J. Austral. Math. Soc. (*Series A*) **52** (1992), 11-25

CONTINUOUS SELECTION THEOREM, COINCIDENCE THEOREM AND INTERSECTION THEOREMS CONCERNING SETS WITH H-CONVEX SECTIONS

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(Received 31 January 1990; revised 30 June 1990)

Communicated by J. H. Rubinstein

Abstract

A continuous selection and a coincidence theorem are proved in H-spaces which generalize the corresponding results of Ben-El-Mechaiekh-Deguire-Granas, Browder, Ko-Tan, Lassonde, Park, Simon and Takahashi to noncompact and/or nonconvex settings. By applying the two theorems, some intersection theorems concerning sets with H-convex sections are obtained which generalize the corresponding results of Fan, Lassonde and Shih-Tan to H-spaces. Some applications to minimax principle are given.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*) 54 C 65, 54 H 25, 52 A 07.
Keywords and phrases: Continuous selection, coincidence, H-space, contractible, weakly H-convex, H-compact, $H - KKM$, compactly closed (open), upper semi-continuous, minimax principle.

1. Introduction

In our recent papers [7, 9], we have obtained some new matching theorems, fixed point theorems and minimax inequalities. By applying a minimax inequality in [7], some non-convex generalizations of well-known intersection theorems concerning sets with convex sections were proved in [8], but we would have to assume that the product space is a H-space.

In the present paper, we shall first show a continuous selection theorem, an $H-KKM$ theorem and a coincidence theorem which improve and generalize

This project was supported by the National Natural Science Foundation of China.

the corresponding results of Ben-El-Mechaiekh-Deguire-Granas [4], Browder [6], Ding-Tan [10], Ko-Tan [16], Lassonde [17], Park [19], Simon [20], and Takahashi [23] to noncompact and nonconvex settings. Next by applying our earlier results, some intersection theorems concerning sets with H-convex sections are obtained without the assumption that the product space is a H-space. These theorems generalize those of Fan [10, 12], Lassonde [17] and Shih-Tan [22] to noncompact and nonconvex settings. Some applications are given.

2. Preliminaries

Let X be a nonempty set; we shall denote by 2^X the family of all subsets of X and by $\mathscr{F}(X)$ the family of all nonempty finite subsets of X. Also Δ_n is the standard n dimensional simplex with the vertices $e_0, e_1, \dots, e_n$. If J is a nonempty subset of $\{0, \dots, n\}$, Δ_J will denote the convex hull of the vertices $\{e_j : j \in J\}$. Let X and Y be topological spaces and D be a subset of X. D is said to be compactly closed (open) in X if $D \cap C$ is closed (open) in C for each nonempty compact subset C of X. A map $S: D \to 2^Y$ is said to be upper semi-continuous (u.s.c.) if for each $x \in D$ and for each open subset U of Y with $S(x) \subset U$, there exists an open neighborhood V of x in X such that for each $z \in D \cap V$, $S(z) \subset U$. S is said to be compactly valued if for each $x \in D$, $S(x)$ is compact in Y.

The following notions which were introduced by Bardaro-Ceppitelli in [2] were motivated by an earlier work of Horvath [15].

A pair $(X, \{F_A\})$ is called an H-space if X is a topological space (which need not be Hausdorff) and $\{F_A\}$ is a family of nonempty contractible subsets of X indexed by $A \in \mathscr{F}(X)$ such that $F_A \subset F_{A'}$, whenever $A \subset A'$. A subset D of X is said to be (i) H-convex if $F_A \subset D$ for each $A \in \mathscr{F}(D)$; (ii) weakly H-convex if $F_A \cap D$ is contractible for each $A \in \mathscr{F}(D)$ (this is equivalent to saying that $(D, \{F_A \cap D\})$ is an H-space); (iii) H-compact in X if, for each $A \in \mathscr{F}(X)$, there exists a compact, weakly H-convex subset D_A of X such that $D \cup A \subset D_A$. A map $F: X \to 2^X$ is called $H-KKM$ if $F_A \subset \bigcup_{x \in A} F(x)$ for each $A \in \mathscr{F}(X)$.

3. Selection theorem, $H-KKM$ theorem and coincidence theorem

The proof of the following useful result is contained in the proof of [15, Theorem 1] (see also [9]).

Lemma 3.1. *Let X be a topological space. For each nonempty subset J of $\{0, \dots, n\}$, let F_J be a nonempty contractible subset of X. If $J \subset J'$ imply $F_J \subset F_{J'}$, then there exists a continuous map $f: \Delta_n \to X$ such that $f(\Delta_J) \subset F_J$ for each nonempty subset J of $\{0, \dots, n\}$.*

The following lemma is a slight improvement of [15, Corollary I.1] (also see [8]).

Lemma 3.2. *Let $(Y, \{F_A\})$ be an H-space, X be a nonempty subset of Y and $G: X \to 2^Y$ be such that*

(a) *G is an $H-KKM$ map;*
(b) *for each $x \in X$, $G(x)$ is closed and for some $x_0 \in X$, $S(x_0)$ is compact.*

Then $\bigcap_{x \in X} G(x) \neq \varnothing$.

