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The uniform distribution modulo one of certain subsequences of ordinates of zeros of the zeta function

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Abstract

On the assumption of the Riemann hypothesis and a spacing hypothesis for the nontrivial zeros $1/2+i\gamma$ of the Riemann zeta function, we show that the sequence

$$\Gamma_{[a,b]}=\left\{\gamma : \gamma>0 \quad \text{and} \quad \frac{\log\left(|\zeta^{(m_\gamma)}(\frac{1}{2}+i\gamma)|/(\log\gamma)^{m_\gamma}\right)}{\sqrt{\frac{1}{2}\log\log\gamma}}\in[a,b]\right\},$$

where the γ are arranged in increasing order, is uniformly distributed modulo one. Here a and b are real numbers with $a<b$, and m_γ denotes the multiplicity of the zero $1/2+i\gamma$. The same result holds when the γ's are restricted to be the ordinates of simple zeros. With an extra hypothesis, we are also able to show an equidistribution result for the scaled numbers $\gamma(\log T)/2\pi$ with $\gamma\in\Gamma_{[a,b]}$ and $0<\gamma\le T$.

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1. *Introduction*

It is well known that the positive ordinates γ of the nontrivial zeros $\rho=\beta+i\gamma$ of the Riemann zeta function, when arranged in increasing order, are uniformly distributed modulo one. This was proved by Rademacher [7] in the 1950s under the assumption of the Riemann hypothesis. Elliott [3] later pointed out that this could be shown unconditionally. Our aim in this paper is to prove that if the Riemann hypothesis holds and a plausible hypothesis about the spacing of the γ's is true, then the γ are also uniformly distributed modulo one when we restrict to certain subsequences.

Throughout we assume the Riemann hypothesis so that every nontrivial zero of the zeta function has the form $\rho=1/2+i\gamma$. Then $N(T)$, the number of ordinates γ in the interval $(0,T]$ is given by

$$N(T)=\frac{T}{2\pi}\log\frac{T}{2\pi}-\frac{T}{2\pi}+O\left(\frac{\log T}{\log\log T}\right) \qquad (1\cdot1)$$

(see Titchmarsh [8, chapter 14]). Note that unconditionally, the error term is $O(\log T)$. We also assume that for some $0 < \delta \le 1$ the following spacing hypothesis holds for the zeros.

HYPOTHESIS $\mathcal{H}_\delta$ 1. *Let γ^+ be the next larger ordinate of a zero of the zeta function after the ordinate γ with the understanding that $\gamma^+ = \gamma$ if and only if $1/2 + i\gamma$ is a multiple zero. Then there exists a positive constant M such that, uniformly for $0 < \lambda < 1$, we have*

$$\limsup_{T\to\infty} \frac{1}{N(T)} \#\left\{0 < \gamma \le T : 0 \le \gamma^+ - \gamma \le \frac{\lambda}{\log T}\right\} \le M\lambda^\delta.$$

Hypothesis $\mathscr{H}_\delta$ is credible because Hypothesis $\mathscr{H}_1$ follows from Montgomery's pair correlation conjecture which, in turn, implies Hypothesis $\mathscr{H}_\delta$ for every $\delta \in (0, 1]$. Notice also that Hypothesis $\mathscr{H}_\delta$ implies that all but o(N(T)) of the zeros are simple, a fact we shall use later.

Let m_γ denote the multiplicity of the zero $\rho = \frac{1}{2} + i\gamma$, and let $a < b$ be real numbers. The sequences we wish to consider are

$$\Gamma_{[a,b]} = \left\{\gamma > 0 : \frac{\log\left(\left|\zeta^{(m_\gamma)}(\frac{1}{2} + i\gamma)\right|/(\log\gamma)^{m_\gamma}\right)}{\sqrt{\frac{1}{2}\log\log\gamma}} \in [a, b]\right\}$$

and

$$\Gamma^*_{[a,b]} = \left\{\gamma > 0 : m_\gamma = 1 \text{ and } \frac{\log\left(\left|\zeta'(\frac{1}{2} + i\gamma)\right|/\log\gamma\right)}{\sqrt{\frac{1}{2}\log\log\gamma}} \in [a, b]\right\},$$

where the γ are listed in increasing order. Our first theorem, a slight modification of a recent result of Çiçek [2], provides the counting functions of these sequences.

