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ASYMPTOTIC MAXIMUM PRINCIPLES FOR SUBHARMONIC AND PLURISUBHARMONIC FUNCTIONS

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Let Ω be a bounded open set in $\mathbf{R}^n$. An immediate consequence of the maximum principle is that if s is a function continuous on $\bar{\Omega}$ and subharmonic on Ω, then

(1) $\sup_{\Omega} s = \sup_{\partial\Omega} s.$

Of course (1) is no longer true if Ω is not bounded. For example in $\mathbf{C} \sim \mathbf{R}^2$ consider the functions

$$s(z) = |z| \quad \text{in } \Omega = \{\rho < |z| < +\infty\} \quad \text{and}$$

$$s(z) = |e^z| \quad \text{in } \Omega = \{\text{Re } z > 0\}.$$

However, if we restrict the growth of s, then (1) may still hold even if the open set Ω is no longer bounded and such is the theme of Phragmèn-Lindelöf type theorems. If we assume even more, namely, that s is upper-bounded, then we can again infer (1) for unbounded open sets Ω. We shall return to this point later.

In the present note, we wish to prove (1) for an arbitrary subharmonic function s on an open subset Ω of $\mathbf{R}^n$. In particular, we do not assume that s is bounded or even of restricted growth. Rather, we impose restrictions on the (possibly unbounded) set Ω.

The following is a particular case of one of our results, but we state it here as an appetizer before pausing for various preliminaries.

THEOREM 1. *Let Ω be an open set in $\mathbf{R}^n$ and suppose that ∞ is not accessible from Ω. Then, for any function s subharmonic in Ω, we have*

$$\sup_{\Omega} s = \sup_{\partial\Omega} s.$$

Notice that we have not even assumed that s is defined (let alone continuous) on $\partial\Omega$. But then, we adopt the convention that

(2) $\sup_{\partial\Omega} s = \sup_{y \in \partial\Omega} \left\{ \overline{\lim_{x \to y}}\, s(x) \right\}.$

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It follows that if s is upper-bounded on $\partial\Omega$, then s is also upper-bounded in Ω and by the same bound.

The results of this paper generalize a lemma of Stray [9] and the ideas are close in spirit to the proof of Iverson's Theorem that any non-constant entire function has ∞ as an asymptotic value. The present investigation was originally motivated by an explicit approximation problem (see [7]) and Stray himself used his above-mentioned lemma to solve one of the most fundamental problems in the qualitative theory of approximation in the complex domain. Namely, Stray gave a characterization of those functions which can be approximated uniformly by entire functions [6]. As an application of our results, we shall state conditions which are necessary for approximation by other classes of functions, for example, by harmonic or holomorphic functions of several variables.

Henceforth, M will denote a second-countable differentiable connected manifold of real dimension n. If in addition M is endowed with a complex analytic structure, we shall simply say that M is a complex manifold. We denote by $M^* = M \cup \{*\}$ the one-point compactification of M, where $* = *_M$ is the ideal point at infinity for M. For a subset $X \subset M$, we denote the boundary of X in M by $\partial_M X$ and if no confusion is likely, we shall merely write ∂X in place of $\partial_M X$.

Let Ω be an open subset of M and $s: \Omega \to [-\infty, +\infty]$ an u.s.c. (upper semi-continuous) function. We shall say that s satisfies the *classical maximum principle* if s is necessarily constant in the neighborhood of each point at which it attains a local maximum. We shall say that s satisfies the boundary maximum principle on Ω if

$$\sup_{\Omega} s = \sup_{\partial\Omega} \bar{s},$$

where

$$\bar{s}(x) = \begin{cases} s(x), & \text{if } x \in \Omega, \\ \overline{\lim}_{y \to x} s(y), & \text{if } x \in \partial\Omega, \end{cases}$$

is the smallest u.s.c. extension of s to $\Omega \cup \partial\Omega \equiv \bar{\Omega}$.

It is well-known that if $\bar{\Omega}$ is compact in M, then the classical maximum principle implies the boundary maximum principle. Our goal is to characterize (non-compact) open sets Ω for which this implication still holds. We shall obtain a sufficient condition and show that in many cases it is also necessary.

