BULL. AUSTRAL. MATH. SOC. 20E05, 20E25
VOL. 7 (1972), 113-120. (20E30, 20F55)

# Residual properties of free groups II

Stephen J. Pride

In this paper it is proved that non-abelian free groups are residually $\langle x, y \mid x^m = 1, y^n = 1, x^k = y^h\rangle$ if and only if $\min\{(m, k), (n, h)\}$ is greater than $1$, and not both of $(m, k)$ and $(n, h)$ are $2$ (where $0$ is taken as greater than any natural number). The proof makes use of a result, possibly of independent interest, concerning the existence of certain automorphisms of the free group of rank two. A useful criterion which enables one to prove that non-abelian free groups are residually $G$ for a large number of groups $G$ is also given.

## 1. Introduction

For unexplained concepts and notation the reader is referred to [5].

Let $A$ and $B$ be groups. $A$ is said to be *residually* $B$ if and only if for each non-identity element $a$ of $A$ there is a homomorphism $\eta$ which maps $A$ onto $B$ and is such that $\eta(a)$ is not the identity of $B$. If $\mathcal{A}$ is a set of groups then $\mathcal{A}$ is said to be residually $B$ if and only if each element $A$ of $\mathcal{A}$ is residually $B$.

This paper reports on further developments in the program of studying residual properties of free groups. As is well-known (see [3]) non-abelian free groups are residually $B$ if and only if $F_2$ is residually $B$, where $F_2$ is a free group of rank two. Let $\{x, y\}$ be a generating set for $F_2$.

Received 20 March 1972. Communicated by M.F. Newman. Paper I in this series is not referred to in the present paper. The author thanks Dr M.F. Newman for his help in the preparation of this paper.

In [1] Katz and Magnus showed that $F_2$ is residually $\langle x, y \mid x^2 = 1\rangle$, and in [4] Poss proved that $F_2$ is residually $\langle x, y \mid x^n = 1, y^n = 1\rangle$ if $n \geq 6$. In this paper a more general question is considered: for which of the groups

$$\langle x, y \mid x^m = 1, y^n = 1, x^k = y^h\rangle = \|m, n, k, h\|$$

is $F_2$ residually $\|m, n, k, h\|$ ? (Here $m$ and $n$ are non-negative integers, and $k$ and $h$ are integers.) It is possible to adapt the proofs of the results of Katz and Magnus and of Poss to show that $F_2$ is residually $\|m, n, k, h\|$ for a large number of values of the quadruplet $(m, n, k, h)$, but a complete answer does not seem possible using [1] and [4].

THEOREM 1. *The free group* $F_2$ *is residually* $\|m, n, k, h\|$ *if and only if* $\min\{(m, k), (n, h)\}$ *is greater than one, and not both of* $(m, k)$ *and* $(n, h)$ *are equal to two.*

Here 0 should be taken to be greater than any other natural number.

The proof of Theorem 1 makes use of the following theorem, which will be proved in Section 2.

THEOREM 2. *Let* $w$ *be a non-identity element of* $F_2$. *There is an automorphism,* $\phi$, *of* $F_2$ *(depending on* $w$*) such that* $\phi(w)$ *has the form*

$$(*) \qquad y^{\alpha_1}x^{\varepsilon_1}y^{\alpha_2}x^{\varepsilon_2} \ldots y^{\alpha_t}x^{\varepsilon_t}y^{\alpha_{t+1}} ,$$

*where* $t \geq 1$, $\alpha_i$ $(1 \leq i \leq t+1)$ *is an integer with* $|\alpha_i| \leq 2$ *and is non-zero except possibly if* $i$ *is equal to* 1 *or* $t+1$, $|\varepsilon_i| = 1$ $(1 \leq i \leq t)$.

It should be remarked that in their proof that $F_2$ is residually $\|2, 0, 0, 0\|$, Katz and Magnus [1] essentially obtain the following weak version of Theorem 1:

*if* $w$ *is a non-identity element of* $F_2$, *there is an automorphism,* $\phi$, *of* $F_2$ *(depending on* $w$*) such that* $\phi(w)$ *has the form*

https://doi.org/10.1017/S0004972700044877 Published online by Cambridge University Press

$$y^{\beta_1}x^{\varepsilon_1}y^{\beta_2}x^{\varepsilon_2} \ldots y^{\beta_s}x^{\varepsilon_s}y^{\beta_{s+1}} ,$$

*where* $s \geq 1$ , $\beta_i$ $(1 \leq i \leq s+1)$ *is an integer, non-zero except possibly if* $i$ *equals* $1$ *or* $s + 1$ , $|\varepsilon_i| = 1$ $(1 \leq i \leq s)$ .