THEOREM 1·1. *Assume the Riemann hypothesis is true and that Hypothesis $\mathscr{H}_\delta$ holds for some $\delta \in (0, 1]$. Let $\max(|a|, |b|) \ll (\log\log\log T)^{\frac{1}{2}-\epsilon}$, where $\epsilon > 0$. Then for all sufficiently large T,*

$$N_{[a,b]}(T) := \sum_{\substack{0<\gamma\le T\\ \gamma\in\Gamma_{[a,b]}}} 1 = \frac{N(T)}{\sqrt{2\pi}} \int_a^b e^{-x^2/2}\,dx + O\left(N(T)\frac{(\log\log\log T)^2}{\sqrt{\log\log T}}\right). \tag{1·2}$$

For the sequence $\Gamma^*_{[a,b]}$ we have

$$N^*_{[a,b]}(T) := \sum_{\substack{0<\gamma\le T\\ \gamma\in\Gamma^*_{[a,b]}}} 1 = \frac{N(T)}{\sqrt{2\pi}} \int_a^b e^{-x^2/2}dx + o(N(T)). \tag{1·3}$$

Observe that (1·3) follows immediately from (1·2) since Hypothesis $\mathscr{H}_\delta$ implies that all but $o(N(T))$ of the zeros are simple.

Our next theorem is our main uniform distribution result.

THEOREM 1·2. *Assume the Riemann hypothesis and that Hypothesis $\mathscr{H}_\delta$ is true for some $0 < \delta \le 1$. Let a and b be either fixed, or functions of T for which* $\max(|a|, |b|) \ll$

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$(\log\log\log T)^{\frac{1}{2}-\epsilon}$, *where* $\epsilon > 0$, *and* $\int_a^b e^{-x^2/2}dx \gg 1$. *Then the sequences* $\Gamma_{[a,b]}$ *and* $\Gamma^*_{[a,b]}$ *are uniformly distributed modulo one.*

The average gap between the ordinates $\gamma \in (0, T]$ is $2\pi/\log T$ by (1·1). Thus the numbers $\gamma(\log T)/2\pi$ have average spacing one. Not surprisingly, it is more difficult to prove that these numbers are equidistributed modulo one. In fact, it is not known. This is also true for the numbers $\gamma(\log T)/2\pi$ with $\gamma \in (0, T]$ and $\gamma \in \Gamma_{[a,b]}$ or $\Gamma^*_{[a,b]}$. However, if we assume the following further conjecture of the second author (confer [5]), we can show uniform distribution in all three cases.

CONJECTURE 1. *For* $x, T \geq 2$ *and any fixed* $\epsilon > 0$,

$$\sum_{0<\gamma\leq T} x^{i\gamma} \ll Tx^{-\frac{1}{2}+\epsilon} + T^{\frac{1}{2}}x^{\epsilon}.$$

Theorem 4 of [5] provides evidence for Conjecture 1. It says that if $\psi(y) = \sum_{n\leq y}\Lambda(n)$, where $\Lambda(n)$ is the von Mangoldt function, then Conjecture 1 implies that

$$\psi(y+h) - \psi(y) = h + O(h^{\frac{1}{2}}y^{\epsilon}) \tag{1·4}$$

for $1 \leq h \leq y$ and $\epsilon > 0$. Conversely, (1·4) implies a weighted form of Conjecture 1, namely,

$$\sum_{\gamma} x^{i\gamma}\left(\frac{\sin \gamma/2T}{\gamma/2T}\right)^2 \ll Tx^{-\frac{1}{2}+\epsilon} + T^{\frac{1}{2}}x^{\epsilon}.$$

Using Conjecture 1, one may prove the following two theorems. In both, $\{x\}$ denotes the fractional part of x.

THEOREM 1·3. *Assume the Riemann hypothesis and Conjecture 1. If* $[\alpha, \beta]$ *is a subinterval of [0,1], then*

$$\sup_{\alpha,\beta}\left|\sum_{\substack{0<\gamma\leq T\\ \{\gamma(\log T)/2\pi\}\in[\alpha,\beta]}} 1 \; - \; (\beta-\alpha)N(T)\right| = o(N(T)). \tag{1·5}$$

THEOREM 1·4. *Assume the Riemann hypothesis, Hypothesis* $\mathscr{H}_\delta$ *for some* $0 < \delta \leq 1$*, and Conjecture 1. Let a and b be either fixed, or functions of T for which* $\max(|a|, |b|) \ll (\log\log\log T)^{\frac{1}{2}-\epsilon}$, *where* $\epsilon > 0$, *and* $\int_a^b e^{-x^2/2}dx \gg 1$. *Then if* $[\alpha, \beta]$ *is a subinterval of [0,1],*

$$\sup_{\alpha,\beta}\left|\sum_{\substack{0<\gamma\leq T,\, \gamma\in\Gamma_{[a,b]}\\ \{\gamma(\log T)/2\pi\}\in[\alpha,\beta]}} 1 \; - \; (\beta-\alpha)N_{[a,b]}(T)\right| = o(N_{[a,b]}(T)). \tag{1·6}$$

This also holds with $\Gamma_{[a,b]}$ *replaced by* $\Gamma^*_{[a,b]}$.

The method we use to prove Theorems 1·1, 1·2 and 1·4 builds on techniques used in the first author's recent thesis to prove a discrete analogue of Selberg's central limit theorem (see [2] and Lemma 2·1 below).