The ideal point $* = *_M$ is said to be *accessible* from a set $\Omega \subset M$ if there is a continuous path, $\lambda:[0, +\infty] \to \Omega$, such that

$$\lim_{t \to \infty} \lambda(t) = *_M.$$

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We may now state our main result which generalizes Stray's lemma [**6**].

THEOREM 2. *Let Ω be an open subset of M and suppose that $*_M$ is not accessible from Ω. Let s be* u.s.c. *in Ω and satisfy the classical maximum principle. Then, s also satisfies the boundary maximum principle*

(i) $\sup_{\Omega} s = \sup_{\partial_M \Omega} \bar{s}$,

and moreover, if Ω is not relatively compact, then

(ii) $\displaystyle\varlimsup_{\substack{x \to *_M \\ x \in \Omega}} s(x) = \varlimsup_{\substack{x \to *_M \\ x \in \partial_M \Omega}} \bar{s}(x).$

In particular, since subharmonic and plurisubharmonic functions satisfy the classical maximum principle, we have the following special but most important cases.

COROLLARY 1. *Let Ω be an open subset of Riemannian (respectively complex) manifold M and suppose that $*_M$ is not accessible from Ω. Then, each function s subharmonic (respectively plurisubharmonic) on Ω satisfies* (i) *and* (ii).

Remark 1. There is a sort of converse to the Theorem. Suppose $*_M$ is accessible from Ω, say there is a path $\gamma:[0, \infty) \to \Omega$ converging to $*_M$. Then set

$$s(x) = 1 - e^{-t} \quad \text{if } \gamma(t) = x,$$

and $s(x) = 0$ if x is not on the path γ. Then s does not satisfy the boundary maximum principle, that is, both (i) and (ii) fail.

There is also a sort of converse to Corollary 1. Suppose M is a domain of $\mathbf{R}^n$ or a Riemann surface. Then if Ω is any open subset of M from which $*_M$ is accessible, we shall construct a harmonic function s on Ω such that

$$\sup_{\Omega} s > \sup_{\partial_M \Omega} s.$$

Indeed, there exists a simple path γ in Ω which tends to $*_M$. We may construct a neighbourhood V of γ, $\gamma \subset V \subset \bar{V} \subset \Omega$, such that setting $F = \gamma \cup \Omega \backslash V$, we have that $\Omega^* \backslash F$ is connected and locally connected. Set $u = 1$ on γ and $u = 0$ on $\Omega \backslash V$. Then by an approximation theorem of Gauthier-Goldstein-Ow [**6**] (in case M is a domain in $\mathbf{R}^n$) and Bagby [**1**] (in case M is a Riemann surface), there is a harmonic function s on all of Ω such that

$$|u - s| < \frac{1}{2}$$

on F. Clearly, the conclusion of Corollary 1 fails for s on Ω.

An analogous approximation theorem for general Riemannian manifolds is not yet available, however this problem is currently being investigated [2].

Remark 2. In Corollary 1 we wonder whether or not it is superfluous to have mentioned plurisubharmonic functions on complex manifolds. It may (or may not) be that given any plurisubharmonic function s on a complex manifold M, there is a Riemannian structure on M with respect to which s is subharmonic. Of course, this is locally true. Given a function s on a manifold M, Calabi [4] has considered the problem of whether there exists a Riemannian structure for which s is harmonic. We are unaware, however, whether the analogous problem has been treated for subharmonicity.

Remark 3. In view of (i) above, one may be tempted by the beautiful conjecture that for a subharmonic function, the sup may be calculated on the set of accessible boundary points. Thus, the "accessible boundary" would be a sort of "boundary" in the sense of function algebras.

For a bounded subharmonic function on a bounded domain Ω in $\mathbf{R}^n$, this is indeed the case, for the set I of inaccessible boundary points is of harmonic measure zero (see [3]). One way to see this is to bear in mind that the harmonic measure of I evaluated at a point $x \in \Omega$ can be thought of as the probability that a Brownian path originating at x first hit $\partial\Omega$ in I. However, it is not only improbable, but indeed impossible, for such a Brownian path to hit any point $y \in I$, for Brownian motion is continuous and so such a y would ipso facto be accessible.