For the proof of Theorem 1, it is no loss of generality to assume that $(n, h) \geq (m, k)$ , since $\|m, n, k, h\|$ and $\|n, m, h, k\|$ are isomorphic. Suppose that $(n, h) \geq (m, k) > 1$ and not both of $(n, h)$ and $(m, k)$ are equal to two. Let $N[m, n, k, h]$ be the normal subgroup of $F_2$ generated by $\{x^m, y^n, x^k y^{-h}\}$ , so that $\|m, n, k, h\|$ is the factor group of $F_2$ by $N[m, n, k, h]$ . If $w$ is a non-identity element of $F_2$ then by Theorem 2 there is an automorphism, $\phi$ , of $F_2$ such that $\phi(w)$ has the form (*). It follows from Theorem 4.1 of [2] that $\phi(w)$ does not belong to $N[(m, k), (n, h), 0, 0]$ . Thus, if $\rho$ is the natural homomorphism of $F_2$ onto $\|m, n, k, h\|$ and $\pi$ is the homomorphism of $\|m, n, k, h\|$ onto $\|(m, k), (n, h), 0, 0\|$ defined by

$$\pi(gN[m, n, k, h]) = gN[(m, k), (n, h), 0, 0] \quad (g \in F_2) ,$$

then $\pi\rho\phi(w)$ is not the identity of $\|(m, k), (n, h), 0, 0\|$ . Thus $\rho\phi(w)$ is not the identity of $\|m, n, k, h\|$ . This establishes that $F_2$ is residually $\|m, n, k, h\|$ .

Now suppose that $(m, k) = 1$ or $(m, k) = (n, h) = 2$ . Then $\|m, n, k, h\|$ is either cyclic or metabelian and so does not generate the variety of all groups. Consequently $F_2$ is not residually $\|m, n, k, h\|$

Before giving a proof of Theorem 2 some general remarks relating to the proof of the 'if' part of Theorem 1 will be made.

A normal subgroup, $N$ , of $F_2$ is said to have the *trivial intersection property* (TI-property) if

$$\bigcap_{\phi \in \mathrm{aut}(F_2)} \phi(N) = 1 ,$$

or equivalently, if $N$ contains no non-trivial characteristic subgroup of

https://doi.org/10.1017/S0004972700044877 Published online by Cambridge University Press

$F_2$ . Clearly $F_2$ is residually $F_2/N$ . Examples of normal subgroups with the TI-property are provided by the groups $N[p, q, 0, 0]$ where $\min\{p, q\} \geq 2$ and not both of $p$ and $q$ are $2$ . It can also be shown (for instance by using Theorem 2 [or the weak version of it due to Katz and Magnus] and Exercise 12, Section 4.4 of [2]) that if $r \geq 1$ and $|p|, |q| > 1$ then the normal subgroup of $F_2$ generated by $\{y^r x^p y^{-r} x^q\}$ has the TI-property.

It is obvious that any normal subgroup of $F_2$ contained in a normal subgroup with the TI-property also has the TI-property. This gives a simple method for showing that $F_2$ is residually the group $G$ .

CRITERION. *Let* $G$ *be a group with presentation*

$$\langle x, y \mid r_1 = 1, r_2 = 1, \ldots \rangle$$

*(where the number of relations can be finite or infinite). Then* $F_2$ *is residually* $G$ *if there is a normal subgroup,* $N$ *, of* $F_2$ *with the TI-property such that*

$$\{r_1, r_2, \ldots\} \subseteq N .$$

It should be noticed that the proof of the 'if' part of Theorem 1 is just an application of this criterion with $N = N[(m, k), (n, h), 0, 0]$ .

## 2. Proof of Theorem 2

As is well-known (see Theorem 4.2 of [2]) every non-identity element of $F_2$ is conjugate to an element of one of the following types:

(i) $x^k$ , $k \neq 0$ ;

(ii) $y^k$ , $k \neq 0$ ;

(iii) $y^{\lambda_1} x^{\mu_1} y^{\lambda_2} x^{\mu_2} \ldots y^{\lambda_r} x^{\mu_r}$ , where $r \geq 1$ and $\lambda_i$ and $\mu_i$ $(1 \leq i \leq r)$ are non-zero integers.

It may therefore be assumed that $w$ is of type (i), (ii) or (iii).

The result is clear unless $w$ is of type (iii). To deal with this case it is convenient to introduce the concept of an *ending* of an element of $F_2$.

DEFINITION. Let $u$ be a non-identity element of $F_2$. The non-identity element $v$ of $F_2$ is said to be an *ending* for $u$ if and only if there is an element $g$ of $F_2$ such that $u = gv$, where $g$ is either empty or has the property that if the first symbol of $v$ is $x$ or $x^{-1}$ ($y$ or $y^{-1}$), then the last symbol of $g$ is $y$ or $y^{-1}$ ($x$ or $x^{-1}$).