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Throughout we write $e(u)=e^{2\pi iu}$. We let C denote a positive constant that may be different at different occurrences, and we let $\mathbb{1}_{[a,b]}$ denote the indicator function of the interval $[a, b]$.

2. *Proof of Theorem 1·1*

Theorem 1·1 follows easily from the next lemma.

LEMMA 2·1. *Assume the Riemann hypothesis and that Hypothesis $\mathscr{H}_\delta$ holds for some $\delta \in (0, 1]$. Let m_γ denote the multiplicity of the zero $\rho = 1/2 + i\gamma$. Then for all sufficiently large T,*

$$\#\left\{0<\gamma\le T : \frac{\log\left(\left|\zeta^{(m_\gamma)}(\tfrac12+i\gamma)\right|/(\log T)^{m_\gamma}\right)}{\sqrt{\tfrac12\log\log T}} \in [a,b]\right\} = \frac{N(T)}{\sqrt{2\pi}}\int_a^b e^{-x^2/2}\,dx + O\left(N(T)\frac{(\log\log\log T)^2}{\sqrt{\log\log T}}\right). \tag{2·1}$$

This follows from the proof of Theorem 1·4 combined with corollary 2·3 of Çiçek [2].

Note that

$$N_{[a,b]}(T) = \sum_{0<\gamma\le T} \mathbb{1}_{[a,b]}\left(\frac{\log\left(\left|\zeta^{(m_\gamma)}(\tfrac12+i\gamma)\right|/(\log\gamma)^{m_\gamma}\right)}{\sqrt{\tfrac12\log\log\gamma}}\right).$$

Thus, to prove (1·2), we need to show that we may replace $\log\gamma$ and $\log\log\gamma$ here by $\log T$ and $\log\log T$, respectively, at the cost of a reasonable error term. To see this, note that by (1·1), the terms in the sum with $0<\gamma\le T/\log T$ contribute at most $O(T)$, hence

$$N_{[a,b]}(T) = \sum_{T/\log T<\gamma\le T} \mathbb{1}_{[a,b]}\left(\frac{\log\left(\left|\zeta^{(m_\gamma)}(\tfrac12+i\gamma)\right|/(\log\gamma)^{m_\gamma}\right)}{\sqrt{\tfrac12\log\log\gamma}}\right) + O(T). \tag{2·2}$$

For $T/\log T<\gamma\le T$ we easily find that

$$\frac{\log\left(\left|\zeta^{(m_\gamma)}(\tfrac12+i\gamma)\right|/(\log\gamma)^{m_\gamma}\right)}{\sqrt{\tfrac12\log\log\gamma}} = \frac{\log\left(\left|\zeta^{(m_\gamma)}(\tfrac12+i\gamma)\right|/(\log T)^{m_\gamma}\right)+O(m_\gamma\log\log T/\log T)}{\sqrt{\tfrac12\log\log T}} \times\left(1+O\left(\frac{1}{\log T}\right)\right).$$

Using (1·1) again, we see that $m_\gamma \ll \log T/\log\log T$. Thus, if we impose the condition that $\max(|a|,|b|) \ll (\log\log\log T)^{\frac12-\epsilon}$, then when the expression on the left lies in the interval $[a, b]$, the right-hand side equals

$$\frac{\log\left(\left|\zeta^{(m_\gamma)}(\tfrac12+i\gamma)\right|/(\log T)^{m_\gamma}\right)}{\sqrt{\tfrac12\log\log T}} + O\left(\frac{1}{\sqrt{\log\log T}}\right).$$

Using this with (2·1), we see that replacing $\log\gamma$ and $\log\log\gamma$ in (2·2) by $\log T$ and $\log\log T$, respectively, changes (2·2) by no more than $O(N(T)(\log\log\log T)^2/\sqrt{\log\log T})$. Hence

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$$N_{[a,b]}(T) = \sum_{T/\log T<\gamma\leq T} \mathbb{1}_{[a,b]}\left(\frac{\log\left(\left|\zeta^{(m_\gamma)}(\frac{1}{2}+i\gamma)\right|/(\log T)^{m_\gamma}\right)}{\sqrt{\frac{1}{2}\log\log T}}\right) + O\left(N(T)\frac{(\log\log\log T)^2}{\sqrt{\log\log T}}\right).$$

Extending the sum back to the full range $0<\gamma\leq T$ and using (2·1) again, we see that for sufficiently large T,

$$N_{[a,b]}(T) = \frac{N(T)}{\sqrt{2\pi}}\int_a^b e^{-x^2/2}\,dx + O\left(N(T)\frac{(\log\log\log T)^2}{\sqrt{\log\log T}}\right),$$

provided $\max(|a|,|b|) \ll (\log\log\log T)^{\frac{1}{2}-\epsilon}$. This proves (1·2). It has already been noted that (1·3) follows from (1·2) and Hypothesis $\mathscr{H}_\delta$, so the proof of Theorem 1·1 is complete.