Theorem 3.1. *Let X be a compact topological space and $(Y, \{F_A\})$ be an H-space. Suppose that $S, T: X \to 2^Y$ are such that*

(a) *for each $x \in X$, $S(x) \neq \varnothing$ and $F_A \subset T(x)$ for each $A \in \mathscr{F}(S(x))$;*
(b) *for each $y \in Y$, $S^{-1}(y) = \{x \in X : y \in S(x)\}$ is open in X.*

Then T has a continuous selection $g: X \to Y$ and there exists a finite set $A \in \mathscr{F}(Y)$ such that $g(X) \subset F_A$.

Proof. By (a), we must have $X = \bigcup_{y \in Y} S^{-1}(y)$. From (b) and the compactness of X it follows that there exists a finite set

$$A = \{y_0, \dots, y_n\} \in \mathscr{F}(Y)$$

such that $X = \bigcup_{i=0}^{n} S^{-1}(y)$. For each nonempty subset J of $\{0, \dots, n\}$, we define $F_J = F_{\{y_j\}_{j \in J}}$. Since $(Y, \{F_A\})$ is an H-space, F_J is a contractible subset of Y and $F_J \subset F_{J'}$ whenever $J \subset J'$. By Lemma 3.1, there is a continuous map $f: \Delta_n \to Y$ such that $f(\Delta_J) \subset F_J$ for each nonempty subset J of $\{0, \dots, n\}$. Let $\{\alpha_i\}_{i=0}^{n}$ be a continuous partition of unity subordinate to the open covering $\{S^{-1}(y_i)\}_{i=0}^{n}$. Define a map $\psi: X \to \Delta_n$ by

$$\psi(x) = \sum_{i=0}^{n} \alpha_i(x) e_i.$$

For each $x \in X$, let $J(x) = \{i \in \{0, \dots, n\} : \alpha_i(x) \neq 0\}$, then we have $\psi(x) \in \Delta_{J(x)}$ so that

$$f \circ \psi(x) \in f(\Delta_{J(x)}) \subset F_{J(x)} \subset F_A.$$

Since $x \in S^{-1}(y_j)$ for each $j \in J(x)$, it follows that $y_j \in S(x)$ for all $j \in J(x)$. By (a), we obtain $F_{J(x)} \subset T(x)$ so that $f \circ \psi(x) \in T(x)$ for each $x \in X$. Hence $g = f \circ \psi$ is a continuous selection of T and there exists a finite set $A \in \mathscr{F}(Y)$ such that $g(X) \subset F_A$.

It would be of some interest to compare Theorem 3.1 with [15, Theorem 3].

Now we shall prove the following $H-KKM$ theorem.

THEOREM 3.2. *Let X be a nonempty subset of an H-space $(Y, \{F_A\})$, Z be a topological space and $G: X \to 2^Z$ be such that*

(a) *for each $x \in X$, $G(x)$ is compactly closed in Z;*

(b) *there exists a compactly valued u.s.c. map $S: Y \to 2^Z$ such that the map $F: X \to 2^Y$ defined by $F(x) = S^{-1}(G(x))$ is $H-KKM$;*

(c) *there exists an H-compact subset L of Y and a nonempty compact subset of Z such that for each $B \in \mathscr{F}(X)$ and for each $z \in S(L_B)\backslash K$, there is an $x \in L_B \cap X$ such that $x \notin G(x) \cap S(L_B)$. Then $K \cap (\bigcap_{x \in X} G(x)) \neq \varnothing$.*

PROOF. For each $x \in X$, let $G_1(x) = G(x) \cap K$, then $G_1(x)$ is closed in K by (a). We shall prove that the family $\{G_1(x) : x \in X\}$ has the finite intersection property. Let $B \in \mathscr{F}(X)$ be arbitrary fixed; then by (c), L_B is a compact, weakly H-convex subset of Y with $L \cup B \subset L_B$ such that for each $z \in S(L_B)\backslash K$, there is an $x \in L_B \cap X$ satisfying $z \notin G(x) \cap S(L_B)$. Now we define the map $G_2: L_B \cap \to 2^{L_B}$ by

$$G_2(x) = F(x) \cap L_B = S^{-1}(G(x)) \cap L_B.$$

Then we have the following properties.

(1) By the weak H-convexity of L_B, $(L_B, \{F_A \cap L_B\})$ is an H-space.

(2) For each $A \in \mathscr{F}(L_B \cap X) \subset \mathscr{F}(X)$, we have $F_A \subset \bigcup_{x \in A} F(x)$ by (b) so that $F_A \cap L_B \subset \bigcup_{A \in A}(F(x) \cap L_B) = \bigcup_{x \in A} G_2(x)$. Thus G_2 is also an $H-KKM$ map.

(3) Since S is compactly valued u.s.c. and L_B is compact in Y, it follows that $S(L_B)$ is compact in Z so that for each $x \in X$, $G(x) \cap S(L_B)$ is closed in Z by (a). By the upper semi-continuity of S, $S^{-1}(G(x) \cap S(L_B))$ is a closed subset of X. Hence, for each $x \in L_B \cap X$,

$$G_2(x) = S^{-1}(G(x)) \cap L_B = S^{-1}(G(x) \cap S(L_B)) \cap L_B$$

is compact in L_B.

