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A MULTIPLICITY THEOREM FOR A VARIABLE EXPONENT DIRICHLET PROBLEM

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Abstract. We consider a nonlinear Dirichlet problem driven by the $p(\cdot)$-Laplacian. Using variational methods based on the critical point theory, together with suitable truncation techniques and the use of upper-lower solutions and of critical groups, we show that the problem has at least three nontrivial solutions, two of which have constant sign (one positive, the other negative). The hypotheses on the nonlinearity incorporates in our framework of analysis, both coercive and noncoercive problems.

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1. Introduction. Let $Z \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary ∂Z. In this paper, we study the existence of multiple nontrivial solutions for the Dirichlet problem

$$\begin{cases} -\Delta_{p(z)}\, x(z) = m(z)|x(z)|^{r-2}x(z) + f(z, x(z)) \text{ a.e. on } Z, \\ x|_{\partial Z} = 0. \end{cases} \tag{1}$$

In problem (1), we assume that $p \in C^1(\bar{Z})$ (i.e., it is continuously differentiable), $1 < r < p^- = \min_{\bar{Z}} p$ and f is a Carathéodory function (i.e., it is measurable in $z \in Z$ and continuous in $x \in X$). The differential operator $-\Delta_{p(z)}\, x = -\operatorname{div}\left(||Dx||^{p(z)-2}Dx\right)$, is called the $p(\cdot)$-Laplacian. When $p(z) \equiv p$ (a constant), it becomes the usual p-Laplacian differential operator. The goal of this paper, is to prove a “three nontrivial solutions theorem” for problem (1).

Recently there have been some three solutions theorems for problems driven by the regular p-Laplacian. We mention the works of Bartsch-Liu [**3**], Carl-Perera [**4**], Guo-Liu [**11**], Liu [**14**], Liu-Liu [**15**], Papageorgiou-Papageorgiou [**18**], Zhang-Chen-Li [**21**], and Zhang-Liu [**22**]. In all these works, the Euler functional of the problem is coercive. Here, the hypotheses on the nonlinearity $f(z, x)$, can incorporate in our framework of analysis problems with a noncoercive Euler functional. Our method of proof is variational, based on critical point theory, coupled with the use of ordered pairs of upper-lower solutions and with suitable truncation techniques. We also use critical groups to distinguish between critical points.

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Multiplicity results for problems driven by variable $p(\cdot)$-Laplacian, were obtained recently by Fan-Zhang **[8]** and Zang **[20]**, but under symmetry conditions on the nonlinearity $f(z,\cdot)$ (namely, they assumed that $x \mapsto f(z,x)$ is an odd function). We should also mention the work of Mihailescu **[17]**, where the author considers a specific right hand side nonlinearity of the form $g(z,x) = A|x|^{\alpha-2}x + B|x|^{\beta-2}x$ with $1 < \alpha < p^- = \min_{\bar{Z}} p < p^+ = \max_{\bar{Z}} p < \beta < \min\{N, \frac{Np^-}{N-p^-}\}$ and $A, B > 0$, and proves the existence of a $\lambda^* > 0$ such that the problem has at least two distinct nontrivial weak solutions when $A, B \in (0, \lambda^*)$. Our multiplicity result here extends that of Mihailescu **[17]**.

Differential equations and variational problems with $p(\cdot)$-growth conditions arise naturally in nonlinear elasticity theory and in the theory of electrorheological fluids. For details, we refer to Acerbi-Mingione **[1]** and Ruzička **[19]**.

The paper is organized as follows. In Section 2, we present some background material concerning variable exponent Lebesgue and Sobolev spaces. In Section 3, we state our hypotheses on the data of problem (1) and we formulate the main multiplicity result of this paper (the three nontrivial solutions theorem). In Section 4, we establish the existence of two nontrivial smooth solutions of constant sign. Finally, in Section 5, we prove the full multiplicity theorem.

2. Mathematical Background. Let $K_+(\bar{Z}) = \left\{p \in C(\bar{Z}) : p(z) > 1 \text{ for all } z \in \bar{Z}\right\}$ and for $p \in K_+(\bar{Z})$, denote

$$p^- = \min_{\bar{Z}} p \quad \text{and} \quad p^+ = \max_{\bar{Z}} p.$$

By $L^0(Z)$ we denote the space of all Lebesgue measurable $\mathbb{R}$-valued functions defined on Z. Recall that two functions in $L^0(Z)$ are considered to be the same element if an only if they are equal almost everywhere on Z. For $p \in K_+(\bar{Z})$, we define the variable exponent Lebesgue space $L^{p(\cdot)}(Z)$ by

$$L^{p(\cdot)}(Z) = \left\{x \in L^0(Z) : \int_Z |x(z)|^{p(z)}\, dz < \infty\right\}.$$

This space is equipped with the so-called "Luxemburg norm", defined by

$$||x||_{p(\cdot)} = \inf\left\{\lambda > 0 : \int_Z \left|\frac{x(z)}{\lambda}\right|^{p(z)} dz \leq 1\right\}.$$

These spaces extend classical Lebesgue spaces and share many of their basic properties. So, they are reflexive, separable and uniformly convex Banach spaces (recall, we assume $p(z) > 1$ for all $z \in \bar{Z}$) and the space $C(\bar{Z})$ is dense in $L^{p(\cdot)}(Z)$. The dual of $L^{p(\cdot)}(Z)$ is the space $L^{q(\cdot)}(Z)$, where $\frac{1}{p(z)} + \frac{1}{q(z)} = 1$ and for any $x \in L^{p(\cdot)}(Z)$ and any $u \in L^{q(\cdot)}(Z)$, we have Holder's inequality

$$\left|\int_Z xu\, dz\right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-}\right) \|x\|_{p(\cdot)} \|u\|_{q(\cdot)}.$$

Moreover, we have the usual inclusion between Lebesgue spaces, namely $p_1, p_2 \in K_+(\bar{Z})$ with $p_1(z) \leq p_2(z)$ for all $z \in \bar{Z}$, then $L^{p_2(\cdot)}(Z) \subseteq L^{p_1(\cdot)}(Z)$ and the embedding is continuous.