3. *Proof of Theorem 1·2*

We assume the Riemann hypothesis, Hypothesis $\mathscr{H}_\delta$, and that

$$\max(|a|,|b|) \ll (\log\log\log T)^{\frac{1}{2}-\epsilon} \text{ with } \epsilon>0.$$

Our assumption that a and b are either fixed, or functions of T for which $\int_a^b e^{-x^2/2}dx \gg 1$ means, by Theorem 1·1, that $N_{[a,b]}(T) \gg N(T)$. Hence, by Weyl's criterion [**10**], the sequence $\Gamma_{[a,b]}$ is uniformly distributed modulo one if, for each fixed positive integer ℓ,

$$\sum_{0<\gamma\leq T} e(\ell\gamma)\ \mathbb{1}_{[a,b]}\left(\frac{\log\left(\left|\zeta^{(m_\gamma)}(\frac{1}{2}+i\gamma)\right|/(\log \gamma)^{m_\gamma}\right)}{\sqrt{\frac{1}{2}\log\log \gamma}}\right) = o(N(T)) \tag{3·1}$$

as $T\to\infty$. By the same argument we used in the last section, replacing $\log\gamma$ and $\log\log\gamma$ here by $\log T$ and $\log\log T$, respectively, changes the sum by at most $o(N(T)$. Thus, it suffices to show that

$$\sum_{0<\gamma\leq T} e(\ell\gamma)\ \mathbb{1}_{[a,b]}\left(\frac{\log\left(\left|\zeta^{(m_\gamma)}(\frac{1}{2}+i\gamma)\right|/(\log T)^{m_\gamma}\right)}{\sqrt{\frac{1}{2}\log\log T}}\right) = o(N(T)).$$

Now let $P(\gamma) = \sum_{p\leq X^2} \frac{1}{p^{1/2+i\gamma}}$, where p runs over the primes. The gist of corollary 2·3 in [**2**] and some of the analysis following it, is that $P(\gamma)$ is on average a good approximation to

$$\frac{\log\left(\left|\zeta^{(m_\gamma)}(\frac{1}{2}+i\gamma)\right|/(\log T)^{m_\gamma}\right)}{\sqrt{\frac{1}{2}\log\log T}},$$

provided X is sufficiently large. Indeed, from the discussion in sections 5·5 and 6 of [**2**] it follows that

$$\sum_{0<\gamma\leq T} \mathbb{1}_{[a,b]}\left(\frac{\log\left(\left|\zeta^{(m_\gamma)}(\frac{1}{2}+i\gamma)\right|/(\log T)^{m_\gamma}\right)}{\sqrt{\frac{1}{2}\log\log T}}\right) = \sum_{0<\gamma\leq T} \mathbb{1}_{[a,b]}\left(\frac{\Re P(\gamma)}{\sqrt{\frac{1}{2}\log\log T}}\right) + O\left(N(T)\frac{(\log\log\log T)^2}{\sqrt{\log\log T}}\right).$$

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This is the key result that allows us to prove our theorem. An immediate consequence is that

$$\mathbb{1}_{[a,b]}\left(\frac{\log\left(\left|\zeta^{(m_\gamma)}(\frac{1}{2}+i\gamma)\right|/(\log T)^{m_\gamma}\right)}{\sqrt{\frac{1}{2}\log\log T}}\right)=\mathbb{1}_{[a,b]}\left(\frac{\Re P(\gamma)}{\sqrt{\frac{1}{2}\log\log T}}\right)$$

for all but $O\big(N(T)(\log\log\log T)^2/\sqrt{\log\log T}\big)=o(N(T))$ values of γ in $(0, T]$. Therefore,

$$\sum_{0<\gamma\le T} e(\ell\gamma)\mathbb{1}_{[a,b]}\left(\frac{\log\left(\left|\zeta^{(m_\gamma)}(\frac{1}{2}+i\gamma)\right|/(\log T)^{m_\gamma}\right)}{\sqrt{\frac{1}{2}\log\log T}}\right)$$
$$=\sum_{0<\gamma\le T} e(\ell\gamma)\mathbb{1}_{[a,b]}\left(\frac{\Re P(\gamma)}{\sqrt{\frac{1}{2}\log\log T}}\right)+O\left(N(T)\frac{(\log\log\log T)^2}{\sqrt{\log\log T}}\right). \tag{3·2}$$

Writing

$$A=A(T)=a\sqrt{\frac{1}{2}\log\log T}\quad\text{and}\quad B=B(T)=b\sqrt{\frac{1}{2}\log\log T},$$

we see that to prove our theorem we must show that

$$\sum_{0<\gamma\le T} e(\ell\gamma)\mathbb{1}_{[A,B]}(\Re P(\gamma))=o(N(T)) \tag{3·3}$$

for each positive integer ℓ. To do this, we replace the characteristic function $\mathbb{1}_{[A,B]}$ by an approximation. Let $\Omega>0$ and set

$$F_\Omega(x)=\Im\int_0^\Omega G\Big(\frac{\omega}{\Omega}\Big)\exp(2\pi i x\omega)\frac{d\omega}{\omega}, \tag{3·4}$$

where

$$G(u)=\frac{2u}{\pi}+2u(1-u)\cot(\pi u)\quad\text{for}\quad u\in[0,1].$$

Then

$$\operatorname{sgn}(x)=F_\Omega(x)+O\left(\frac{\sin^2(\pi\Omega x)}{(\pi\Omega x)^2}\right), \tag{3·5}$$