By Lemma 3.2, $\bigcap_{x\in L_B\cap X} G_2(x) \neq \varnothing$. Take any $y \in \bigcap_{x\in L_B\cap X} G_2(x)$, then we have

$$S(y)\cap\left(\bigcap_{x\in L_B\cap X}(G(x)\cap S(L_B))\right)\neq\varnothing.$$

By (c), we must have

$$S(y)\cap\left(\bigcap_{x\in L_B\cap X}(G(x)\cap S(L_B))\right)\subset S(y)\cap\left(\bigcap_{x\in L_B\cap X}(G(x)\cap K)\right)$$
$$\subset S(y)\cap\left(\bigcap_{x\in B}(G(x)\cap K)\right)=S(y)\cap\left(\bigcap_{x\in B}G_1(x)\right)\subset\bigcap_{x\in B}G_1(x).$$

It follows that $\bigcap_{x\in B} G_1(x) \neq \varnothing$. By the compactness of K, $\bigcap_{x\in X} G_1(x) \neq \varnothing$, that is, $K\cap(\bigcap_{x\in X} G(x)) \neq \varnothing$.

REMARK 3.1. If S is a single-valued continuous map, Theorem 3.2 reduces to [10, Theorem 1] and in turn generalizes [1, Theorem 1]. It is easy to see that condition (c) of Theorem 3.2 is equivalent to the condition:

(c_1) there exists an H-compact subset L of Y and a nonempty compact subset K of Z such that for each $B\in\mathscr{F}(X)$,

$$\bigcap_{x\in L_B\cap X}(G(x)\cap S(L_B))\subset K.$$

We also note that under hypothesis (a) of Theorem 3.2, condition (c_1) is implied by the condition: there exists an H-compact subset L of Y such that $\bigcap_{x\in L\cap X} G(x)$ is compact in Z. Since every convex space is an H-space [17], Theorem 3.2 generalizes [17, Theorem I] (which is equivalent to [19, Theorem 6] to an H-space with a weaker assumption.

In the following we shall prove a coincidence theorem.

THEOREM 3.3. *Let X be a nonempty subset of an H-space $(Y, \{F_A\})$, Z be a topological space and $A, B\colon X\to 2^Z$ be such that*

(a) *for each $z\in Z$, $B^{-1}(z)\neq\varnothing$ and $F_D\subset A^{-1}(z)$ for each $D\in\mathscr{F}(B^{-1}(z))$;*

(b) *for each $x\in X$, $B(x)$ is compactly open in Z;*

(c) *there exists an H-compact subset L of Y and a nonempty compact subset K of Z such that for each $B\in\mathscr{F}(X)$ and for each $z\in Z\backslash K$, there is an $x\in L_B\cap X$ such that $z\in B(x)$.*

Then for any compactly valued u.s.c. map $S\colon Y\to 2^Z$, there exists an $x_0\in X$ such that $S(x_0)\subset A(x_0)$.

PROOF. Define a map $G: X \to 2^X$ by

$$G(x) = Z \backslash B(x) \qquad \text{for each } x \in X.$$

Then we have the following properties.

(1) For each $x \in X$, $G(x)$ is compactly closed by (b).

(2) By (c), there exist an H-compact subset L of Y and a nonempty compact subset K of Z such that for each $B \in \mathscr{F}(X)$ and for each $z \in Z \backslash K$, there is an $x \in L_B \cap X$ such that $z \notin G(x)$ so that $z \notin G(x) \cap S(L_B)$ for any compactly valued u.s.c. map $S: Y \to 2^Z$.

Now for any given compactly valued u.s.c. map $S: Y \to 2^Z$, define a map $F: X \to 2^Y$ by

$$F(x) = S^{-1}(G(x)) \qquad \text{for each } x \in X.$$

If F is an $H-KKM$ map, it follows from Theorem 3.2 that

$$\bigcap_{x \in X} G(x) = \bigcap_{x \in X} (Z \backslash B(x)) = Z / \bigcup_{x \in X} B(x) \neq \varnothing.$$

But condition (a) implies $Z = \bigcup_{x \in X} B(x)$, we obtain a contradiction so that F is not an $H-KKM$ map. Therefore there exists $D \in \mathscr{F}(X)$ and $x_0 \in F_D$ such that $x_0 \notin \bigcup_{x \in D} F(x) = \bigcup_{x \in D} S^{-1}(G(x))$. It follows that $S(x_0) \cap (\bigcup_{x \in D} G(x)) = S(x_0) \cap (\bigcup_{x \in D} (Z \backslash B(x)) = \varnothing$. Thus, $S(x_0) \subset B(x)$ for all $x \in D$ so that for any given $z \in S(x_0)$, we have $D \in \mathscr{F}(B^{-1}(z))$. By (a), $F_D \subset A^{-1}(z)$. It follows that $x_0 \in A^{-1}(z)$ and so $z \in A(x_0)$. From the arbitrariness of $z \in S(x_0)$ it follows that $S(x_0) \subset A(x_0)$.

REMARK 3.2. We note that condition (c) of Theorem 3.3 is equivalent to the following condition:

(c$'$) there exists an H-compact subset L of Y and a compact subset K of Z such that

$$Z \backslash \bigcup_{x \in L_B \cap X} B(x) \subset K.$$

Theorem 3.3 improves and generalizes [4, Theorem 1, 6, Theorem 1, 16, Theorem 3.1, 17, Theorem 1.1, 19, Theorem 6, 20, Theorem 4.3 and 23, Theorem 2 and 5].