The variable exponent Sobolev space $W^{1,p(\cdot)}(Z)$, is defined by

$$W^{1,p(\cdot)}(Z) = \left\{ x \in L^{p(\cdot)}(Z) : ||Dx|| \in L^{p(\cdot)}(Z) \right\}.$$

This space is equipped with the norm

$$\|x\|_{W^{1,p(\cdot)}(Z)} = \|x\|_{p(\cdot)} + \|Dx\|_{p(\cdot)} \tag{2}$$

for all $x \in W^{1,p(\cdot)}(Z)$. We denote by $\mathrm{W}_0^{1,p(\cdot)}(Z)$ the closure of $C_c^\infty(Z)$ in $W^{1,p(\cdot)}(Z)$ and the critical Sobolev exponent is defined by

$$p^*(z) = \begin{cases} \frac{Np(z)}{N-p(z)} & \text{if } p(z) < N, \\ \infty & \text{if } p(z) \geq N. \end{cases}$$

The spaces $W^{1,p(\cdot)}(Z)$ and $\mathrm{W}_0^{1,p(\cdot)}(Z)$ are separable, reflexive and uniformly convex Banach spaces and, if $r \in K_+(\bar{Z})$ with $r(z) < p^*(z)$ for all $z \in \bar{Z}$, then the space $W^{1,p(\cdot)}(Z)$ is embedded compactly in $L^{r(\cdot)}(Z)$. Moreover, there exists a constant $c > 0$ such that

$$\|x\|_{p(\cdot)} \leq c\|Dx\|_{p(\cdot)}$$

for all $x \in \mathrm{W}_0^{1,p(\cdot)}(Z)$ (Poincaré's inequality). This implies that $\|x\|_{W^{1,p(\cdot)}(Z)}$ defined by (2) and $\|Dx\|_{p(\cdot)}$ are equivalent norms on $\mathrm{W}_0^{1,p(\cdot)}(Z)$. Also, $\mathrm{W}_0^{1,p(\cdot)}(Z)^* = W^{-1,q(\cdot)}(Z)$, $\frac{1}{p(z)} + \frac{1}{q(z)} = 1$ for all $z \in \bar{Z}$.

If $p \in K_+(\bar{Z})$ and $g : Z \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function which satisfies

$$|g(z, x)| \leq a(z) + c|x|^{p(z)-1}$$

for almost all $z \in Z$ and all $x \in \mathbb{R}$, with $a \in L^{q(\cdot)}(Z)$ and $c > 0$, then the Nemytskii operator $N_f : L^{p(\cdot)}(Z) \to L^{q(\cdot)}(Z)$ defined by

$$N_f(x)(\cdot) = f(\cdot, x(\cdot))$$

is continuous and bounded. For further details on variable exponent Lebesgue and Sobolev spaces, we refer to Fan-Zhao citeR9 and Kováčik-Rákosník [**13**].

Let $p \in K_+(\bar{Z})$ and consider the functional $\psi : \mathrm{W}_0^{1,p(\cdot)}(Z) \to \mathbb{R}$ defined by

$$\psi(x) = \int_Z \frac{1}{p(z)} ||Dx||^{p(z)} \, dz.$$

Then $\psi \in C^1(\mathrm{W}_0^{1,p(\cdot)}(Z))$ and the $p(\cdot)$-Laplacian is the derivative of ψ. So, if $A = \psi'$ and by $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair $(\mathrm{W}_0^{1,p(\cdot)}(Z), \mathrm{W}^{-1,p'(\cdot)}(Z))$, then

$$\langle A(x), y \rangle = \int_Z ||Dx||^{p(z)-2} \, (Dx, Dy)_{\mathbb{R}^N} \, dz \tag{3}$$

for all $x, y \in \mathrm{W}_0^{1,p(\cdot)}(Z)$. We know that A is continuous, bounded, strictly monotone (hence maximal monotone) and of type $(S)_+$, (i.e., if $x_n \xrightarrow{w} x$ in $\mathrm{W}_0^{1,p(\cdot)}(Z)$ and

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$\limsup_{n\to\infty}\langle A(x_n), x_n - x\rangle \leq 0$, then $x_n \to x$ in $W_0^{1,p(\cdot)}(Z)$). For details, see Fan-Zhang **[8]**.

In our analysis of problem (1), we will also use the space

$$C_0^1(\bar{Z}) = \left\{x \in C^1(\bar{Z}) : x|_{\partial Z} = 0\right\}.$$

This is an ordered Banach space, with positive (order) cone

$$C_+ = \{x \in C_0^1(\bar{Z}) : x(z) \geq 0 \text{ for all } z \in \bar{Z}\}.$$

This cone has a nonempty interior, given by

$$\operatorname{int} C_+ = \left\{x \in C_+ : x(z) > 0 \text{ for all } z \in Z \text{ and } \frac{\partial x}{\partial n}(z) < 0 \text{ for all } z \in \partial Z\right\}.$$

Here by $n(\cdot)$ we denote the outward unit normal on ∂Z.

Let Y be a Hausdorff topological space and V a subspace of it. For every integer $n \geq 0$, by $H_n(Y, V)$ we denote the n^{th}-singular relative homology group with integer coefficients. Let X be a Banach space and $\varphi \in C^1(X)$ a functional satisfying the PS-condition. The critical groups of φ at an isolated critical point $x_0 \in X$ with $\varphi(x_0) = c$, are defined by

$$C_n(\varphi, x_0) = H_n(\varphi^c \cap \mathcal{U}, (\varphi^c \cap \mathcal{U})\backslash\{x_0\})$$

for all $n \geq 0$, where $\varphi^c = \{x \in X : \varphi(x) \leq c\}$ and $\mathcal{U}$ is a neighborhood of $x_0 \in X$ such that $K \cap \varphi^c \cap \mathcal{U} = \{x_0\}$ with $K = \{x \in X : \varphi'(x) = 0\}$ (the critical set of φ). From the excision property of singular homology theory, we see that the above definition of critical groups is independent of $\mathcal{U}$ (see Chang **[5]** and Mawhin-Willem **[16]**).