(see [9, pp. 26–29]). It follows that

$$\mathbb{1}_{[A,B]}(x)=\frac{1}{2}F_\Omega(x-A)-\frac{1}{2}F_\Omega(x-B)+O\left(\frac{\sin^2(\pi\Omega(x-A))}{(\pi\Omega(x-A))^2}\right)+O\left(\frac{\sin^2(\pi\Omega(x-B))}{(\pi\Omega(x-B))^2}\right). \tag{3·6}$$

This is the desired approximation of $\mathbb{1}_{[A,B]}$. Here we take $x=\Re P(\gamma)=\Re\sum_{p\le X^2}\frac{1}{p^{1/2+i\gamma}}$ and

$$X=T^{\frac{1}{(\log\log T)^{20}}},\qquad \Omega=(\log\log T)^2. \tag{3·7}$$

Now, it was shown in the course of the proof of proposition 5·5 in [2] (with slightly different notation and parameters) that

$$\sum_{0<\gamma\leq T} \frac{\sin^2\left(\pi\Omega(\Re P(\gamma)-A)\right)}{\left(\pi\Omega(\Re P(\gamma)-A)\right)^2} \ll \frac{N(T)}{\Omega},$$

and similarly for the sum with A replaced by B. Thus, by (3·6),

$$\begin{aligned}&\sum_{0<\gamma\leq T} e(\ell\gamma)\mathbb{1}_{[A,B]}(\Re P(\gamma))\\ &= \frac{1}{2}\sum_{0<\gamma\leq T} e(\ell\gamma)F_\Omega(\Re P(\gamma)-A) - \frac{1}{2}\sum_{0<\gamma\leq T} e(\ell\gamma)F_\Omega(\Re P(\gamma)-B) + O\left(\frac{N(T)}{\Omega}\right).\end{aligned} \qquad (3·8)$$

From this and (3·3) we see that it suffices to prove that

$$\sum_{0<\gamma\leq T} e(\ell\gamma)F_\Omega(\Re P(\gamma)-A) = o(N(T)) \qquad (3·9)$$

for each positive integer ℓ, and similarly for the sum with A replaced by B.

To this end we use (3·4) to write

$$\sum_{0<\gamma\leq T} e(\ell\gamma)F_\Omega(\Re P(\gamma)-A) = \sum_{0<\gamma\leq T} e(\ell\gamma)\Im\int_0^\Omega G\left(\frac{\omega}{\Omega}\right)e^{-2\pi iA\omega}\exp\left(2\pi i\omega\Re P(\gamma)\right)\frac{d\omega}{\omega}. \qquad (3·10)$$

By Taylor's theorem, for any positive integer K,

$$\exp\left(2\pi i\omega\Re P(\gamma)\right) = 1 + \sum_{1\leq k<K}\frac{(2\pi i\omega\Re P(\gamma))^k}{k!} + O\left(\frac{(2\pi\omega|\Re P(\gamma)|)^K}{K!}\right).$$

Inserting this in (3·10) and taking

$$K = 2\left[(\log\log T)^6\right], \qquad (3·11)$$

where $[\,x]$ denotes the greatest integer less than or equal to x, we obtain

$$\begin{aligned}\sum_{0<\gamma\leq T} e(\ell\gamma)F_\Omega(\Re P(\gamma)-A) &= F_\Omega(A)\sum_{0<\gamma\leq T} e(\ell\gamma)\\ &+ \sum_{0<\gamma\leq T} e(\ell\gamma)\Im\int_0^\Omega G\left(\frac{\omega}{\Omega}\right)e^{-2\pi iA\omega}\sum_{1\leq k<K}\frac{(2\pi i\omega)^k}{k!}(\Re P(\gamma))^k\frac{d\omega}{\omega}\\ &+ O\left(\sum_{0<\gamma\leq T}|\Re P(\gamma)|^K\int_0^\Omega G\left(\frac{\omega}{\Omega}\right)\frac{(2\pi\omega)^K}{K!}\frac{d\omega}{\omega}\right).\end{aligned} \qquad (3·12)$$

We estimate the sums over γ on the right-hand side of the equation by means of the following result, which is an immediate consequence of an unconditional theorem and its corollary in [5] (see [4] also).