Thus $y^2x$ is an ending for $x^{-2}y^2x$, whereas $yx$ is not. It is obvious that an element may have several endings. The phrases '$v$ is an ending for $u$' and '$u$ ends in $v$' will be used synonymously.

Let $n$ be an integer with $n > 0$, and let $\phi_n$ be the automorphism of $F_2$ defined by

$$x \mapsto (yx)^n y^2 x ,$$

$$y \mapsto \left((yx)^n y^2 x\right)^n (yx)^n y .$$

The following assertion will be proved by induction on $r$.

(††) If $w = y^{\lambda_1}x^{\mu_1} \ldots y^{\lambda_r}x^{\mu_r}$ is an element of $F_2$ of type (iii), and $n$ is an integer satisfying

$$n > |\mu_i| , \quad i = 1, 2, \ldots, r ,$$

then $\phi_n(w)$ has the form (*) and ends in one of the following:

$$x ,$$

$$y\left((yx)^n y^2 x\right)^{\mu_r} \quad (\text{if } \mu_r < 0),$$

$$\left(x^{-1}y^{-1}\right)^{n+1}\left((yx)^n y^2 x\right)^{\mu_r} \quad (\text{if } \mu_r < 0),$$

$$y^{-1}\left(x^{-1}y^{-1}\right)^n\left(x^{-1}y^{-2}\left(x^{-1}y^{-1}\right)^n\right)^{n-\mu_r} ,$$

$$x^{-1}y^{-2}\left(x^{-1}y^{-1}\right)^{n-1}\left(x^{-1}y^{-2}\left(x^{-1}y^{-1}\right)^n\right)^{n+1-\mu_r} .$$

https://doi.org/10.1017/S0004972700044877 Published online by Cambridge University Press

It will be convenient in the following to denote the elements

$$y\left((yx)^n y^2 x\right)^{\mu_r},$$

$$(x^{-1}y^{-1})^{n+1}\left((yx)^n y^2 x\right)^{\mu_r},$$

$$y^{-1}(x^{-1}y^{-1})^n\left(x^{-1}y^{-2}(x^{-1}y^{-1})^n\right)^{n-\mu_r},$$

$$x^{-1}y^{-2}(x^{-1}y^{-1})^{n-1}\left(x^{-1}y^{-2}(x^{-1}y^{-1})^n\right)^{n+1-\mu_r},$$

by $\gamma_r$, $\delta_r$, $o_r$, $\omega_r$ respectively. (Whenever $\gamma_r$ or $\delta_r$ is mentioned it is to be understood that $\mu_r < 0$.) It will also be convenient to denote the element $(yx)^n y^2 x$ by $p$, and the element $((yx)^n y^2 x)^n (yx)^{n-1} y^2 x$ by $q$. Notice that $p$ and $q$ are both of the form (*) and both have $y$ as first symbol and $x$ as last symbol.

If $\lambda$ and $\mu$ are non-zero integers, then it is not difficult to verify that $\phi_n(y^\lambda x^\mu)$ is equal to

(1) $p^{n+1}(yx)^{n-1}y^2xq^{\lambda-1}p^{\mu-1}$ if $\lambda > 0, \mu > 0$,

(2) $p^n(yx)^n yp^\mu$ if $\lambda = 1, \mu < 0$,

(3) $p^{n+1}(yx)^{n-1}y^2xq^{\lambda-2}p^{n-1}(yx)^n yp^\mu$ if $\lambda > 1, \mu < 0$,

(4) $y^{-1}(x^{-1}y^{-1})^n(p^{-1})^{n-\mu}$ if $\lambda = -1$,

(5) $y^{-1}(x^{-1}y^{-1})^n(p^{-1})^{n-1}(q^{-1})^{-\lambda-2}x^{-1}y^{-2}(x^{-1}y^{-1})^{n-1}(p^{-1})^{n+1-\mu}$ if $\lambda < -1$.

Using (1)-(5) it is easy to check that (††) holds when $r = 1$.