3. Hypotheses and the Main Result. As we already mentioned in the Introduction, $Z \subseteq \mathbb{R}^N$ is a bounded domain with a C^2-boundary ∂Z. We consider an exponent $p(\cdot)$ which belongs in $C^1(\bar{Z})$ (i.e., it is continuously differentiable on $\bar{Z}$). We set $p^- = \min_{\bar{Z}} p$, $p^+ = \max_{\bar{Z}} p$ and we assume that $1 \leq r < p^-$ (see (1)). We also make the following hypotheses concerning the weight function m and the nonlinearity f:

H(m): $m \in L^\infty(Z)_+$ and $m \not\equiv 0$.

H(f): $f : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that $f(z, 0) = 0$ a.e. on Z and

(i) for all $x \in \mathbb{R}$, $z \mapsto f(z, x)$ is measurable;

(ii) for almost all $z \in Z$, $x \mapsto f(z, x)$ is continuous;

(iii) for almost all $z \in Z$ and all $x \in \mathbb{R}$, we have

$$|f(z, x)| \leq a(z) + c|x|^{q-1}$$

with $a \in L^\infty(Z)_+$, $c > 0$, $p^+ < q < (p^*)^-$;

(iv) there exists $\vartheta \in \left(p^+, (p^*)^-\right)$ such that

$$\limsup_{x\to 0} \frac{f(z, x)}{|x|^{\vartheta-2}x} < +\infty$$

uniformly for a.a. $z \in Z$;

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(v) for almost all $z \in Z$ and all $x \neq 0$, $f(z,x)x > 0$ (strict sign condition) and there exists $\delta_0 \in (0,1)$ such that, for almost all $z \in Z$, $x \mapsto f(z,x)$ is increasing on $[-\delta_0, \delta_0]$;

The main result of this paper, asserts that problem (1) has at least three nontrivial solutions. More precisely, we have the following result.

THEOREM 1. *If hypotheses $H(m)$ and $H(f)$ hold, then there exists $\lambda_0^* > 0$ such that for $0 < ||m||_\infty < \lambda_0^*$, problem 1 has at least three distinct nontrivial solutions*

$$x_0 \in \operatorname{int} C_+, v_0 \in -\operatorname{int} C_+, y_0 \in C_0^1(\bar{Z}) \text{ and } v_0 < y_0 < x_0.$$

REMARK. Our result improves considerably the multiplicity theorem of Mihailescu [**17**]. There, $m(z) \equiv A > 0$ for all $z \in Z$ (i.e., m is constant), $f(z,x) = B|x|^{\beta-2}x$ with $B > 0$, $p^+ < b < \min\{N, \frac{Np^-}{N-p^-}\}$ and the author proves the existence of only two nontrivial weak solutions. He obtains weak solutions, because he requires that $p \in K_+(\bar{Z})$. Here by strengthening the requirement on the exponent (we assume $p \in C^1(\bar{Z})$), we are able to guarantee that the solutions we obtain are smooth.

4. Two Solutions for the Main Problem. In this Section, using the method of upper-lower solutions, together with suitable truncation techniques and variational arguments, we produce two smooth solutions of constant sign.

To this end, we consider the truncation functions $\tau_\pm : \mathbb{R} \to \mathbb{R}$ defined by

$$\tau_+(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases} \text{ and } \tau_-(x) = \begin{cases} x & \text{if } x < 0 \\ 0 & \text{if } x \geq 0 \end{cases}.$$

We set $f_\pm(z,x) = f(z, \tau_\pm(x))$. Evidently, both are Carathéodory functions. Set $F_\pm(z,x) = \int_0^x f_\pm(z,s)\,ds$, the primitives of $f_\pm$. We consider the functionals $\varphi_\pm : W_0^{1,p(\cdot)}(Z) \to \mathbb{R}$ defined by

$$\varphi_\pm(x) = \int_Z \frac{1}{p(z)} ||Dx||^{p(z)}\,dz - \frac{1}{r}\int_Z m(z)|x^\pm(z)|^r\,dz - \int_Z F_\pm(z, x(z))\,dz$$

for all $x \in W_0^{1,p(\cdot)}(Z)$. Here $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$. From what was said in Section 2, we see that $\varphi_\pm \in C^1(W_0^{1,p(\cdot)}(Z))$.

We consider the following auxiliary Dirichlet problem:

$$\begin{cases} -\Delta_{p(z)}\,x(z) = m(z)\,x^+(z)^{r-1} + f_+(z, x(z)) \text{ a.e. on } Z, \\ x|_{\partial Z} = 0. \end{cases} \tag{4}$$

We will produce a positive strict upper solution $\overline{x} \in W_0^{1,p(\cdot)}(Z)$ for problem (4). Recall that $\overline{x} \in W_0^{1,p(\cdot)}(Z)$ is an upper solution of problem (4), if $\overline{x}|_{\partial Z} \geq 0$ and

$$\int_Z ||D\overline{x}||^{p(z)-2}\,(D\overline{x}(z), Du(z))_{\mathbb{R}^N}\,dz \geq \int_Z \left[m(z)\overline{x}^+(z)^{r-1} + f_+(z, \overline{x}(z))\right]u(z)\,dz$$

for all $u \in C_c^\infty(Z)$, $u \geq 0$. We say that $\overline{x}$ is a strict upper solution for problem (4), if it is an upper solution for problem (4) but not a solution for problem (4).

PROPOSITION 2. *If hypotheses H(f) and H(m) hold, then we can find* $\lambda_+^* > 0$ *such that, for* $||m||_\infty < \lambda_+^*$, *problem* (4) *has a strict upper solution* $\overline{x} \in \text{int}\, C_+$.

Proof. By virtue of hypotheses H(f)(iii),(iv),(v) we have

$$0 \leq m(z)x^+ + f_+(z,x) \leq c_1 \left(||m||_\infty^s + |x|^{q-1}\right) \tag{5}$$

for a.a. $z \in Z$ and all $x \in \mathbb{R}$ and some $c_1 > 0$, $s > 1$. Consider the following Dirichlet problem

$$-\Delta_{p(z)}\, e(z) = 1 \text{ a.e. on } Z, \quad \text{and} \quad e|_{\partial Z} = 0. \tag{6}$$

CLAIM 1. *Problem* (6) *has a unique solution* $e \in \text{int}\, C_+$.