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LEMMA 3·1. *Assume the Riemann hypothesis and let $x, T > 1$. Then*

$$\sum_{0<\gamma\le T} x^{i\gamma} = -\frac{T}{2\pi}\frac{\Lambda(x)}{\sqrt{x}} + O(\sqrt{x}\log 2xT\log\log 3x) + O\Big(\log x \min\Big(\frac{T}{\sqrt{x}}, \frac{\sqrt{x}}{\langle x\rangle}\Big)\Big)$$
$$+ O\Big(\log 2T \min\Big(\frac{T}{\sqrt{x}}, \frac{1}{\sqrt{x}\log x}\Big)\Big). \qquad (3·13)$$

Here $\Lambda(x) = \log p$ if x is a positive integral power of a prime p and $\Lambda(x) = 0$ for all other real numbers x, and $\langle x\rangle$ denotes the distance from x to the nearest prime power other than x itself. If $0 < x < 1$, (3·13) also holds provided we replace x on the right-hand side by $1/x$.

When $x > 1$, we will write (3·13) as

$$\sum_{0<\gamma\le T} x^{i\gamma} = M(x) + E_1(x) + E_2(x) + E_3(x), \qquad (3·14)$$

where $M(x)$ and $E_i(x)$, $i = 1, 2, 3$, also depend on T. When $0 < x < 1$, all the x's on the right-hand side of (3·14) are to be replaced by $1/x$. Note that when $x = 1$, $\sum_{0<\gamma\le T} x^{i\gamma} = N(T)$.

Returning to (3·12), observe that by (3·5), $F_\Omega(A) \ll 1$. Furthermore, taking $x = e^{2\pi\ell} > 1$ in (3·13), we find that $\sum_{0<\gamma\le T} e(\ell\gamma) \ll T$. Thus, the first term on the right-hand side of (3·12) is

$$F_\Omega(A) \sum_{0<\gamma\le T} e(\ell\gamma) \ll T. \qquad (3·15)$$

For the final term in (3·12) we use lemma 5·2 of [**2**], which says that

$$\sum_{0<\gamma\le T} |\Re P(\gamma)|^K \ll (cK\Psi)^{K/2} N(T),$$

where $\Psi = \log\log T$. From this and Stirling's approximation, we find that the O-term in (3·12) is

$$\ll N(T)\int_0^\Omega G\Big(\frac{\omega}{\Omega}\Big)\frac{(2\pi\omega)^K}{K!}(cK\Psi)^{K/2}\frac{d\omega}{\omega}$$
$$\ll N(T)\int_0^\Omega G\Big(\frac{\omega}{\Omega}\Big)\frac{\omega(2\pi e)^K\omega^{K-1}}{K^K}(cK\Psi)^{K/2}\frac{d\omega}{\omega}.$$

By (3·7) and (3·11), and since G is bounded, this is

$$\ll N(T)\int_0^\Omega G\Big(\frac{\omega}{\Omega}\Big)\Big(\frac{c\,\Omega\sqrt{\Psi}}{\sqrt{K}}\Big)^K d\omega \ll \frac{N(T)}{2^K}\int_0^\Omega G\Big(\frac{\omega}{\Omega}\Big)\,d\omega \ll T.$$

Combining this and (3·15), we may rewrite (3·12) as

$$\sum_{0<\gamma\le T} e(\ell\gamma) F_\Omega(\Re P(\gamma) - A)$$
$$= \int_0^\Omega G\Big(\frac{\omega}{\Omega}\Big)\sum_{1\le k<K}\Im\big(e^{-2\pi iA\omega}\, i^k\big)\frac{(2\pi\omega)^k}{k!}\sum_{0<\gamma\le T} e(\ell\gamma)(\Re P(\gamma))^k\frac{d\omega}{\omega} + O(T). \qquad (3·16)$$

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To estimate the right-hand side we next bound the sums

$$S(k) = \sum_{0<\gamma\le T} e(\ell\gamma)(\Re P(\gamma))^k. \tag{3·17}$$

By the binomial theorem

$$S(k) = \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\sum_{0<\gamma\le T} e(\ell\gamma)\left(\sum_{p\le X^2}\frac{1}{p^{1/2+i\gamma}}\right)^j\left(\sum_{p\le X^2}\frac{1}{p^{1/2-i\gamma}}\right)^{k-j}.$$

Let $a_r(p_1 \dots p_r)$ denote the number of permutations of the primes $p_1,\dots,p_r$, which might or might not be distinct. Also, for the rest of the paper, n will always denote a product of j primes, each of which is at most X^2, while m denotes a product of $k-j$ primes, again each of size at most X^2. We may thus write

$$\begin{aligned}S(k) &= \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\sum_{0<\gamma\le T} e(\ell\gamma)\sum_{n}\frac{a_j(n)}{n^{1/2+i\gamma}}\sum_{m}\frac{a_{k-j}(m)}{m^{1/2-i\gamma}}\\ &= \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\sum_{n}\frac{a_j(n)}{\sqrt{n}}\sum_{m}\frac{a_{k-j}(m)}{\sqrt{m}}\sum_{0<\gamma\le T}\left(\frac{me^{2\pi\ell}}{n}\right)^{i\gamma}.\end{aligned}$$