On the other hand, one can construct a domain Ω and an unbounded subharmonic function s on Ω such that s remains bounded on the accessible boundary. Indeed, let Ω be the domain in $\mathbf{C}$ which is bounded by the curves

$$x = -1, \quad x = 0, \quad y = (1/x)\sin(1/x) \quad \text{and}$$

$$y = (1/x)\sin(1/x) + 1.$$

Let φ be a conformal mapping of Ω onto the right half-plane such that the prime end $x = 0$ is mapped to the point at infinity. Then $s = \operatorname{Re} \varphi$ vanishes at all accessible boundary points of Ω, but of course s is not upper-bounded.

Remark 4. As mentioned in the introduction, one can obtain certain maximum principles if one assumes (which we have not) that s is upper-bounded. The following is a nice example of such a result.

MAXIMUM PRINCIPLE ([8, p. 368]). *Let Ω be a domain with smooth boundary $\partial\Omega \neq \emptyset$ in a parabolic Riemannian manifold M. If s is a function continuous on $\bar{\Omega}$, harmonic on Ω, and upper-bounded, then* (1) *holds for s.*

In the above theorem, s is assumed to be upper-bounded. However, in our results, if s is bounded on $\partial\Omega$, we may infer that s is bounded, and by the same bound.

The above maximum principle can be viewed as a particular instance of the more general principle that if $*_M$ is of harmonic measure zero with respect to Ω, then (1) holds for upper-bounded subharmonic functions s on Ω.

If Ω is the unit ball for instance, in $\mathbf{R}^n$, it is impossible to replace the hypothesis that s be bounded by the hypothesis that s grow slowly. Indeed, Gaidenko [5] has shown that for any positive function $p(r)$ increasing to infinity on $[0, 1)$, there exists a closed set E on the unit sphere in $\mathbf{R}^n$ and a non-constant function u harmonic in the unit ball Ω, continuous on $\bar{\Omega}\backslash E$, zero on $\partial\Omega\backslash E$, and such that $u(x) \leqq p(|x|)$. Set $M = \mathbf{R}^n\backslash E$. Then $u = 0$ on $\partial_M\Omega$, u grows slowly in Ω, but $u \not\equiv 0$. Thus, the boundary maximum principle does not hold for harmonic functions on Ω which grow slowly as $x \rightarrow *_M$.

On the other hand, if Ω is a half-space, then Phragmen-Lindelöf principles yield (1) under the assumption that s be of restricted growth.

Proof of Theorem 2. To prove (i), we may assume that Ω is connected. Since M is second countable, we may choose a Riemannian metric d on M. For any point $x \in M$ and $r > 0$, we denote by

$$B(x, r) = \{y \in M: \quad d(x, y) < r\}$$

and

$$\bar{B}(x, r) = \{y \in M: \quad d(x, y) \leqq r\}$$

the open and closed balls respectively of center x and radius r.

We claim that if $B(x, r) \subset \Omega$, then $\bar{B}(x, \rho)$ is compact for each $0 < \rho < r$. To see this, let $\{K_j\}$ be an exhaustion of M by compact sets with $K_j \subset K_{j+1}^0$, for each j. It is enough to show that $B(x, \rho) \subset K_j$ for some j. Suppose not. Then, for each j, there exists $x_j \in B(x, \rho)\backslash K_j$. Let γ_j be a path from x_{j-1} to x_j within $B(x, \rho)$. We form a path γ by joining γ_1 to $\gamma_2 \ldots \gamma_{j-1}$ to γ_j, etc. We shall modify γ so that it eventually leaves each K_j. Suppose some subsequence $\{\gamma_{j(i)}\}$ of $\{\gamma_j\}$ meets K_0. Let a_i be the last point of $\gamma_{j(i)}$ which meets K_1. We may assume that $\{a_i\}$ converges to some point

$$y_1 \in \bar{B}(x, \rho) \cap \partial K_1.$$

Thus, we may choose $\epsilon_1 > 0$ such that

$$B(y_1, \epsilon_1) \subset B(x, r)\backslash K_0.$$

Now we replace that portion of γ from a_i to a_{i+1} by a path $\gamma_{1,i}$ from a_i to $x_{j(i)}$ and back to a_i and then to a_{i+1} within $B(y_1, \epsilon_1)$. Thus, γ has been modified so that it still has points $x_{j(i)}$ outside of $K_{j(i)}$ and is eventually completely outside of K_0.