Recall that the operator $A : \mathrm{W}_0^{1,p(\cdot)}(Z) \to \mathrm{W}^{-1,p'(\cdot)}(Z)$ defined by (3), corresponding to the $p(\cdot)$-Laplacian, is maximal monotone and it satisfies

$$\langle A(x), x\rangle \geq ||Dx||_{p(\cdot)}^{p^-} \text{ if } ||Dx||_{p(\cdot)} > 1.$$

Therefore, A is coercive too. But a maximal monotone coercive operator is surjective. Hence problem (6) has a solution $e \in \mathrm{W}_0^{1,p(\cdot)}(Z)$, $e \not\equiv 0$, and this solution is unique due to the strict monotonicity of the operator A. Acting with the test function $-e^- \in \mathrm{W}_0^{1,p(\cdot)}(Z)$ and using Proposition 2.3 of Fan-Zhang [**8**], we obtain

$$||De^-||_{p(\cdot)} = 0,$$

and this by Poincaré's inequality implies $e^- = 0$, hence $e \geq 0$ and $e \not\equiv 0$. From Theorem 4.1 of Fan-Zhao [**9**], we have $e \in L^\infty(Z)$. Then, we can apply Theorem 1.1 of Fan [**7**] and conclude that $e \in C_+$. Note that

$$-\Delta_{p(z)}\, e(z) \geq 0 \text{ a.e. on } Z.$$

Hence, by virtue of Proposition 3.1 of Fan [**6**] (the nonlinear strong maximum principle for variable exponents), we obtain that $e \in \text{int}\, C_+$. This proves Claim 1.

CLAIM 2. *There exists* $\lambda_+^* > 0$ *such that for every* $m \in L^\infty(Z)_+$ *with* $0 < ||m||_\infty < \lambda_+^*$, *we can find* $\eta_1 \equiv \eta_1(m) > 0$ *such that*

$$c_1||m||_\infty^s + c_1\,(\eta_1||e||_\infty)^{q-1} < \bar{\eta}_1, \tag{7}$$

where $\bar{\eta}_1 = \min\{\eta_1^{p^- -1}, \eta_1^{p^+ -1}\}$.

We argue by contradiction. So, suppose that we can not find $\eta_1 > 0$ for which (7) holds. This means that there exists a sequence $\{m_n\}_{n\geq 1} \subset L^\infty(Z)_+$ such that $||m_n||_\infty \to 0$ and, for all $\eta_1 > 0$,

$$\begin{aligned} &\bar{\eta}_1 \leq c_1||m_n||_\infty^s + c_1\,(\eta_1||e||_\infty)^{q-1}, \\ \Rightarrow\ &\bar{\eta}_1 \leq c_1\eta_1^{q-1}||e||_\infty^{q-1}, \\ \Rightarrow\ &1 \leq c_1 \min\{\eta_1^{q-p^-}, \eta_1^{q-p^+}\}||e||_\infty^{q-1}. \end{aligned} \tag{8}$$

Recall that $q > p^+ \geq p^-$. Hence, if in (8) we let $\eta_1 \downarrow 0$, we reach a contradiction. This proves Claim 2.

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We set $\overline{x} = \eta_1 e \in \operatorname{int} C_+$. Then for all $u \in W_0^{1,p(\cdot)}(Z)$, $u \geq 0$, and integrating by parts, we have

$$\begin{aligned}
\langle A(\overline{x}), x\rangle &= \int_Z ||D\overline{x}||^{p(z)-2} (D\overline{x}, Du)_{\mathbb{R}^N}\, dz \qquad (9)\\
&= \int_Z (-\Delta_{p(z)}\, \overline{x}) u\, dz\\
&= \int_Z \eta_1^{p(z)-1} (-\Delta_{p(z)}\, e) u\, dz\\
&= \int_Z \eta_1^{p(z)-1} u\, dz \quad \text{(see (6))}\\
&\geq \int_Z \overline{\eta}_1\, u\, dz \quad (\text{since } u \geq 0)\\
&> \int_Z \left(c_1 ||m||_\infty^s + c_1\, (\eta_1 ||e||_\infty)^{q-1}\right) u\, dz \quad \text{(see (7))}\\
&\geq \int_Z (m(z)\overline{x}(z) + f_+(z, \overline{x}(z)))\, u(z)\, dz \qquad (10)
\end{aligned}$$

(see (5) and recall $\overline{x} = \eta_1\, e$). Since $u \in C_c^\infty(Z)_+$ is arbitrary, from (10) we infer that $\overline{x} \in \operatorname{int} C_+$ is a strict upper solution for problem (1). □

Similarly, if we consider the auxiliary problem

$$\begin{cases} -\Delta_{p(z)}\, x(z) = -m(z)x^-(z)^{r-1} + f_-(z, x(z)) \text{ a.e. on } Z,\\ x|_{\partial Z} = 0, \end{cases} \qquad (11)$$

then we produce a strict lower solution $\underline{x} \in -\operatorname{int} C_+$ for problem (11). In fact, arguing as in the proof of Proposition 2 with $\overline{\eta}_1 = \max\{\eta_1^{p^- -1}, \eta_1^{p^+ -1}\}$, we obtain the following result.

