \geq t_4$; hence

$$\operatorname*{Max}_{t_0 \leq t \leq T} F\big(y(t)\big) \leq \operatorname*{Max}_{t_0 \leq t \leq t_4} z(t) + T \leq K_1 + T. \tag{14}$$

Combining (13) and (14) in (12), we have

$$\frac{1}{T}\int_{t_0}^{T} A^2(s)\,ds \leq 3(C_0^2 + K_0^2) + \frac{3K_0 c}{T}(K_1 + T), \tag{15}$$

where $K_1 = \operatorname{Max}_{t_0 \leq t \leq t_4} z(t)$. Letting T tend to infinity in (15) we obtained the desired contradiction to (A_4) proving Theorem 2 in the case. Next, we return to (10) and take integral average as follows

$$\begin{aligned} \frac{1}{T}\int_{t_0}^{T} A(s)\,ds = C_0\Big(1 - \frac{t_0}{T}\Big) - \frac{1}{T}\big(z(T) - z(t_0)\big) \\ - \frac{1}{T}\int_{t_0}^{T}\int_{t_0}^{T} f'(y)z'^2. \end{aligned} \tag{16}$$

Since $f'(y)z'^2 \in L^1[t_0, \infty)$, we let T tend to infinity in (16) and find on account of (4)

$$\lim_{T\to\infty} \frac{1}{T}\int_{0}^{T} A(s)\,ds = \lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{T} A(s)\,ds = C_0 - K_0,$$

a desired contradiction to (A_3), proving Theorem 1 in this case.

We now return to the case when $f'(y)z'^2 \notin L^1[t_0, \infty)$. Here the integral average of $\int_{t_0}^{t} f'(y)z'^2$ also diverges. It therefore follows (16) and $z(t) > 0$ that $\lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{T} A(s)\,ds = -\infty$, which is incompatible with (A_2). This proves both Theorems 1 and 2 in case $f'(y)z'^2 \notin L^1[t_0, \infty)$.

Since (A_5) implies (A_4) so Corollary 1 follows from Theorem 2. When $f(y) = |y|^\gamma \operatorname{sgn} y$ then (F_3) is satisfied with $c = \frac{1-\gamma}{\gamma} > 0$, thus Corollary 3 follows from Theorem 2.

§**3.** We now return to the proofs of Theorems 3 and 4 for equation (1) in the superlinear case. Define $w(t) = G\big(y(t)\big)$ and use (1) to obtain

$$w'' = a(t) + f'(y)w'^2, \quad t \geq t_0, \tag{17}$$

which is similar to (3) in the sublinear case. Once again we consider the two cases when $f'(y)w'^2 \in L^1[t_0, \infty)$ and $f'(y)w'^2 \notin L^1[t_0, \infty)$. In the first instance, the proof follows much the same as given in the previous section. Here $f'(y)w'^2 \in L^1[t_0, \infty)$ implies that $w(t) = o(t)$ as $t \longrightarrow \infty$ as in (4). Repeating the proof from (5) to (15), we obtain a contradiction to (A_4) proving Theorem 4. On the other hand, we conclude similarly from (16) that $\lim_{T\to\infty} \frac{1}{T}\int_0^T A$ exists as a finite limit which contradicts (A_3), proving Theorem 3 when $f'(y)w'^2 \in L^1[t_0, \infty)$.

The more difficult case for the superlinear equation is when $f'(y)z'^2 \notin L^1[t_0, \infty)$. Here we adopt a simplified version of Butler's proof given in our earlier paper [17]. By (F_4) with $d > 1$, we can choose μ, μ_1 such that $0 < \mu < \mu_1 < 1$ and $\mu d > 1$. Now, we claim

$$\limsup_{t\to\infty}\Big\{\mu_1 \int_{t_0}^t \int_{t_0}^s f'(y)w'^2 - w(t)\Big\} > 0. \tag{18}$$

Denote the double integral in (18) by $\Phi(t)$, hence $\Phi'(t) = \int_{t_0}^t f'(y)w'^2 \longrightarrow \infty$ as $t \longrightarrow \infty$ monotonically and $\Phi''(t) = f'(y)w'^2 \geq 0$. Suppose that (18) does not hold; then there exists $t_1 \geq t_0$ such that

$$\Phi(t) \leq \frac{1}{\mu_1} w(t), \quad t \geq t_1. \tag{19}$$

Using (19), we can estimate as follows

$$\begin{aligned} \sqrt{\frac{\Phi''(t)}{\Phi'(t)}}\sqrt{\frac{\Phi'(t)}{\Phi(t)}} &\geq \frac{\sqrt{f'(y)}w'(t)}{\sqrt{w(t)}}\sqrt{\mu_1} \\ &\geq \sqrt{\mu_1 d}\frac{w'(t)}{w(t)} \end{aligned} \tag{20}$$