4. Intersection theorems concerning sets with H-convex sections

In this section, we always assume that every H-space $(X, \{F_A\})$ has the following property: for each $A \in \mathscr{F}(X)$, F_A is H-compact in X. Clearly,

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each convex space X is an H-space [17] with the property that $F_A = \mathrm{co}(A)$ for each $A \in \mathscr{F}(X)$.

The following notations are used throughout this section. Let $(X_i, \{F_{A_i}\})$, $i = 1, \dots, n$, be n (≥ 2) H-spaces and $X = \prod_{i=1}^n X_i$. For each $i \in \{1, \dots, n\}$, let $\widehat{X}_i = \prod_{j \neq i} X_j$. Also $\hat{x}_i$ denotes an element of $\widehat{X}_i$. For each $i = 1, \dots, n$, $X_i \times \widehat{X}_i = X$ and $(x_i, \hat{x}_i)$ denotes an element of X (with the appropriate ordering).

We shall prove the following intersection theorems.

THEOREM 4.1. *Let $(X_i, \{F_{A_i}\})$, $i = 1, \dots, n$, be n (≥ 2) H-spaces and $X = \prod_{i=1}^n X_i$. If $M_1, \dots, M_n, N_1, \dots, N_n$ are $2n$ subsets of X such that*

(a) *for each $i \in \{1, \dots, n\}$ and for each $x_i \in X_i$, the section $M_i(x_i) = \{\hat{y}_i \in \widehat{X}_i : (x_i, \hat{y}_i) \in M_i\}$ is compactly open in $\widehat{X}_i$;*

(b) *for each $i \in \{1, \dots, n\}$ and for each $\hat{y}_i \in \widehat{X}_i$, the section $M_i(\hat{y}_i) = \{x_i \in X_i : (x_i, \hat{y}_i) \in M_i\} \neq \varnothing$ and $F_{D_i} \subset N_i(\hat{y}_i) = \{x_i \in X_i : (x_i, \hat{y}_i) \in N_i\}$ for each $D_i \in \mathscr{F}(M_i(\hat{y}_i))$;*

(c) *for at least $(n-1)$ indices i, there exists an H-compact subset L_i of X_i such that $\widehat{X}_i \setminus \bigcup_{x_i \in L_i} M_i(x_i)$ is compact in $\widehat{X}_i$. Then $\bigcap_{i=1}^n N_i \neq \varnothing$.*

PROOF. We may assume without loss of generality that condition (c) holds for $i = 2, \dots, n$. By (b), we have

$$\widehat{X}_i = \bigcup_{x_i \in X_i} M(x_i) \quad \text{for each } i = 1, \dots, n. \tag{4.1}$$

From (a), (c) and (4.1) it follows that for each $i = 2, \dots, n$, there exists a finite set $B_i = \{x_i^1, \dots, x_i^{k_i}\} \in \mathscr{F}(X_i)$ such that

$$\widehat{X}_i \setminus \bigcup_{x_i \in L_i} M_i(x_i) \subset \bigcup_{j=1}^{k_i} M_i(x_i^j).$$

Thus, we have

$$\widehat{X}_i \subset \bigcup_{x_i \in L_i \cup \{x_i^1, \dots, x_i^{k_i}\}} M_i(x_i). \tag{4.2}$$

Since L_i is H-compact in X_i, there exists a compact, weakly H-convex subset C_i of X_i with $L_i \cup B_i \subset C_i$ and (4.2) imply

$$\widehat{X}_i \subset \bigcup_{x_i \in C_i} M_i(x_i). \tag{4.3}$$

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Now we define the maps $M_1, N_1 \colon \prod_{i=2}^n C_i \to 2^{X_1}$ as follows: for each $\hat{y}_1 \in \prod_{i=2}^n C_i$,

$$M_1(\hat{y}_1) = \{x_1 \in X_1 : (x_1, \hat{y}_1) \in M_1\}$$

and

$$N_1(\hat{y}_1) = \{x_1 \in X_1 : (x_1, \hat{y}_1) \in N_1\}.$$

By (b), for each $\hat{y}_1 \in \prod_{i=2}^n C_i$, $M_1(\hat{y}_1) \neq \varnothing$ and $F_{D_1} \subset N_1(\hat{y}_1)$ for each $D_1 \in \mathscr{F}(M_1(\hat{y}_1))$. For each $x_1 \in X_1$,

$$M_1^{-1}(x_1) = \left\{\hat{y}_1 \in \prod_{i=2}^n C_i : (x_1, \hat{y}_1) \in M_1\right\} = \prod_{i=2}^n C_i \cap M_1(x_1)$$

is open in $\prod_{i=2}^n C_i$ by (a). It follows from Theorem 3.1 that there is a continuous map $g \colon \prod_{i=2}^n C_i \to X_1$ and $A_1 \in \mathscr{F}(X_1)$ such that $g(\hat{y}_1) \in N_1(\hat{y}_1)$ for each $y_1 \in \prod_{i=2}^n C_i$ and $g(\prod_{i=2}^n C_i) \subset F_{A_1}$. By the assumption that F_{A_1} is H-compact, there exists a compact weakly H-convex subset C_1 of X_1 with $F_{A_1} \subset C_1$. Hence, we have $g(\prod_{i=2}^n C_i) \subset C_1$ and $(g(\hat{y}_1), \hat{y}_1) \in N_1$ for each $\hat{y}_1 \in \prod_{i=2}^n C_i$.