In a similar way, we can modify γ further along so that it is eventually outside of K_1. By induction we may in fact modify γ so that it is eventually outside of each K_j. But now γ tends to $*_M$ in Ω which contradicts our assumptions. This proves our assertion that $\bar{B}(x, \rho)$ is compact for $\rho < r$ if $B(x, r) \subset \Omega$.

For any point $x \in \Omega$, we denote by $r(x)$ the radius of the largest ball centered at x and contained in Ω. Since $*_M$ is not accessible from Ω, an argument similar to the construction of γ above shows that Ω cannot be all of M. Hence $r(x)$ is finite for each $x \in \Omega$.

To prove (i) it is sufficient to prove that the left side is no greater than the right side, for the opposite inequality is trivial. Fix, then, a point $x_0 \in \Omega$. Let B_0 be the ball of radius $r(x_0)/2$ centered at x_0. Since s satisfies the classical maximum principle, s attains its maximum on $\bar{B}_0$ at some point $x_1 \in \partial B_0$ which we may assume to be at a minimal distance from $\partial\Omega$. In the same way, we denote by B_1 the ball of radius $r(x_1)/2$ centered at x_1. We may choose a point $x_2 \in \partial B_1$ at which s attains its maximum for $\bar{B}_1$ and such that $d(x_2, \partial\Omega)$ is minimal. In this manner, we construct a sequence $\{x_j\}$ such that for each j, B_j is the ball centered at x_j of radius $r(x_j)/2$, $x_{j+1} \in \partial B_j$, and

$$\max_{\bar{B}_j} s = s(x_{j+1}),$$

and $d(x_{j+1}, \partial\Omega)$ is minimal.

We show that the sequence $\{x_j\}$ cannot have an accumulation point in Ω. Indeed, if x_∞ were such a point, then we would have a subsequence $\{x_{j(k)}\}$ converging to x_∞. Suppose s is eventually constant on the subsequence $\{x_{j(k)}\}$. Then s is eventually constant on the sequence $\{x_j\}$ itself and so

$$d(x_j, \partial\Omega) \to 0$$

which contradicts x_∞ being a limit point in Ω. Thus, we may assume that s is not eventually constant on $\{x_{j(k)}\}$ and since s is non-decreasing on this sequence, it follows from semi-continuity that

$$s(x_j) < s(x_\infty)$$

for all j. However, for large k, $x_\infty \in B_{j(k)}$, and so

$$s(x_\infty) \leqq s(x_{j(k)+1}),$$

which is a contradiction. Thus, the sequence $\{x_j\}$ has no accumulation point in Ω.

Suppose $\{x_j\}$ has an accumulation point $y \in \partial\Omega$. Then

(3) $s(x_0) \leqq \overline{\lim}_{x \to y} = \bar{s}(y).$

It remains to consider the case where $\{x_j\}$ has neither accumulation points in Ω nor on $\partial_M\Omega$. Then $x_j \to *_M$. Choose a path γ_j from x_j to x_{j+1} with

$$l(\gamma_j) \leqq r_j/2 + 1/j,$$

where $l(\gamma_j)$ denotes the length of γ_j and $r_j = r(x_j)$. Suppose for a subsequence (which we continue to denote by $\{x_j\}$) there is a point $y_j \in \gamma_j \cap K$, where K is a fixed compact subset of M. Then either

$$(4) \quad \begin{cases} d(x_j, y_j) \leqq r_j/4 + 1/(2j) & \text{or} \\ d(x_{j+1}, y_j) \leqq r_j/4 + 1/(2j). \end{cases}$$