Since $\Phi(t_0) = 0$, so there exists $t_2 \geq t_0$ such that $\Phi(t_2) = 1$. Let $\bar{t} = \max(t_1, t_2)$, then $\Phi(\bar{t}) \geq 1$. Now integrate (20) from $\bar{t}$ to t and apply Schwarz's Inequality to the left hand side of (20) to obtain

$$\Big\{\log\frac{\Phi'(t)}{\Phi'(\bar{t})}\log\frac{\Phi(t)}{\Phi(\bar{t})}\Big\}^{\frac{1}{2}} \geq \sqrt{\mu_1 d}\log\frac{w(t)}{w(\bar{t})}. \tag{21}$$

The right hand side of (20) can be estimated from below by (19) as follows:

$$\begin{aligned} \sqrt{\mu_1 d}\log\frac{w(t)}{w(\bar{t})} &\geq \sqrt{\mu_1 d}\log\frac{\mu_1\Phi(t)}{w(\bar{t})} \\ &= \sqrt{\mu_1 d}\{\log\Phi(t) + \log\mu_1 - \log w(\bar{t})\} \\ &\geq \sqrt{\mu d}\{\log\Phi(t) - \log\Phi(\bar{t})\} \end{aligned} \tag{22}$$

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where $\mu_1 > \mu$ and t sufficiently large, say $t \geq t_3 \geq \bar{t}$. Combining (21) and (22), we find

$$\log \frac{\Phi'(t)}{\Phi'(\bar{t})} \geq \mu d \log \frac{\Phi(t)}{\Phi(\bar{t})},$$

whence

$$\Phi'(t)\Phi^{-\mu d}(t) \geq \frac{\Phi'(\bar{t})}{\Phi^{\mu d}(\bar{t})} > 0. \tag{23}$$

Since $\mu d > 1$, a quadrature of (23) gives the desired contradiction, so (18) must hold. Thus there exists a sequence $\{t_k\}$ tending to infinity as $k \longrightarrow \infty$ such that

$$\lim_{k \longrightarrow \infty} \{\mu_1 \Phi(t_k) - w(t_k)\} > 0. \tag{24}$$

Integrating (17) once, we find

$$w'(t) = w'(t_0) + A(t) - A(t_0) + \int_{t_0}^{t} f'(y) w'^2,$$

which can be further integrated from t_0 to t_k and yields

$$w(t_k) = w(t_0) + B_0(t_k - t_0) + \int_{t_0}^{t_k} A(s)\, ds + \Phi(t_k). \tag{25}$$

where $B_0 = w'(t_0) - A(t_0)$. Dividing (25) by t_k and regrouping give

$$\frac{1}{t_k}\Big(-\Phi(t_k) + w(t_k)\Big) = \frac{w(t_0)}{t_k} + B_0\Big(1 - \frac{t_0}{t_k}\Big) + \frac{1}{t_k}\int_{t_0}^{t_k} A. \tag{26}$$

We note from (24) that for sufficiently large k, $\mu_1 \Phi(t_k) \geq w(t_k)$. Substituting this into (26), we have

$$\frac{1}{k}\Big(-\Phi(t_k) + \mu_1 \Phi(t_k)\Big) \geq \frac{w(t_0)}{t_k} + B_0\Big(1 - \frac{t_0}{t_k}\Big) + \frac{1}{t_k}\int_{t_0}^{t_k} A. \tag{27}$$

Let k tend to infinity in (27) above, and note that $\mu_1 < 1$, we find

$$\lim_{k \to \infty} \frac{1}{t_k} \int_{t_0}^{t_k} A(s)\, ds = -\infty,$$

which contradicts (A_2). This completes the proof for both Theorem 3 and Theorem 4 in the case when $f'(y)w'^2 \notin L^1[t_0, \infty)$.