Let $C = \prod_{i=1}^n C_i$ and $\widehat{C}_i = \prod_{j \neq i} C_j$. For each $i \in \{2, \dots, n\}$, we define the maps $M_i, N_i \colon C_i \to 2^{\widehat{C}_i}$ by

$$M_i(x_i) = \{\hat{y}_i \in \widehat{C}_i : (x_i, \hat{y}_i) \in M_i\}$$

and

$$N_i(x_i) = \{\hat{y}_i \in \widehat{C}_i : (x_i, \hat{y}_i) \in N_i\}$$

for each $x_i \in C_i$. Then, for each $x_i \in C_i$, $M_i(x_i)$ is open in $\widehat{C}_i$ by (a) and for each $\hat{y}_i \in \widehat{C}_i$, $M_i^{-1}(\hat{y}_i) = \{x_i \in C_i : (x_i, \hat{y}_i) \in M_i\} = C_i \cap M_i(\hat{y}_i) \neq \varnothing$ and $F_{D_i} \subset N_i^{-1}(\hat{y}_i)$ for each $D_i \in \mathscr{F}(M_i^{-1}(\hat{y}_i))$ by (b) and (4.3). From Theorem 3.3 with $X = Y = C_i$ and $Z = \widehat{C}_i = K$ it follows that for any compactly valued u.s.c. map $S \colon C_i \to 2^{\widehat{C}_i}$ there is an $x_i \in C_i$ such that $S(x_i) \subset N_i(x_i)$.

Now, let $p_i \colon \widehat{C}_1 \to C_i$, $i = 2, \dots, n$ and $q_i \colon \widehat{C}_i \to C_1$, $i = 1, \dots, n$ be the projective maps, then p_i, q_i are continuous open maps. We consider the following map

$$q_i^{-1} \circ g \circ p_i^{-1} \colon C_i \to 2^{\widehat{C}_i}, \quad i = 2, \dots, n.$$

Since p_i and q_i are continuous open maps and g is continuous, it is easy to see that $q_i^{-1} \circ g \circ p_i^{-1}$ is compactly valued and u.s.c. on C_i. Thus for $i = 2, \dots, n$, there exists $x_i \in C_i$ such that

$$q_i^{-1} \circ g \circ p_i^{-1}(x_i) \subset N_i(x_i). \tag{4.4}$$

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Let $\hat{x}_1 = (x_2, \dots, x_n)$ and $g(\hat{x}_1) = x_1$, then

$$x = (x_1, \dots, x_n) \in N_1 .$$

Since, for $i = 2, \dots, n$,

$$x_1 = g(\hat{x}_1) \in g(C_2 \times \cdots \times C_{i-1} \times \{x_i\} \times C_i \times \cdots \times C_n)$$

and

$$q_i^{-1} \circ g \circ p_i^{-1} = g(C_2 \times \cdots \times C_{i-1} \times \{x_i\} \times C_i \times \cdots \times C_n) \\ \times C_2 \times \cdots \times C_{i-1} \times C_1 \times \cdots \times C_n ,$$

we must have

$$\hat{x}_i = \prod_{j \neq i} x_i \in q_i^{-1} \circ g \circ p_i^{-1}(x_i) \subset N_i(x_i) \text{ for } i = 2, \dots, n .$$

Hence $x = (x_1, \dots, x_n) \in N_i$ for all $i = 1, \dots, n$ so that $\prod_{i=1}^n N_i \neq \varnothing$.

Remark 4.1. Theorem 4.1 generalizes [17, Theorem 1.9] to $2n$ sets and H-spaces with weaker assumptions. We observe that condition (c) of Theorem 4.1 is implied by the following condition:

(c_1) at least $(n-1)$ of the X_i's (say $X_2, \dots, X_n$) are compact. Indeed, in the case, (c) is satisfied by $L_i = X_i$ for $i = 2, \dots, n$, because by (b) the set $\widehat{X}_i \backslash \bigcup_{x_i \in X_i} M_i(x_i) = \varnothing$. Thus Theorem 4.1 also generalizes [11, Theorem 1] to H-spaces. It would be of some interest to compare Theorem 4.1 with [3, Theorem 2].

Theorem 4.2. *Let $(X_i, \{F_{A_i}\})$, $i = 1, \dots, n$, be (≥ 2) H-spaces and $X = \prod_{i=1}^n X_i$. If $M_1, \dots, M_n, N_1, \dots, N_n$ are $2n$ subsets of X such that*

(a) *for each $i \in \{1, \dots, n\}$ and for each $x_i \in X_i$, the section $M_i(x_i)$ is compactly open in $\widehat{X}_i$;*

(b) *for each $i \in \{1, \dots, n\}$ and for each $\hat{y}_i \in \widehat{X}_i$, the section $M_i(\hat{y}_i) \neq \varnothing$ and $F_{D_i} \subset N_i(\hat{y}_i)$ for each $D_i \in \mathscr{F}(M_i(\hat{y}_i))$;*

(c) *for at least $(n-1)$ indices i, there exists an H-compact subset L_i of X_i and a compact subset $\widehat{K}_i$ of $\widehat{X}_i$ such that $L_i \cap M_i(\hat{y}_i) \neq \varnothing$ for each $\hat{y}_i \in \widehat{X}_i \backslash \widehat{K}_i$.*

Then $\bigcap_{i=1}^n N_i \neq \varnothing$.