On the other hand

$$r_j \leqq d(x_j, y_j) + r(y_j) \leqq (r_j/2 + 1/j) + r(y_j),$$

and hence

(5) $r_j/2 \leqq 1/j + r(y_j).$

Now we may assume that y_j converges to some point $a \in K$. Suppose $a \in \Omega$. Then (4) and (5) yield

$$\underline{\lim}_{j \to \infty} d(x_j, a) \leqq r(a)/2.$$

But then for infinitely many j, $x_j \in \bar{B}(a, r(a)/2)$ which is compact. This is impossible and so $a \in \partial_M\Omega$. In fact the previous argument shows that no subsequence of y_j converges to a point of Ω. From (5) we have that $r_j \to 0$ for the sequence of x_j with γ_j meeting K. From (4), $\{x_j\}$ has an accumulation point on K. This is impossible since $x_j \to *_M$.

This contradiction came from assuming that γ_j meets K for infinitely many j. Thus the path γ formed by joining the γ_j's is eventually outside of K. Since K was an arbitrary compact subset of M, it follows that $\gamma \to *_M$. This contradicts our hypotheses, and so the only possibility is (3). This concludes the proof of (i).

To prove (ii) assume the contrary. Let $K_1 \subset K_2 \subset \dots$ be an exhaustion of M by compact sets such that for each $x \in K_j$ there is a neighbourhood $U = U_x$ such that $U \setminus K_j$ has only finitely many components. Such an exhaustion always exists; for example, an exhaustion by sets having smooth boundaries has this property.

If (ii) fails then there is some j_0 such that

$$\overline{\lim_{\substack{x \to *_M \\ x \in \Omega}}} s(x) = \overline{\lim_{\substack{x \to *_M \\ x \in \Omega \setminus K_{j_0}}}} s(x) > \sup_{x \in \partial_M\Omega \setminus (K_{j_0})^0} \bar{s}(x) = m.$$

This inequality remains true if we replace j_0 by any $j > j_0$.

First step. Fix $j \geqq j_0$. Let $\mathscr{A}_j$ be the collection of all connected components A of $\Omega \setminus K_j$ with

$$\sup_A s > m.$$

The family $\mathscr{A}_j$ cannot be empty, and we claim that it is finite. From (i) it follows that for each $A \in \mathscr{A}_j$, there is a point $x \in \partial_M A$ such that $\bar{s}(x) > m$. This point must lie on ∂K_j. If $\mathscr{A}_j$ were infinite, then there would be a limit point of such points on $\partial_M \Omega \cap \partial K_j$, which is impossible by the u.s.c. of $\bar{s}$ (and since $\sup \bar{s} = m$ on $\partial_M \Omega \setminus K_j$).

Second step. Each set $A \in \mathscr{A}_k$ is contained in some set $A' \in \mathscr{A}_j$ for $j < k$. Since each $\mathscr{A}_j$ is finite, we get a sequence of sets

$$A_1 \supset A_2 \supset \dots, \quad A_i \in \mathscr{A}_{j_0+i}.$$

Choose now any point $x_i \in A_i$ and connect the points x_i, x_{i+1} in A_i. Thus we get a path to $*_M$. This contradiction finishes the proof.

We shall now apply the above asymptotic maximum principle to obtain some information in approximation theory.

In the following discussion, M will denote (as in Corollary 1) a Riemannian (respectively complex) manifold and H will denote any subsheaf of the sheaf of germs of complex-valued harmonic (respectively pluriharmonic) functions on M such that sections of H are closed under local uniform convergence. For example, H could be the sheaf of germs of complex-valued or of real-valued harmonic functions on a Riemannian manifold or of pluriharmonic or holomorphic functions on a complex manifold. The results are valid in all cases.

Let $F \subset M$ be closed in the topology of M. We denote by $A_M(F)$ the functions on F which are uniform limits on F of sequences with elements from $H(M)$. Stray [**9**] has given a description of $A_M(F)$ in the holomorphic case for a rather general class of open subsets M of C. For other sheaves H and for more general M, the problem is open. However, we shall make a few remarks about the general situation.

