MAPPINGS FOR THE FIRST ORDER ASTEROIDAL RESONANCE

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Abstract. We construct a two step algebraic mapping from Sessin's simplified model for the first order resonance. The orbits obtained with this mapping are compared to the ones calculated with the exact solution. We also derive a reduced Hamiltonian. A plane Poincaré mapping, using delta periodic function, is constructed and compared to the reduced Hamiltonian contour curves showing the splitting of the separatrix due to delta perturbation technique.

Key words: Mapping, sympletic integrator, resonance, asteroid.

1. The Two step Mapping

Sessin and Ferraz-Mello, 1984, integrable Hamiltonian for the first order resonance can be written as:

$$F(\phi,\theta,k,h) = \frac{F_{02}}{L_0^2}\phi^2 - \varepsilon\left[\left(\frac{A_1}{\sqrt{L_0}}k - A_2k_j\right)cos\theta + \left(\frac{A_1}{\sqrt{L_0}}h - A_2h_j\right)sin\theta\right] \quad (1)$$

where $\theta = p\lambda - (p+1)\lambda_j$ is the critical angle, λ and λ_j are the mean longitudes of the asteroid and Jupiter, respectively, k, h, k_j and h_j are the usual Poincaré variables, A_1 and A_2 are combinations of the Laplace coefficients calculated at a_0 , $F_{02} = 2p^2a_j^2/a_0^2$, $L_0 = \sqrt{\mu a_0}$, and a_0 is the resonant semi-major axis. The subscript j refers to Jupiter. ε is the mass ratio small parameter. This Hamiltonian has a first integral besides the energy as it was shown by Sessin and Ferraz-Mello, 1984. Applying Chirikov's, 1972, technique following Wisdom, 1982 and 1983, taking the Keplerian motion perturbed by the resonant terms, we obtain the following two step mapping (Stuchi, 1991):

step one :

$$k_1 = k_0 \quad (2a)$$

$$h_1 = h_0 + \varepsilon\frac{2\pi}{\Omega}\gamma\left(\frac{A_1}{\sqrt{L_0}}\right)cos\theta_0 \quad (2b)$$

$$\phi_1 = \phi_0 + \varepsilon\frac{2\pi}{\Omega}\gamma\left(\frac{A_1k_0}{\sqrt{L_0}} - A_2k_j\right)sin\theta_0 \quad (2c)$$

$$\theta_1 = \theta_0 - 2\gamma\frac{F_{02}}{L_0^2}\phi_1\left(\frac{\pi}{2\Omega}\right) \quad (2d)$$

S. Ferraz-Mello (ed.), Chaos, Resonance and Collective Dynamical Phenomena in the Solar System, 391–394.

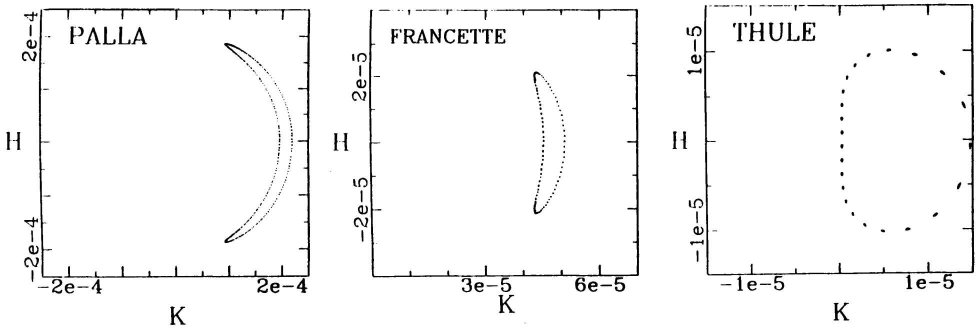

Fig. 1.
Projection on the (K,H) plane of the orbits obtained with the two-step mapping with $e_j = 0$ and $\Omega = 10n_j$.

step two :

$$k_2 = k_1 + \varepsilon\frac{2\pi}{\Omega}\gamma\left(\frac{A_1}{\sqrt{L_0}}\right) sin\theta_1 \tag{2e}$$

$$h_2 = h_1 \tag{2f}$$

$$\phi_2 = \phi_1 + \varepsilon\frac{2\pi}{\Omega}\gamma\left(\frac{A_1 h_1}{\sqrt{L_0}} - A_2 h_j\right) cos\theta_1 \tag{2g}$$

$$\theta_2 = \theta_1 - 2\gamma\frac{F_{02}}{L_0^2}\phi_2\left(\frac{\pi}{2\Omega}\right) \tag{2h}$$

where γ is a scale parameter.

It is easy to show that an equivalent one step sympletic integrator can be derived using Channel and Scovel, 1989 method. Both (2) and this equivalent mapping are first order sympletic integrators and give the same result. In fact, we obtained a good agreement when comparing orbits obtained with (2) and the exact solutions of (1), mainly when the mapping period (or step) is taken as one tenth of Jupiter's period and $e_j = 0$. This is shown in figure 1.