Proof. We shall show that condition (c) is equivalent to condition (c) of Theorem 4.1 and hence Theorem 4.2 follows from Theorem 4.1. Suppose that condition (c) of Theorem 4.1 holds. Let $\widehat{X}_i \backslash \bigcup_{x_i \in L_i} M(x_i) = \widehat{K}_i$, then $\widehat{K}_i$ is a compact subset of $\widehat{X}_i$ and for each $\hat{y}_i \in \widehat{X}_i \backslash \widehat{K}_i$, $\hat{y}_i \in \bigcup_{x_i \in L_i} M_i(x_i)$.

Thus, there exists $x_i \in L_i$ such that $(x_i, \hat{y}_i) \in M_i$, that is $x_i \in L_i \cap M_i(\hat{y}_i)$ and hence $L_i \cap M_i(\hat{y}_i) \neq \varnothing$. Therefore condition (c) of Theorem 4.2 holds. If condition (c) of Theorem 4.2 holds, then for each $\hat{y}_i \in \widehat{X}_i \setminus \bigcup_{x_i \in L_i} M_i(x_i)$, $\hat{y}_i \notin M_i(x_i)$ for all $x_i \in L_i$ so that $x_i \notin M_i(\hat{y}_i)$ for all $x_i \in L_i$. Thus $L_i \cap M_i(\hat{y}_i) = \varnothing$. It follows that $\hat{y}_i \in \widehat{K}_i$ and

$$\widehat{X}_i \setminus \bigcup_{x_i \in L_i} M_i(x_i) \subset \widehat{K}_i.$$

By (a), $\widehat{X}_i \setminus \bigcup_{x_i \in L_i} M_i(x_i)$ is closed in $\widehat{K}_i$ so that it is compact in $\widehat{X}_i$. This proves that condition (c) of Theorem 4.1 holds.

THEOREM 4.3. *Let $(X_i, \{F_{A_i}\})$, $i = 1, \dots, n$, be n (≥ 2) H-spaces and $X = \prod_{i=1}^n X_i$. If $M_1, \dots, M_n, N_1, \dots, N_n$ are $2n$ subsets of X such that*

(a) *for each $i \in \{1, \dots, n\}$ and for each $x_i \in X_i$, the section $M_i(x_i)$ is compactly open in $\hat{X}_i$;*

(b) *for each $i \in \{1, \dots, n\}$ and for each $\hat{y}_i \in \widehat{X}_i$, the section $M_i(\hat{y}_i) \neq \varnothing$ and $F_{D_i} \subset N_i(\hat{y}_i)$ for each $D_i \in \mathscr{F}(M_i(\hat{y}_i))$;*

(c) *there exists a compact subset K of X such that for each $i = 1, \dots, n$, the projection L_i of K on X_i is H-compact in X_i and such that $K \cap (\prod_{i=1}^n M_i(y_i)) \neq \varnothing$ for each $y \in X \setminus K$.*

Then $\bigcap_{i=1}^n N_i \neq \varnothing$.

PROOF. For each $i = 1, \dots, n$, let L_i and $\widehat{K}_i$ be the projections of K on X_i and $\widehat{X}_i$, respectively, then L_i is H-compact in X_i by the assumption and $\widehat{K}_i$ is a compact subset of $\widehat{X}_i$. The condition (c) of Theorem 4.3 imply that for each $i = 1, \dots, n$, $L_i \cap M_i(\hat{y}_i) \neq \varnothing$ for each $\hat{y}_i \in \widehat{X}_i \setminus \widehat{K}_i$. By Theorem 4.2, $\bigcap_{i=1}^n N_i \neq \varnothing$.

REMARK 4.3. Theorem 4.3 generalizes [12, Theorem 11] in several ways. We note that if condition (b) of Theorem 4.3 is replaced by the following condition:

(b_1) for each $i \in \{1, \dots, n\}$ and for each $y_i \in X_i$, the section $M_i(y_i) \neq \varnothing$ and for at least q (≥ 2) indices i, $F_{D_i} \subset N_i(\hat{y}_i)$ for each $D_i \in \mathscr{F}(M_i(\hat{y}_i))$ and for each $\hat{y}_i \in \widehat{X}_i$.

Then at least q of the sets $N_1, \dots, N_n$ have a nonempty intersection by applying Theorem 4.3 for the q H-spaces satisfying condition (b_1). Thus Theorem 4.3 also generalizes [13, Theorem 15].

5. Some applications to the von Neumann Minimax Theorem

For convenience, we state the special case $n = 2$ of Theorem 4.1.

THEOREM 5.1. *Let* $(X, \{F_A\})$ *and* $(Y, \{F_A\})$ *be two H-spaces and let* M_1, M_2, N_1, N_2 *be subsets of* $X \times Y$. *Suppose that*

(a) *for each* $x \in X$, *the section* $M_1(x) = \{y \in Y : (x, y) \in M_1\}$ *is compactly open in* Y, *the section* $M_2(x) = \{y \in Y : (x, y) \in M_2\} \neq \varnothing$ *and* $F_A \subset N_2(x)$ *for each* $A \in \mathscr{F}(M_2(x))$;

(b) *for each* $y \in Y$, *the section* $M_2(y) = \{x \in X : (x, y) \in M_2\}$ *is compactly open in* X, *the section* $M_1(y) = \{x \in X : (x, y) \in M_1\} \neq \varnothing$ *and* $F_A \subset N_1(y)$ *for each* $A \in \mathscr{F}(M_1(y))$;

(c) *there exists an H-compact subset* X_0 *of* X *such that the intersection* $\bigcap_{x \in X_0}(Y \backslash M_1(x))$ *is compact in* Y.