Proposition 3. *If hypotheses $H(f)$ and $H(m)$ hold, then we can find $\lambda_-^* > 0$ such that for $0 < ||m||_\infty < \lambda_-^*$, problem* (11) *has a strict lower solution $\underline{x} \in -\operatorname{int} C_+$.*

Using $\overline{x} \in \operatorname{int} C_+$ and $\underline{x} \in -\operatorname{int} C_+$ produced in Propositions 2 and 3 respectively, we will generate two smooth solutions of constant sign. For this purpose, we introduce some additional truncations of the identity and of the nonlinearity f. So, we define

$$\tau_+(z, x(z)) = \begin{cases} 0 & \text{if } x(z) < 0,\\ x(z) & \text{if } 0 \leq x(z) \leq \overline{x}(z),\\ \overline{x}(z) & \text{if } x(z) > \overline{x}(z), \end{cases}$$

and

$$\hat{f}_+(z, x(z)) = \begin{cases} 0 & \text{if } x(z) < 0,\\ f_+(z, x(z)) & \text{if } 0 \leq x(z) \leq \overline{x}(z),\\ f_+(z, \overline{x}(z)) & \text{if } x(z) > \overline{x}(z). \end{cases}$$

Clearly, both τ_+ and $\hat{f}_+$ are Carathéodory functions. We set

$$\hat{\tau}_+(x)(\cdot) = \tau_+(\cdot, x(\cdot))$$

for all $x \in W_0^{1,p(\cdot)}(Z)$, and

$$\hat{F}_+(z,x) = \int_0^x \hat{f}_+(z,s)\,ds$$

for all $x \in \mathbb{R}$. We know that $\hat{\tau}_+(x) \in W_0^{1,p(\cdot)}(Z)$ for all $x \in W_0^{1,p(\cdot)}(Z)$ (see, for example, Gasinski-Papageorgiou [**10**]). Also, we introduce the functional $\hat{\varphi}_+ : W_0^{1,p(\cdot)}(Z) \to \mathbb{R}$ defined by

$$\hat{\varphi}_+(x) = \int_Z \frac{1}{p(z)}||Dx||^{p(z)}\,dz - \frac{1}{r}\int_Z m\hat{\tau}_+(x)^r\,dz - \int_Z \hat{F}_+(z,x)\,dz$$

for all $x \in W_0^{1,p(\cdot)}(Z)$. We also consider the Euler functional $\varphi : W_0^{1,p(\cdot)}(Z) \to \mathbb{R}$ defined by

$$\varphi(x) = \int_Z \frac{1}{p(z)}||Dx||^{p(z)}\,dz - \frac{1}{r}\int_Z m(z)\,|x(z)|^r\,dz - \int_Z F(z,x)\,dz$$

for all $x \in W_0^{1,p(\cdot)}(Z)$, where $F(z,x) = \int_0^x f(z,s)\,ds$. Evidently $\hat{\varphi}_+$, $\varphi \in C^1(W_0^{1,p(\cdot)}(Z))$.

PROPOSITION 4. *If hypotheses $H(f)$ and $H(m)$ hold and $0 < ||m||_\infty < \lambda_+^*$, then problem* (1) *has a solution $x_0 \in \operatorname{int} C_+$ which is a local minimizer of φ.*

Proof. Clearly $\hat{\varphi}_+$ is coercive and exploiting the compact embedding of the Sobolev space $W_0^{1,p(\cdot)}(Z)$ into $L^{p(\cdot)}(Z)$, we can easily verify that $\hat{\varphi}_+$ is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $x_0 \in W_0^{1,p(\cdot)}(Z)$ such that

$$\begin{aligned} &\hat{\varphi}_+(x_0) = \inf\left\{\hat{\varphi}_+(x) : x \in W_0^{1,p(\cdot)}(Z)\right\} = \hat{m}_+,\\ \Rightarrow\ &\hat{\varphi}_+'(x_0) = 0,\\ \Rightarrow\ &A(x_0) = m\,\hat{\tau}_+(x_0)^{r-1} + \widehat{\mathcal{N}}_+(x_0), \end{aligned} \tag{12}$$

where $\widehat{\mathcal{N}}_+(x)(\cdot) = \hat{f}_+(\cdot, x(\cdot))$ for all $x \in W_0^{1,p(\cdot)}(Z)$.

Recall that $\overline{x} \in \operatorname{int} C_+$ is a strict upper solution for problem (4). So, we have

$$A(\overline{x}) > m\,\hat{\tau}_+(\overline{x})^{r-1} + \widehat{\mathcal{N}}_+(\overline{x}) \text{ in } W^{-1,p'(\cdot)}(Z). \tag{13}$$

From (12) and (13), it follows that

$$A(\overline{x}) - A(x_0) > m\left(\hat{\tau}_+(\overline{x})^{r-1} - \hat{\tau}_+(x_0)^{r-1}\right) + \widehat{\mathcal{N}}_+(\overline{x}) - \widehat{\mathcal{N}}_+(x_0) \tag{14}$$

in $W^{-1,p'(\cdot)}(Z)$. Hence, if on (14) we act with the test function $(x_0 - \overline{x})^+ \in W_0^{1,p(\cdot)}(Z)_+$, and since

$$\tau_+(z, x_0(z)) = \overline{x}(z) \text{ and } \hat{f}_+(z, x_0(z)) = f_+(z, \overline{x}(z)) \text{ a.e. on } \{x_0 > \overline{x}\},$$

we have

$$\begin{aligned} &< A(\overline{x}) - A(x_0), (x_0 - \overline{x})^+ > \\ &= \int_{\{x_0 > \overline{x}\}} \left(||D\overline{x}||^{p(z)-2} D\overline{x} - ||Dx_0||^{p(z)-2} Dx_0, Dx_0 - D\overline{x} \right)_{\mathbb{R}^N} dz = 0. \end{aligned} \tag{15}$$

Due to the strict monotonicity of the operator A, from (15) we infer that

$$|\{x_0 > \overline{x}\}|_N = 0,$$

where $|\cdot|_N$ denotes the Lebesgue measure on $\mathbb{R}^N$. Hence, $x_0 \leq \bar{x}$. Also, if on (12) we act with the test function $-x_0^- \in W_0^{1,p(\cdot)}(Z)$, using Proposition 2.3 of Fan-Zhang **[8]** and the Poincaré inequality, we obtain $||x_0^-|| = 0$ and so $x_0 \geq 0$. Therefore,

$$\hat{\tau}_+(x_0) = x_0 \text{ and } \widehat{N}_+(x_0) = x_0.$$

This means that (12) becomes

$$A(x_0) = m x_0^{r-1} + \widehat{N}_+(x_0),$$

from which follows that

$$-\Delta_{p(z)} x_0(z) = m(z) x_0(z)^{r-1} + f_+(z, x_0(z)) \text{ a.e. on } Z, \text{ and } x_0|_{\partial Z} = 0. \tag{16}$$