Since $e^{2\pi\ell} = (-1)^{-2i\ell}$ is of the form $\alpha_0^{\beta_0}$ with α_0, β_0 algebraic, $\alpha_0 \ne 0, 1$ and $-2i\ell$ not rational, the Gelfond–Schneider theorem implies that $e^{2\pi\ell}$ is transcendental. Thus, $me^{2\pi\ell}/n$ can neither be a positive integer nor the reciprocal of a positive integer. The M term in integer. The M term in (3·14) is therefore always zero. Hence, we may write

$$\begin{aligned}S(k) &= \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\sum_{n}\frac{a_j(n)}{\sqrt{n}}\sum_{\substack{m\\ me^{2\pi\ell}/n>1}}\frac{a_{k-j}(m)}{\sqrt{m}}\left(\sum_{i=1}^{3}E_i\left(\frac{me^{2\pi\ell}}{n}\right)\right)\\ &\quad+\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\sum_{n}\frac{a_j(n)}{\sqrt{n}}\sum_{\substack{m\\ me^{2\pi\ell}/n<1}}\frac{a_{k-j}(m)}{\sqrt{m}}\left(\sum_{i=1}^{3}E_i\left(\frac{n}{me^{2\pi\ell}}\right)\right)\\ &=: S_1(k)+S_2(k).\end{aligned} \tag{3·18}$$

To estimate $S_1(k)$, we insert the bounds for E_1, E_2, and E_3 from (3·13) in to obtain

$$S_1(k) = \mathcal{E}_1(k)+\mathcal{E}_2(k)+\mathcal{E}_3(k), \tag{3·19}$$

where

$$\mathcal{E}_1(k) = \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\sum_{n}\frac{a_j(n)}{\sqrt{n}}\sum_{\substack{m\\ me^{2\pi\ell}/n>1}}\frac{a_{k-j}(m)}{\sqrt{m}}\sqrt{\frac{me^{2\pi\ell}}{n}}\log\left(\frac{me^{2\pi\ell}T}{n}\right)\log\log\left(\frac{3me^{2\pi\ell}}{n}\right),$$

$$\mathcal{E}_2(k) = \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\sum_{n}\frac{a_j(n)}{\sqrt{n}}\sum_{\substack{m\\ me^{2\pi\ell}/n>1}}\frac{a_{k-j}(m)}{\sqrt{m}}\frac{\log(me^{2\pi\ell}/n)}{\sqrt{me^{2\pi\ell}/n}}\min\left(T, \frac{me^{2\pi\ell}/n}{\langle me^{2\pi\ell}/n\rangle}\right),$$

$$\mathcal{E}_3(k) = \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\sum_{n}\frac{a_j(n)}{\sqrt{n}}\sum_{\substack{m\\ me^{2\pi\ell}/n>1}}\frac{a_{k-j}(m)}{\sqrt{m}}\frac{\log T}{\sqrt{me^{2\pi\ell}/n}}\min\left(T, \frac{1}{\log(me^{2\pi\ell}/n)}\right).$$

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First consider $\mathcal{E}_1(k)$. Since $e^{2\pi\ell}$ is fixed, we see that

$$\mathcal{E}_1(k) = \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\sum_n \frac{a_j(n)}{n}\sum_{\substack{m \\ me^{2\pi\ell}/n>1}} a_{k-j}(m)\log\left(\frac{mT}{n}\right)\log\log\left(\frac{3m}{n}\right)$$
$$\ll \frac{\log T\log\log T}{2^k}\sum_{j=0}^{k}\binom{k}{j}\sum_n a_j(n)\sum_{\substack{m \\ me^{2\pi\ell}/n>1}} a_{k-j}(m).$$

Here we have dropped the n in the denominator in the sum over n and used (3·7) and (3·11) to deduce that $m \le X^{2k} \le X^{2K} \le T$. Next, from the definitions of $a_j(n)$ and $a_{k-j}(m)$ and by the binomial theorem, we see that

$$\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\sum_n a_j(n)\sum_{\substack{m \\ me^{2\pi\ell}/n>1}} a_{k-j}(m)$$
$$\ll \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\left(\sum_{p\le X^2} 1\right)^j\left(\sum_{p\le X^2} 1\right)^{k-j} = \pi(X^2)^k, \tag{3·20}$$

where $\pi(X^2)$ denotes the number of primes up to X^2. By the prime number theorem $\pi(X^2) \ll X^2/\log X$, so

$$\mathcal{E}_1(k) \ll \log T\log\log T\frac{X^{2k}}{(\log X)^k}.$$

To estimate

$$\mathcal{E}_2(k) = \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\sum_n \frac{a_j(n)}{\sqrt{n}}\sum_{\substack{m \\ me^{2\pi\ell}/n>1}} \frac{a_{k-j}(m)}{\sqrt{m}}\frac{\log(me^{2\pi\ell}/n)}{\sqrt{me^{2\pi\ell}/n}}\min\left(T, \frac{me^{2\pi\ell}/n}{\langle me^{2\pi\ell}/n\rangle}\right), \tag{3·21}$$