Corollary 2 follows from Theorem 4 in the same manner as in Section 2. For equation (2) with $\gamma > 1$, condition (F_4) becomes

$$f'(y)G(y) \geq \gamma(\gamma - 1)^{-1} = d > 1,$$

and hence Corollary 4 follows.

§**4.** We conclude in this section with several remarks indicating the usefulness of these extensions and the boundary of known results.

REMARK 1. Both Corollary 2 and Corollary 4 imply Butler's theorem [2] that the Wintner's oscillation condition (A_1) remains valid for the Emden-Fowler equation (2) in the superlinear case, *i.e.* $\gamma > 1$.

REMARK 2. To illustrate the usefulness of condition (A_1) involving integral averages, one need only to consider $a(t) = 1 + \sin t + t\cos t$. Here $A(t) = t + t\sin t$ which clearly fails (A_0) but satisfies (A_1).

REMARK 3. On the other hand, let $a(t) = t\cos t$ so $A(t) = t\sin t + \cos t$. Here $A(t)$ fails to satisfy (A_1), but satisfies conditions (A_2) and (A_3) so that equation (1) subject to (F_2) and (F_4) is oscillatory.

REMARK 4. Let $0 < \alpha < 1$. Define $a(t) = t^\alpha \cos t + \alpha t^{\alpha-1}\sin t$ so $A(t) = t^\alpha \sin t$. Again $A(t)$ fails condition (A_1). Although $A(t)$ satisfies (A_2), it fails (A_3). Nonetheless, condition (A_4) is satisfied so we have again oscillation in the superlinear case.

REMARK 5. We now consider the sublinear function $f(y) = |y|^\lambda \operatorname{sgn} y + y$, $0 < \lambda < 1$, (see also [10]). Clearly $f(y)$ satisfies (F_1). Furthermore, $f'(y) = \lambda|y|^{\lambda-1} + 1 \geq 1 > 0$ for all y. Note that for $|y| \leq 1$ we have

$$F(y) = \int_0^{|y|} \frac{dv}{v^\lambda + v} \geq \frac{1}{2}\int_0^{|y|} \frac{dv}{v^\lambda} = \frac{|y|^{1-\lambda}}{1-\lambda},$$

and for $|y| \geq 1$,

$$F(y) = \int_0^{|y|} \frac{dv}{v^\lambda + v} \geq \int_0^{1} \frac{dv}{v^\lambda + v} \geq \frac{1}{2(1-\lambda)}.$$

Hence, for $|y| \leq 1$

$$\begin{aligned} f'(y)F(y) &\geq \frac{|y|^{1-\lambda}}{2(1-\lambda)}(\lambda|y|^{\lambda-1} + 1) \\ &= \frac{1}{2(1-\lambda)}(\lambda + |y|^{1-\lambda}) \geq \frac{\lambda}{2(1-\lambda)}, \end{aligned}$$

and for $|y| \geq 1$

$$f'(y)F(y) \geq \frac{1}{2(1-\lambda)}(\lambda|y|^{\lambda-1} + 1) > \frac{1}{2(1-\lambda)}.$$

In any case, we have $f'(y)F(y) \geq \frac{\lambda}{2(1-\lambda)} > 0$; hence condition ($F_3$) is applicable.

REMARK 6. The sublinear condition (F_1) does not have any requirement about the behaviour of $f(y)$ at infinity. Likewise, the sublinear condition (F_2) does not impose integrability of $f(y)$ near $y = 0$. However, should $\lim_{|y|\to\infty} F(y)$ or $\lim_{|y|\to 0} G(y)$ exist as finite numbers then condition (F_3) or condition (F_4) imply that $f'(y) \geq k > 0$ is bounded below uniformly for all y. In this case the oscillation of equation (1) reduces to that of the linear equation, see [14] and also [8, Theorem A]. As a typical example, we consider

$f(y) = y(|y|^{\frac{1}{2}} + |y|^{-\frac{1}{2}})$ which satisfies both (F_1) and (F_2); hence it is both superlinear and sublinear. On the other hand, $f'(y) = \frac{3}{2}|y|^{\frac{1}{2}} + \frac{1}{2}|y|^{-\frac{1}{2}}$ is positive for all y and indeed have its minimum value $\sqrt{3}$ at $|y| = \frac{1}{3}$. In this case, both (F_3) and (F_4) are therefore satisfied.

We refer the reader to [20], [21] for some other oscillation results for equation (1) by successful application of these nonlinearity conditions.