Next we show that $x_0 \neq 0$. Let $v \in C_+ \backslash \{0\}$. Since $\overline{x} \in \operatorname{int} C_+$, for $t \in (0, 1)$ small, we will have $0 \leq t\, v(z) \leq \overline{x}(z)$ for all $z \in \bar{Z}$. Then

$$\begin{aligned} \hat{\varphi}_+(tv) &= \int_Z \frac{t^{p(z)}}{p(z)} ||Dv||^{p(z)} dz - \frac{t^r}{r} \int_Z m\, \hat{\tau}_+(v)^r dz - \int_Z \hat{F}_+(z, tv)\, dz \\ &\leq \frac{t^{p^-}}{p^-} \int_Z ||Dv||^{p(z)} dz - \frac{t^r}{r} \int_Z m\, v^r dz, \end{aligned} \tag{17}$$

since $\hat{F}_+(z, tv) \geq 0$. Because $1 \leq r < p^-$, from (17) we infer that for t small enough, we will have

$$\begin{aligned} &\hat{\varphi}_+(tu_1) < 0, \\ &\Rightarrow \hat{\varphi}_+(x_0) = \hat{m}_+ \leq \hat{\varphi}_+(tv) < 0 = \hat{\varphi}_+(0), \\ &\Rightarrow x_0 \neq 0. \end{aligned}$$

From (16), as before, using the nonlinear regularity theory for variable exponents, we have $x_0 \in C_+$. Moreover, by virtue of hypothesis H(f)(v), we have

$$\Delta_{p(z)} x_0(z) \leq 0 \text{ a.e. on } Z.$$

Invoking Proposition 3.1 of Fan **[6]** (the nonlinear strong maximum principle for variable exponents), we obtain $x_0 \in \operatorname{int} C_+$. Moreover, from (16) and recalling that

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$x_0 \geq 0$, we have

$$\begin{aligned}
-\Delta_{p(z)}\, x_0(z) &= m(z)x_0(z)^{r-1} + f(z, x_0(z)) \\
&\leq c_1 \left(||m||_\infty^s + x_0(z)^{q-1}\right) \text{ (see (5))} \\
&\leq c_1 \left(||m||_\infty^s + \overline{x}(z)^{q-1}\right) \text{ (since } x_0 \leq \overline{x}) \\
&\leq c_1 \left(||m||_\infty^s + (\eta_1 ||e||_\infty)^{q-1}\right) \text{ (since } \overline{x} = \eta_1 e) \\
&< \bar{\eta}_1 \text{ (see (7))} \\
&\leq -\Delta_{p(z)}\, \overline{x}(z) \text{ a.e. on } Z.
\end{aligned}$$

Since $x_0, \overline{x} \in \operatorname{int} C_+$, invoking Theorem 3.2 of Fan [**6**], we infer that

$$\overline{x} - x_0 \in \operatorname{int} C_+.$$

This, together with the fact that $x_0 \in \operatorname{int} C_+$, imply that we can find $r > 0$ small such that

$$\hat{\varphi}_+|_{\bar{B}_r^{C_0^1}(x_0)} = \varphi|_{\bar{B}_r^{C_0^1}(x_0)},$$

where $\bar{B}_r^{C_0^1}(x_0) = \left\{x \in C_0^1(\bar{Z}) : \|x - x_0\|_{C_0^1(\bar{Z})} \leq r\right\}$. It follows that $x_0 \in \operatorname{int} C_+$ is a local $C_0^1(\bar{Z})$-minimizer of φ. Applying Theorem 3.1 of Fan [**6**], we conclude that x_0 is a local $W_0^{1,p(\cdot)}(Z)$-minimizer of φ. □

We conduct a similar analysis on the negative semiaxis. So, we define

$$\tau_-(z, x(z)) = \begin{cases} \underline{x}(z) & \text{if } x(z) < \underline{x}(z), \\ x(z) & \text{if } \underline{x}(z) \leq x(z) \leq 0, \\ 0 & \text{if } x(z) > 0, \end{cases}$$

and

$$\hat{f}_-(z, x(z)) = \begin{cases} f_-(z, \underline{x}(z)) & \text{if } x(z) < \underline{x}(z), \\ f_-(z, x(z)) & \text{if } \underline{x}(z) \leq x \leq 0, \\ 0 & \text{if } x(z) > 0. \end{cases}$$

Both are Carathéodory functions. Also, we set

$$\hat{\tau}_-(x)(\cdot) = \tau_-(\cdot, x(\cdot)) \text{ for all } x \in W_0^{1,p(\cdot)}(Z)$$

and $\hat{F}_-(z, x) = \int_0^x \hat{f}_-(z, s)\, ds$ for all $x \in \mathbb{R}$. Then, we introduce the C^1-functional $\hat{\varphi}_- : W_0^{1,p(\cdot)}(Z) \to \mathbb{R}$ defined by

$$\hat{\varphi}_-(x) = \int_Z \frac{1}{p(z)} ||Dx||^{p(z)}\, dz - \frac{1}{r} \int_Z m\, |\hat{\tau}_-(x)|^r\, dz - \int_Z \hat{F}_-(z, x(z))\, dz$$

for all $x \in W_0^{1,p(\cdot)}(Z)$. Then, reasoning as in the proof of Proposition 4, we obtain the following result.

PROPOSITION 5. *If hypotheses $H(f)$ and $H(m)$ hold and $0 < ||m||_\infty < \lambda_-^*$, the problem* (1) *has a solution $v_0 \in -\operatorname{int} C_+$ which is a local minimizer of φ.*

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5. Proof of Theorem 1. In this section, we prove Theorem 1.

Proof. From Propositions 4 and 5, we already have two solutions $x_0 \in \text{int}\, C_+$ and $v_0 \in -\text{int}\, C_+$. So, it remains to produce a third nontrivial solution for problem (1).