Then the intersection $N_1 \cap N_2$ *is nonempty.*

REMARK 5.1. If the coercive condition (c) is replaced by the following condition:

(c_1) there exists an H-compact subset Y_0 of Y such that the intersection $\bigcap_{y \in Y_0}(X \backslash M_2(y))$ is compact in X, then the inclusion of Theorem 5.1 still holds. We also note that if at least one of X or Y is compact, then condition (c) of Theorem 5.1 holds. Theorem 5.1 improves and generalizes [22, Theorem 2] and Ha's result [14] in several ways.

THEOREM 5.2. *Let* $(X, \{F_A\})$ *and* $(Y, \{F_A\})$ *be two H-spaces and* $f, s, t, g: X \times Y \to \mathbb{R}$ *and* $\lambda \in \mathbb{R}$ *be such that*

(a) $s \leq t$ *on* $X \times Y$;

(b) *for each* $x \in X$, $y \mapsto f(x, y)$ *is lower semi-continuous on each compact subset of* Y *and for each* $y \in Y$, $x \mapsto g(x, y)$ *is upper semi-continuous on each compact subset of* X;

(c) *for each* $x \in X$, $A \in \mathscr{F}(\{y \in Y : g(x, y) < \lambda\})$ *imply* $F_A \subset \{y \in Y : t(x, y) < \lambda\}$ *and for each* $y \in Y$, $A \in \mathscr{F}(\{x \in X : f(x, y) > \lambda\})$ *imply* $F_A \subset \{x \in X : s(x, y) > \lambda\}$;

(d) *there exists an H-compact subset* X_0 *of* X *such that the intersection* $\bigcap_{x \in X_0}(Y \backslash \{y \in Y : f(x, y) > \lambda\})$ *is compact in* Y.

Then either there exists $\hat{y} \in Y$ *such that* $f(x, \hat{y}) \leq \lambda$ *for all* $x \in X$ *or there exists* $\hat{x} \in X$ *such that* $g(\hat{x}, y) \geq \lambda$ *for all* $y \in Y$.

Proof. Suppose that the conclusion does not hold. Let

$$M_1 = \{(x, y) \in X \times Y : f(x, y) > \lambda\},$$
$$M_2 = \{x, y) \in X \times Y : g(x, y) < \lambda\},$$
$$N_1 = \{(x, y) \in X \times Y : s(x, y) > \lambda\},$$
$$N_2 = \{(x, y) \in X \times Y : t(x, y) < \lambda\}.$$

Then for each $x \in X$,

$$M_2(x) = \{y \in Y : g(x, y) < \lambda\} \neq \varnothing$$

and for each $y \in Y$,

$$M_1(y) = \{x \in X : f(x, y) > \lambda\} \neq \varnothing.$$

Moreover,

(i) for each $x \in X$, $M_1(x) = \{y \in Y : f(x, y) > \lambda\}$ is compactly open in Y and for each $y \in Y$, $M_2(y) = \{x \in X : g(x, y) < \lambda\}$ is compactly open in X by (a);

(ii) for each $x \in X$, $F_A \subset N_2(x)$ whenever $A \in \mathscr{F}(M_2(x))$ and for each $y \in Y$, $F_A \subset N_1(y)$ whenever $A \in \mathscr{F}(M_1(y))$ by (c);

(iii) condition (c) of Theorem 5.1 holds by (d).

Thus all hypotheses of Theorem 5.1 are satisfied so that $N_1 \cap N_2 \neq \varnothing$. Take any $(\hat{x}, \hat{y}) \in N_1 \cap N_2$, then $s(\hat{x}, \hat{y}) > \lambda$ which contradicts (a). Therefore the conclusion must hold.

Recall that a real-valued function φ defined on an H-space $(X, \{F_A\})$ is said to be H-quasi-concave if for each real number t, the set $\{x \in X : \varphi(x) > t\}$ is H-convex; φ is said to be H-quasi-convex if $-\varphi$ is H-quasi-concave.

Corollary 5.1. *Let $(X, \{F_A\})$ and $(Y, \{F_A\})$ be two H-spaces and $f, s, t, g: X \times Y \to \mathbb{R}$ be such that*

(a) $f \leq s \leq t \leq g$ *on* $X \times Y$;

(b) *for each $x \in X$, $y \mapsto f(x, y)$ is lower semi-continuous on each compact subset of Y and for each $y \in Y$, $x \mapsto g(x, y)$ is upper semi-continuous on each compact subset of X;*

(c) *for each $x \in X$, $t(x, y)$ is an H-quasi-convex function of y on Y and for each $y \in Y$, $s(x, y)$ is an H-quasi-concave function of x on X;*

(d) *there exists an H-compact subset X_0 of X such that for each $t \in \mathbb{R}$, the intersection $\bigcap_{x \in X_0}(Y \backslash \{y \in Y : f(x, y) > t\})$ is compact in Y.*

Then for each $\lambda \in \mathbb{R}$, either there exists $\hat{y} \in Y$ such that $f(x, \hat{y}) \leq \lambda$ for all $x \in X$ or there exists $\hat{x} \in X$ such that $g(\hat{x}, y) \geq \lambda$ for all $y \in Y$.