REMARK 7. Condition (A_5) was known to be sufficient for oscillation in the sublinear case; see [14] and [18]. On the other hand, Kamenev [7] showed that the iterated condition of (A_5), namely

$$(A_6) \qquad \limsup_{T\to\infty} \frac{1}{T^\alpha} \int_0^T (T-t)^\alpha a(t)\,dt = +\infty,$$

for some $\alpha > 1$ alone is sufficient for oscillation of the linear equation, (whilst (A_5) is not). It was also extended to equation (2) in the sublinear case; see [16]. However, it is not known whether (A_6) plus the following compatible condition

$$(A_7) \qquad \liminf_{T\to\infty} \frac{1}{T^\beta} \int_0^T (T-t)^\beta a(t)\,dt > -\infty,$$

for some $\beta \geq 1$ is sufficient for oscillation of equation (2) in the superlinear case. The special case when $\beta = 0$ in (A_7) has been answered in the affirmative in an earlier paper [15]. (see also [8]).

ADDED IN PROOF. A partial answer to the question given in Remark 7 was given in a recent paper of the author, Differential and Integral Equations, **6**(1993), 83–91.

REFERENCES

1. N P Bhatia, *Some oscillation theorems for second order differential equations*, J Math Anal Appl **15**(1966), 442–446

2. G J Butler, *Integral averages and the oscillation of second order ordinary differential equation*, SIAM J Math Anal **11**(1980), 190–200

3. G J Butler, L H Erbe and A B Mingarelli, *Riccati techniques and variational principles in oscillation theory for linear systems*, Trans Amer Math Soc **303**(1987), 263–282

4. W J Coles, *A nonlinear oscillation theorem*, International Conference on Differential Equations, (ed H A Antosiewicz), Academic Press, New York, 1975, 193–202

5. W B Fite, *Concerning the zeros of the solutions of certain differential equations*, Trans Amer Math Soc **19**(1918), 341–352

6. P Hartman, *On nonoscillatory linear differential equations of second order*, Amer J Math **74**(1952), 389–400

7. I V Kamenev, *Integral criterion for oscillations of linear differential equations of second order*, Math Zametki **23**(1978), 249–251

8. M K Kwong and J S W Wong, *Linearization of second order nonlinear oscillation theorems*, Trans Amer Math Soc **279**(1983), 705–722

9. Ch G Philos, *Oscillation criteria for second order superlinear differential equations*, Canad J Math **41**(1989), 321–340

10. ———, *Integral averages and oscillation of second order sublinear differential equations*, Diff and Integral Equations **4**(1991), 205–213

11. A Wintner, *A criterion of oscillatory stability*, Quarterly J Appl Math **7**(1949), 114–117

12. P Waltman, *An oscillation criterion for a nonlinear second order equation*, J Math Anal Appl **10**(1965), 439–441

13. J S W Wong, *On two theorems of Waltman*, SIAM J Appl Math **14**(1966), 724–728

14. ———, *Oscıllatıon theorems for second order nonlınear dıfferentıal equatıons*, Bull Inst Math Acad Sınıca **3**(1975), 283–309

15. ———, *An oscıllatıon crıterıon for second order nonlınear dıfferentıal equatıons*, Proc Amer Math Soc **98**(1986), 109–112

16. ———, *An oscıllatıon crıterıon for second order sublınear dıfferentıal equatıons*, Conference Proceedıngs, Canad Math Soc **8**(1987), 299–302

17. ———, *Oscıllatıon theorems for second order nonlınear ordınary dıfferentıal equatıons*, Proc Amer Math Soc **106**(1989), 1069–1077

18. ———, *A sublınear oscıllatıon theorem*, J Math Anal Appl **139**(1989), 408–412

19. ———, *An oscıllatıon theorem for second order sublınear dıfferentıal equatıons*, Proc Amer Math Soc **110**(1990), 633–637

20. ———, *Oscıllatıon of sublınear second order dıfferentıal equatıons wıth ıntegral coeffıcıents*, J Math Anal Appl **162**(1991), 476–481

21. ———, *Oscıllatıon crıterıa for second order nonlınear dıfferentıal equatıons wıth ıntegrable coeffıcıents*, Proc Amer Math Soc **115**(1992), 389–395

Chınney Investments Lımıted
1218 Swıre House
Hong Kong

Department of Mathematıcs
Unıversıty of Scıence and Technology
Hong Kong