Now assume that $r$ is greater than $1$. Let $w = y^{\lambda_1}x^{\mu_1} \dots y^{\lambda_r}x^{\mu_r}$ be an element of type (iii), and let $n$ be an integer satisfying

$$n > |\mu_i|, \quad i = 1, 2, \dots, r.$$

Denote $y^{\lambda_1}x^{\mu_1} \dots y^{\lambda_{r-1}}x^{\mu_{r-1}}$ by $w_1$.

https://doi.org/10.1017/S0004972700044877 Published online by Cambridge University Press

By the induction hypothesis $\phi_n(\omega_1)$ has the form (*) and ends in one of $x$, $\gamma_{r-1}$, $\delta_{r-1}$, $\sigma_{r-1}$, $\omega_{r-1}$. If $\phi_n(\omega_1)$ ends in $x$, $\gamma_{r-1}$, or $\delta_{r-1}$ then using (1)-(5) it can be shown, without too much difficulty that $\phi_n(\omega)$ has the form (*) and ends in one of $x$, $\gamma_r$, $\delta_r$, $\sigma_r$, $\omega_r$; when $\phi_n(\omega_1)$ ends in $\sigma_{r-1}$ or $\omega_{r-1}$ the verification is more complicated. Thus, suppose that $\phi_n(\omega_1)$ ends in $\sigma_{r-1}$. Then

$$\phi_n(\omega_1) = g\sigma_{r-1}$$

where $g$ is either empty or has $x$ or $x^{-1}$ as last symbol. Straightforward computations show

$$\phi_n(\omega) = \begin{cases} gyxp^{\mu_{r-1}}(yx)^{n-1}y^2xq^{\lambda_r-1}p^{\mu_r-1} & \text{if } \lambda_r > 0,\ \mu_r > 0,\ \mu_{r-1}+1 > 0\,, \\ gy^{-1}x^{-1}yxq^{\lambda_r-1}p^{\mu_r-1} & \text{if } \lambda_r > 0,\ \mu_r > 0,\ \mu_{r-1}+1 = 0\,, \\ gy^{-1}(x^{-1}y^{-1})^n(p^{-1})^{-\mu_{r-1}-2}x^{-1}y^{-2}x^{-1}yxq^{\lambda_r-1}p^{\mu_r-1} & \\ & \text{if } \lambda_r > 0,\ \mu_r > 0,\ \mu_{r-1}+1 < 0\,, \\ gyxp^{\mu_{r-1}-1}(yx)^nyp^{\mu_r} & \text{if } \lambda_r = 1,\ \mu_r < 0,\ \mu_{r-1} > 0\,, \\ gy^{-1}(x^{-1}y^{-1})^n(p^{-1})^{-\mu_{r-1}-1}x^{-1}y^{-1}p^{\mu_r} & \\ & \text{if } \lambda_r = 1,\ \mu_r < 0,\ \mu_{r-1} < 0\,, \\ gyxp^{\mu_{r-1}}(yx)^{n-1}y^2xq^{\lambda_r-2}p^{n-1}(yx)^nyp^{\mu_r} & \\ & \text{if } \lambda_r > 1,\ \mu_r < 0,\ \mu_{r-1}+1 > 0\,, \\ gy^{-1}x^{-1}yxq^{\lambda_r-2}p^{n-1}(yx)^nyp^{\mu_r} & \text{if } \lambda_r > 1,\ \mu_r < 0,\ \mu_{r-1}+1 = 0\,, \\ gy^{-1}(x^{-1}y^{-1})^n(p^{-1})^{-\mu_{r-1}-2}x^{-1}y^{-2}x^{-1}yxq^{\lambda_r-2}p^{n-1}(yx)^nyp^{\mu_r} & \\ & \text{if } \lambda_r > 1,\ \mu_r < 0,\ \mu_{r-1}+1 < 0\,, \\ gy^{-1}(x^{-1}y^{-1})^n(p^{-1})^{n-\mu_{r-1}-1}(q^{-1})^{-\lambda_r-1}x^{-1}y^{-2}(x^{-1}y^{-1})^{n-1}(p^{-1})^{n+1-\mu_r} & \\ & \text{if } \lambda_r < 0\,. \end{cases}$$

Hence $\phi_n(w)$ has the form (*) and ends in one of $x$, $\gamma_r$, $\delta_r$, $\omega_r$.

The case when $\phi_n(w_1)$ ends in $\omega_{r-1}$ is similar to the case just considered, and details will be omitted.

## References

[1] Robert A. Katz and Wilhelm Magnus, "Residual properties of free groups", *Comm. Pure Appl. Math.* 22 (1969), 1-13.

[2] Wilhelm Magnus, Abraham Karrass, Donald Solitar, *Combinatorial group theory* (Interscience [John Wiley & Sons], New York, London, Sydney, 1966).

[3] Ada Peluso, "A residual property of free groups", *Comm. Pure Appl. Math.* 19 (1966), 435-437.

[4] Samuel Poss, "A residual property of free groups", *Comm. Pure Appl. Math.* 23 (1970), 749-756.

[5] Joseph J. Rotman, *The theory of groups: An introduction* (Allyn and Bacon, Boston, 1965).

Department of Mathematics,
Institute of Advanced Studies,
Australian National University,
Canberra, ACT.