If e_j is varied from zero to its usual value .048 the orbit gets fuzzier but its average, as indeed expected, agrees with the averaged hamiltonian (1) as shown in figure 2. We note that we are plotting the variables (K,H) instead of (k,h).

2. The Reduced Hamiltonian

Hamiltonian (1) can be written in the canonical set of variables (G, θ', K, H) as

$$F = \frac{1}{2}A\left(G + \frac{H^2 + K^2}{2}\right)^2 - \tau K \tag{3}$$

where $\theta' = \theta + \omega_j$, A and τ are related to A_1 A_2 and F_{02}.

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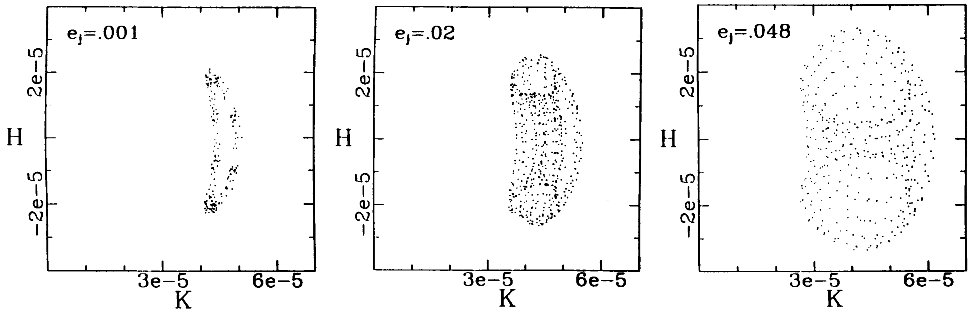

Fig. 2.
Projection of Francette (3:2) on the (K,H) plane when e_j is varied from .001 to its actual value.

According to Whittaker's method (Guckenheimer & Holmes, 1983) Hamiltonian (3) can be reduced to a one degree of freedom Hamiltonian parametrized by the energy E, since G is a second integral of motion

$$G(H,K;E) = -\frac{H^2+K^2}{2} \pm A'\sqrt{E+\tau K} \tag{4}$$

where $A' = \sqrt{2/A}$. It is important to notice that $\dot{\theta} = \partial F/\partial G$ does not become zero for all orbits inside the branch of the separatrix described by G taken with the minus sign. This branch contains the hyperbolic fixed point.

The reduced Poincaré mapping is derived from

$$\frac{dK}{d\theta'} = H, \quad \frac{dH}{d\theta'} = -K - \frac{A'\tau}{2}(E+\tau K)^{-\frac{1}{2}} \tag{5}$$

when the solution is evaluated at $\theta' = 2K\pi$. The analytical solution of these equations are not easily attained. Therefore, we again use the periodic delta function to derive a plane Poincaré mapping which is given by

$$K_{n+1} = K_n cos\alpha + \left(H_n - \frac{A'\tau}{4}(E+\tau K_n)^{-\frac{1}{2}}\right) sin\alpha \tag{6a}$$

$$H_{n+1} = -K_n sin\alpha + \left(H_n - \frac{A'\tau}{4}(E+\tau K_n)^{-\frac{1}{2}}\right) cos\alpha - \frac{A'\tau}{4}(E+\tau K_{n+1})^{-\frac{1}{2}} \tag{6b}$$

where $\alpha = \frac{2\pi}{\Omega}$ is the mapping period and Ω is taken as 6.77582 i.e., the value which makes the equations for fixed points of (5) and (6) compatible. Figure 3 shows the orbits calculated with mapping (6) for the 3/2 case with $E = 0.0015$ (we note that (K,H) have a different scale from section 1) and enlarged views of a neighborhood of the hyperbolic fixed point for the lowest energy values at which homoclinic tangles are observed. It is clear that in the 2/1 resonance this occurs at a smaller value than the 3/2 case. In both cases the stochastic layer increases with the value of the energy which acts as perturbation parameter.

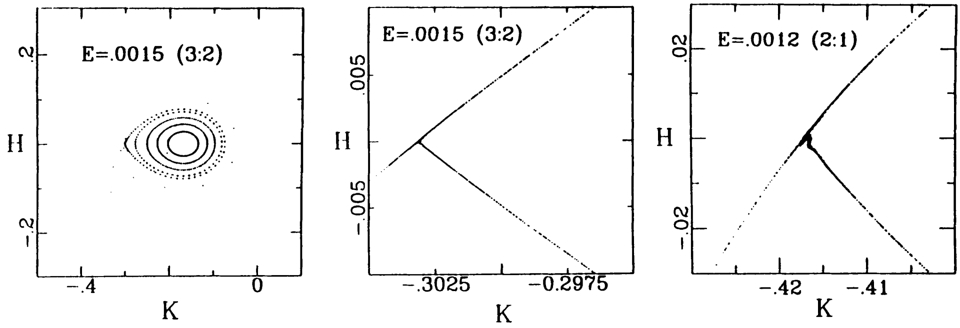

Fig. 3.

a) Orbits of a 3/2 asteroid obtained with the reduced mapping; b) the enlarged view of the separatrix near the hyperbolic fixed point of a); c) the same as b) for a 2:1 asteroid.

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