We consider the truncation of the identity and of the nonlinearity f at the points $\{v_0, x_0\}$. So, we introduce the functions

$$\tau_0(z,x) = \begin{cases} v_0(z) & \text{if } x < v_0(z), \\ x & \text{if } v_0(z) \le x \le x_0(z), \\ x_0(z) & \text{if } x > x_0(z), \end{cases}$$

and

$$\hat{f}_0(z,x) = \begin{cases} f(z, v_0(z)) & \text{if } x < v_0(z), \\ f(z,x) & \text{if } v_0(z) \le x \le x_0(z), \\ f(z, x_0(z)) & \text{if } x > x_0(z). \end{cases}$$

Both are Carathéodory functions. We set

$$\hat{\tau}_0(x)(\cdot) = \tau_0(\cdot, x(\cdot)) \text{ for all } x \in W_0^{1,p(\cdot)}(Z)$$

and $\hat{F}_0(z,x) = \int_0^x \hat{f}_0(z,s)\,ds$ for all $x \in \mathbb{R}$. Then, we introduce the C^1-functional $\hat{\varphi}_0 : W_0^{1,p(\cdot)}(Z) \to \mathbb{R}$ defined by

$$\hat{\varphi}_0(x) = \int_Z \frac{1}{p(z)} ||Dx||^{p(z)}\,dz - \frac{1}{r}\int_Z m\,|\hat{\tau}_0(x)|^r\,dz - \int_Z \hat{F}_0(z, x(z))\,dz$$

for all $x \in W_0^{1,p(\cdot)}(Z)$.

In parallel, we also consider the truncations of the identity and of the nonlinearity f at $\{0, x_0\}$ and at $\{v_0, 0\}$. So, we introduce the functions

$$\tau_0^+(z,x) = \begin{cases} 0 & \text{if } x < 0, \\ x & \text{if } 0 \le x \le x_0(z), \\ x_0(z) & \text{if } x > x_0(z), \end{cases} \quad \text{and } \tau_0^-(z,x) = \begin{cases} v_0(z) & \text{if } x < v_0(z), \\ x & \text{if } v_0(z) \le x \le 0, \\ 0 & \text{if } x > 0, \end{cases}$$

$$\hat{f}_0^+(z,x) = \begin{cases} 0 & \text{if } x < 0, \\ f(z,x) & \text{if } 0 \le x \le x_0(z), \\ f(z, x_0(z)) & \text{if } x > x_0(z), \end{cases}$$

and

$$\hat{f}_0^-(z,x) = \begin{cases} f(z, v_0(z)) & \text{if } x < v_0(z), \\ f(z,x) & \text{if } v_0(z) \le x \le 0, \\ 0 & \text{if } x > 0. \end{cases}$$

These are Carathéodory functions. We set

$$\hat{\tau}_0^\pm(x)(\cdot) = \tau_0^\pm(\cdot, x(\cdot)) \text{ for all } x \in W_0^{1,p(\cdot)}(Z)$$

and $\hat{F}_0^\pm(z,x) = \int_0^x \hat{f}_0^\pm(z,s)\,ds$ for all $x \in \mathbb{R}$. Then, we introduce the C^1-functionals $\hat{\varphi}_0^\pm : W_0^{1,p(\cdot)}(Z) \to \mathbb{R}$ defined by

$$\hat{\varphi}_0^\pm(x) = \int_Z \frac{1}{p(z)} ||Dx||^{p(z)}\,dz - \frac{1}{r}\int_Z m\,|\hat{\tau}_0^\pm(x)|^r\,dz - \int_Z \hat{F}_0^\pm(z, x(z))\,dz$$

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for all $x \in W_0^{1,p(\cdot)}(Z)$. Finally we consider the following order intervals

$$T_0 = [v_0, x_0] = \left\{x \in W_0^{1,p(\cdot)}(Z) : v_0(z) \le x(z) \le x_0(z) \text{ a.e. on } Z\right\},$$
$$T_0^+ = [0, x_0] = \left\{x \in W_0^{1,p(\cdot)}(Z) : 0 \le x(z) \le x_0(z) \text{ a.e. on } Z\right\},$$
$$T_0^- = [v_0, 0] = \left\{x \in W_0^{1,p(\cdot)}(Z) : v_0(z) \le x(z) \le 0 \text{ a.e. on } Z\right\}.$$

CLAIM 3. *The critical points of $\hat{\varphi}_0$ are in T_0, and of $\hat{\varphi}_0^\pm$ are in $T^\pm$ respectively.*

We do the proof for $\hat{\varphi}_0$, the proofs for $\hat{\varphi}_0^\pm$ being similar.

So, let $x \in W_0^{1,p(\cdot)}(Z)$ be a critical point of $\hat{\varphi}_0$. Then

$$\hat{\varphi}_0'(x) = 0,$$
$$\Rightarrow A(x) = m\,\hat{\tau}_0(x)^{r-1} + \widehat{\mathcal{N}}_0(x), \tag{18}$$

where $\widehat{\mathcal{N}}_0(x)(\cdot) = \hat{f}_0(\cdot, x(\cdot))$ for all $x \in W_0^{1,p(\cdot)}(Z)$. On (18), we act with the test function $(x - x_0)^+ \in W_0^{1,p(\cdot)}(Z)$. Then, from the definition of $\hat{\tau}_0$ and $\widehat{\mathcal{N}}_0$, we have

$$\begin{aligned}\langle A(x), (x - x_0)^+\rangle &= \int_Z m\,\hat{\tau}_0(x)^{r-1}(x - x_0)^+\,dz + \int_Z \hat{f}_0(z, x)(x - x_0)^+\,dz\\ &= \int_Z m x_0^{r-1}(x - x_0)^+\,dz + \int_Z f(z, x_0)(x - x_0)^+\,dz\\ &= \langle A(x_0), (x - x_0)^+\rangle,\end{aligned}$$

where the last equality is a result of the fact that $x_0 \in \operatorname{int} C_+$ is a solution of problem (1). Hence,

$$\langle A(x) - A(x_0), (x - x_0)^+\rangle = 0.$$

This, by virtue of the strict monotonicity of A, implies that $x \le x_0$. Similarly, we show that $v_0 \le x$. Therefore, $x \in T_0$. Similarly for $\hat{\varphi}_0^\pm$. This proves the Claim 3.