Following Stray, we denote by $\Omega(F)$ the set of all $x \in M \setminus F$ such that there is no continuous path $\gamma:[0, +\infty) \to M \setminus F$ such that

$$\gamma(0) = x \quad \text{and} \quad \gamma(t) \to *_M \quad \text{as } t \to +\infty.$$

Stray made use of the "asymptotic hull" $\tilde{F} = F \cup \Omega(F)$ to obtain some deep results in approximation theory. In this context, a key lemma is the following result of Stray [**9**] which we state in a more general form.

COROLLARY 2. *Let F and K be respectively closed and compact subsets of a manifold M and suppose $w \in H(U)$ for some open $U \supset \widetilde{K \cup F}$. Then,*

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(i) $\sup_{\Omega(F)} |w| \leqq \sup_{F} |w|,$

and moreover, if F is not compact, then

(ii) $\overline{\lim}_{x \to *_M, z \in \Omega(K \cup F)} |w(x)| \leqq \sup_{F} |w|.$

Proof. We set $\Omega = \Omega(K \cup F)$ in Theorem 2. Then, by (i) of Theorem 2,

$$\sup_{\Omega(K \cup F)} |w| = \sup_{\partial_M[\Omega(K \cup F)]} |w| \leqq \sup_{K \cup F} |w|.$$

This proves (i).

From (ii) of Theorem 2,

$$\overline{\lim}_{x \to *_M, x \in \Omega(K \cup F)} |w(x)| \leqq \sup_{\partial_M[\Omega(K \cup F)] \setminus K} |w| \leqq \sup_{F} |w|.$$

This establishes (ii).

This corollary has itself a corollary giving a necessary condition for uniform approximation by global sections.

COROLLARY 3. *A continuous function w on a closed subset F of a manifold M can be uniformly approximated on F by functions in $H(M)$ only if w extends to a function $\tilde{w}$ continuous on $\tilde{F}$ whose restriction to $(\tilde{F})^0$ is in $H((\tilde{F})^0)$.*

Proof. If $\{w_n\}$ is a sequence of functions in $H(M)$ which converges uniformly on F to w, then $\{w_n\}$ is uniformly Cauchy on F. It follows from Corollary 2 (i) with $K = \emptyset$, that $\{w_n\}$ is also uniformly Cauchy on $\tilde{F}$. Thus $\{w_n\}$ converges uniformly on $\tilde{F}$ to a function $\tilde{w}$ which is the proclaimed extension.

Classical inverse theorems of approximation tell us that the faster a function can be approximated by nice functions, the smoother that function must be. The smoothness is sometimes stated as the possibility of extending the function holomorphically to a larger set. In this light, the above corollary may be viewed as a guarantee that a given function can be holomorphically (or pluriharmonically, etc.) extended based on the mere possibility of approximation, without regard to speed. But for this extension to be non-trivial we must impose conditions on the initial set on which the approximation takes place, namely, $F \neq \tilde{F}$.

The above corollary is related to the inverse theorems of approximation much as the asymptotic maximum principle is related to Phragmen-Lindelöf theorems. The inverse theorems and Phragmen-Lindelöf theorems are quantitative results involving speed of approximation and speed of growth respectively. The above corollary and the asymptotic maximum principle are qualitative results which disregard speed. However, geometric hypotheses are imposed.

Added in proof. After the present paper was submitted, we learned of a paper by R. Sh. Sahakian, [*On a generalization of the maximum principle* (Russian), Izv. Akad. Nauk Arm. SSR Mat. *22* (1987), 94-101], which presents a less general version of our result.

THEOREM OF SAHAKIAN. *Let Ω be a domain of the extended complex plane $\bar{\mathbf{C}}$ and e a closed subset of $\partial\Omega$. If e is not accessible from Ω, then*

$$(6) \quad \sup_{\Omega} s = \sup_{\partial\Omega\setminus e} \bar{s}$$

for each continuous subharmonic function s on Ω. Conversely, if e is accessible from Ω, then there exists a function f, holomorphic in Ω, such that (6) *fails for $s = |f|$.*

The first part is a particular case of our Corollary 1 (setting $M = \bar{\mathbf{C}}\setminus e$). The second part is proved in an analogous fashion to our Remark 1.

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