REMARK 5.2. Theorem 5.2 and Corollary 5.1 improve and generalize [5, Theorem 5.4]. It would be of some interest to compare Theorem 5.2 and Corollary 5.1 with [8, Theorem 4 and Corollary 4].

THEOREM 5.3. *Let* $(X, \{F_A\})$ *and* $(Y, \{F_A\})$ *be two* H*-spaces and* $f, s, t, g: Y \times Y \to \mathbb{R}$ *be such that*

(a) $s \le t$ *on* $X \times Y$;

(b) *for each* $x \in X$, $y \mapsto f(x, y)$ *is lower semi-continuous on each compact subset of* Y *and for each* $y \in Y$, $x \mapsto g(x, y)$ *is upper semi-continuous on each compact subset of* X;

(c) *for each* $\gamma \in \mathbb{R}$ *and for each* $x \in X$, $F_A \subset \{y \in Y : t(x, y) < \gamma\}$ *whenever* $A \in \mathscr{F}(\{y \in Y : g(x, y) < \gamma)$, *and for each* $\gamma \in \mathbb{R}$ *and for each* $y \in Y$, $F_A \subset \{x \in X : s(x, y) > \gamma\}$ *whenever* $A \in \mathscr{F}(\{x \in X : f(x, y) > \gamma\})$;

(d) *there exists an* H*-compact subset* L *of* X *and a compact subset* K *of* Y *such that*

$$\inf_{y \in Y} \sup_{x \in X} f(x, y) \le \inf_{y \in Y \setminus K} \sup_{x \in L} f(x, y).$$

Then the following minimax inequality holds,

$$\alpha = \inf_{y \in Y} \sup_{x \in X} f(x, y) \le \sup_{x \in X} \inf_{y \in Y} g(x, y) = \beta.$$

PROOF. Without loss of generality, we may assume that $\alpha \neq -\infty$ and $\beta \neq +\infty$. Assume to the contrary that $\alpha > \beta$. Choose a real number λ such that $\alpha > \lambda > \beta$. Let

$$\begin{aligned} M_1 &= \{(x, y) \in X \times Y : f(x, y) > \lambda\}, \\ M_2 &= \{(x, y) \in X \times Y : g(x, y) < \lambda\}, \\ N_1 &= \{(x, y) \in X \times Y : s(x, y) > \lambda\}, \\ N_2 &= \{(x, y) \in X \times Y : t(x, y) < \lambda\}. \end{aligned}$$

Then $\alpha > \lambda$ implies that for each $y \in Y$, $M_1(y) \neq \varnothing$; and $\lambda > \beta$ implies that for each $x \in X$, $M_2(x) \neq \varnothing$. The condition (d) implies that $\bigcap_{x \in L}(Y \setminus M_1(x)) \subset K$ and each $M_1(x)$ is compactly open in Y by (b), thus $\bigcap_{x \in L}(Y \setminus M_1(x))$ is compact in Y. The other conditions of Theorem 5.1 are easily verified. By Theorem 5.1, $N_1 \cap N_2 \neq \varnothing$ so that there exists $(\hat{x}, \hat{y}) \in X \times Y$ such that $s(\hat{x}, \hat{y}) > \lambda$ and $t(\hat{x}, \hat{y}) < \lambda$ which contradicts (a). This completes the proof.

COROLLARY 5.2. *Let* $(X, \{F_A\})$ *an* $d(Y, \{F_A\})$ *be two-*H*-spaces and* $f, s, t, g : X \times Y \to \mathbb{R}$ *be such that*

(a) $f \le s \le t \le g$ *on* $X \times Y$;

(b) *for each* $x \in X$, $y \mapsto f(x, y)$ *is lower semi-continuous on each compact subset of* Y *and for each* $y \in Y$, $x \mapsto g(x, y)$ *is upper semi-continuous on each compact subset of* X;

(c) *for each* $x \in X$, $t(x, y)$ *is an* H*-quasi-convex function of* y *on* Y *for each* $y \in Y$, $s(x, y)$ *is an* H*-quasi-concave function of* x *on* X;

(d) *there exists an* H*-compact subset* L *of* X *and a compact subset* K *of* Y *such that*

$$\inf_{y\in Y} \sup_{x\in X} f(x, y) \le \inf_{y\in Y\setminus Y} \sup_{x\in L} f(x, y).$$

Then the following minimax inequality holds,

$$\alpha = \inf_{y\in Y} \sup_{x\in X} f(x, y) \le \sup_{x\in X} \inf_{y\in Y} g(x, y) = \beta.$$

REMARK 5.3. Theorem 5.3 and Corollary 5.2 generalizes [22, Theorem 4(2), 3, Corollary 5.5] and Liu's result [18] in several ways. When $f = s = t = g$, the conclusion of Corollary 5.2 (respectively Theorem 5.3) implies the following minimax equality, which generalizes the minimax principle of the von Neumann type due to Sion [21]:

$$\inf_{y\in Y} \sup_{x\in X} f(x, y) = \sup_{x\in X} \inf_{y\in Y} f(x, y).$$

It would be of some interest to compare the minimax equality with the corresponding result of Barbaro-Ceppitelli in [3].

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