Clearly we may assume that x_0 is the only nontrivial critical point of $\hat{\varphi}_0^+$ and v_0 is the only nontrivial critical point of $\hat{\varphi}_0^-$. Otherwise, by virtue of Claim 3, we already have a third nontrivial solution, which is actually of constant sign.

CLAIM 4. *Both x_0 and v_0 are local minimizers of $\hat{\varphi}_0$.*

Clearly $\hat{\varphi}_0^+$ is coercive and sequentially weakly lower semicontinuous. So, we can find $u_0 \in W_0^{1,p(\cdot)}(Z)$ such that

$$\hat{\varphi}_0^+(u_0) = \inf\left\{\hat{\varphi}_0^+(x) : x \in W_0^{1,p(\cdot)}(Z)\right\} = \hat{m}_0^+.$$

As in the proof of Proposition 4, we can check that $\hat{m}_0^+ < 0$ and so $u_0 \ne 0$. Because u_0 is a critical point of $\hat{\varphi}_0^+$, it follows that $u_0 = x_0 \in \operatorname{int} C_+$. Then, we can find $r > 0$ small such that

$$\hat{\varphi}_0\big|_{\bar{B}_r^{C_0^1(\bar{Z})}(x_0)} = \hat{\varphi}_0^+\big|_{\bar{B}_r^{C_0^1(\bar{Z})}(x_0)},$$

which implies that x_0 is a local $C_0^1(\bar{Z})$-minimizer of $\hat{\varphi}_0$. Invoking Theorem 3.1 of [**6**], we conclude that x_0 is a local $W_0^{1,p(\cdot)}(Z)$-minimizer of $\hat{\varphi}_0$. Similarly for $v_0 \in -\text{int}\, C_+$. This proves the Claim 4.

Without any loss of generality, we may assume that

$$\hat{\varphi}_0(v_0) \leq \hat{\varphi}_0(x_0).$$

Also, since x_0 is an isolated local minimizer of $\hat{\varphi}_0$, arguing as in Proposition 29 of [**2**], we can find $r > 0$ small such that

$$\hat{\varphi}_0(v_0) \leq \hat{\varphi}_0(x_0). \tag{19}$$

We consider the sets

$$E_0 = \{v_0, x_0\}, E = T_0 = [v_0, x_0] \text{ and } D = \partial B_r(x_0).$$

It is to check that the pair $\{E_0, E\}$ and D, are linking in $W_0^{1,p(\cdot)}(Z)$ (see also [**10**, p.642]). Also, because $\hat{\varphi}_0$ is coercive, we can easily check (using the fact that A is of type $(S)_+$), that it satisfies the PS-condition. So, we can apply the Linking Theorem (see [**10**, p.644]) and obtain $y_0 \in W_0^{1,p(\cdot)}(Z)$, a critical point of $\hat{\varphi}_0$, such that

$$\hat{\varphi}_0(y_0) = \inf_{\gamma\in\Gamma} \max_{t\in[-1,1]} \hat{\varphi}_0(\gamma(t)) > \beta_r \text{ (see(19))}, \tag{20}$$

where

$$\Gamma = \left\{\gamma \in C([-1, 1], W_0^{1,p(\cdot)}(Z)) : \gamma(-1) = v_0 \text{ and } \gamma(1) = x_0\right\}.$$

Therefore, y_0 is a critical point of mountain pass-type and from (19) and (20), we have $y_0 \neq v_0$, $y_0 \neq x_0$. Moreover, Claim 3 implies that $y_0 \in T_0$. So, $v_0 \leq y_0 \leq x_0$.

Since y_0 is a critical point of mountain pass-type, we have

$$C_1(\hat{\varphi}_0, y_0) \neq 0 \tag{21}$$

(see [**5**] and [**16**]). On the other hand, by virtue of hypothesis H(f)(iv), we can find $\beta > 0$ and $\delta \in (0, \delta_0)$ such that

$$0 \leq f(z, x)x \leq \beta|x|^{\vartheta}$$

for a.a. $z \in Z$ and all $|x| \leq \delta$.

If $|x| \leq \delta$ also satisfies $x \leq x_0(z)$, then $f(z, x) = \hat{f}_0(z, x)$ and so

$$0 \leq \hat{f}_0(z, x)x \leq \beta|x|^{\vartheta}.$$

If $|x| \leq \delta$ also satisfies $x > x_0(z)$ (resp. $x < v_0(z)$), then $f(z, x) = \hat{f}_0(z, x_0(z))$ (resp. $f(z, x) = \hat{f}_0(z, v_0(z))$). Hence, using hypotheses H(f)(v), we have

$$\hat{f}_0(z, x)x = f(z, u(z))x \leq f(z, x)x \leq \beta|x|^{\vartheta}$$

for a.a. $z \in Z$ and with $u = x_0$ or v_0. Therefore, we can say that for almost all $z \in Z$ and $|x| \leq \delta$ we have

$$\hat{f}_0(z, x)x \leq \beta|x|^{\vartheta}. \tag{22}$$

Hence, if $\mu \in (r, p^+)$, we have

$$\left(\frac{\mu}{r} - 1\right) |x|^r + \mu \hat{F}_0(z, x) - \hat{f}_0(z, x)x \geq \left(\frac{\mu}{r} - 1\right) |x|^r - \beta |x|^\vartheta \tag{23}$$

for a.a. $z \in Z$ and all $x \in \mathbb{R}$ (see (22) and recall $\hat{F}_0 \geq 0$). Since $r < \vartheta$ and $|x| \leq \delta < 1$, from (23) and by taking δ even smaller if necessary, we have

$$\left(\frac{\mu}{r} - 1\right) |x|^r + \mu \hat{F}_0(z, x) - \hat{f}_0(z, x)x \geq 0.$$

Invoking Proposition 2.1 of [**12**], we have

$$C_k(\hat{\varphi}_0, 0) = 0 \text{ for all } k \geq 0. \tag{24}$$

Comparing (21) and (24), we conclude that $y_0 \neq 0$, since $R < p^-$. Moreover, as before, using the regularity theorem of [**7**], we have $y_0 \in C_0^1(\bar{Z})$ and since $y_0 \in T_0$, it is a solution of problem (1). □

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