we require a lower bound for $\langle me^{2\pi\ell}/n\rangle$. Recall that $\langle x\rangle$ denotes the distance from x to the nearest prime power other than x itself. As $e^{2\pi\ell}$ is transcendental, $me^{2\pi\ell}/n$ is not an integer. Now, for any positive non integral real number x, we have

$$\langle x\rangle \ge \min_{r\in\mathbb{Z}}|x-r| = \min\{x-[x], 1-x+[x]\}.$$

Thus

$$\left\langle\frac{me^{2\pi\ell}}{n}\right\rangle \ge \min\left\{\frac{me^{2\pi\ell}}{n} - \left[\frac{me^{2\pi\ell}}{n}\right], 1-\frac{me^{2\pi\ell}}{n} + \left[\frac{me^{2\pi\ell}}{n}\right]\right\}.$$

We now recall a special case of Baker's theorem [**1**, p. 24]. Since $e^{2\pi\ell} = (-1)^{-2i\ell}$, for a given positive integer ℓ,

$$\left|e^{2\pi\ell} - \frac{p}{q}\right| > q^{-C\log\log q}$$

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for all rationals p/q ($p \geq 0, q \geq 4$), where C is a constant that depends on ℓ. Thus

$$\left|\frac{me^{2\pi\ell}}{n} - \frac{m}{n}\frac{p}{q}\right| > \frac{m}{n} q^{-C\log\log q}. \tag{3·22}$$

If we let $q = m$ and $p = n\left[\frac{me^{2\pi\ell}}{n}\right]$, we obtain

$$\frac{me^{2\pi\ell}}{n} - \left[\frac{me^{2\pi\ell}}{n}\right] \geq \frac{m}{n} m^{-C\log\log m}.$$

Similarly, taking $q = m$ and $p = n\left[\frac{me^{2\pi\ell}}{n}\right] + n$, we see that $\left(1 - \frac{me^{2\pi\ell}}{n} + \left[\frac{me^{2\pi\ell}}{n}\right]\right)$ has the same lower bound. Hence

$$\left\langle\frac{me^{2\pi\ell}}{n}\right\rangle \geq \frac{m}{n} m^{-C\log\log m}, \tag{3·23}$$

provided $m \geq 4$. Now the terms in (3·21) with $1 \leq m \leq 3$ contribute

$$\begin{aligned} &\ll_\ell \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\sum_{n<6e^{2\pi\ell}}\frac{a_j(n)}{n}\sum_{m\leq 3}a_{k-j}(m) \\ &\ll \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\left(\sum_{p\leq 6e^{2\pi\ell}}1\right)^j\left(\sum_{p\leq 3}1\right)^{k-j} \ll e^{Ck}, \end{aligned}$$

where C is a constant depending on ℓ. Using (3·23) to estimate the terms in (3·21) with $m \geq 4$, we find that they contribute

$$\begin{aligned} &\ll \frac{\log T}{2^k}\sum_{j=0}^{k}\binom{k}{j}\sum_{n}a_j(n)\sum_{m}a_{k-j}(m)m^{C\log\log m} \\ &\ll \frac{\log T}{2^k}X^{3Ck\log\log X}\sum_{j=0}^{k}\binom{k}{j}\left(\sum_{p\leq X^2}1\right)^j\left(\sum_{p\leq X^2}1\right)^{k-j} \\ &\ll (\log T)\pi(X^2)^k X^{3Ck\log\log X} \ll (\log T)X^{4Ck\log\log X}. \end{aligned}$$

Thus,

$$\mathcal{E}_2(k) \ll_\ell (\log T)X^{4Ck\log\log X} + e^{Ck} \ll_\ell (\log T)X^{4Ck\log\log X}.$$

Next consider

$$\mathcal{E}_3(k) = \frac{\log T}{2^k}\sum_{j=0}^{k}\binom{k}{j}\sum_{n}a_j(n)\sum_{\substack{m \\ me^{2\pi\ell}/n>1}}\frac{a_{k-j}(m)}{m}\min\left(T, \frac{1}{\log(me^{2\pi\ell}/n)}\right). \tag{3·24}$$

Since $m, n \leq X^{2k}$, by (3·22) with $p = n$, $q = m$ and $m \geq 4$, we have

$$\frac{me^{2\pi\ell}}{n} - 1 > \frac{m}{n}m^{-C\log\log m} \geq X^{-2k-2kC\log\log(X^{2k})} \gg X^{-3kC\log\log X}. \tag{3·25}$$

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Thus,

$$\frac{1}{\log\,(me^{2\pi\ell}/n)} \ll X^{3kC\log\log X}$$

for $m \geq 4$. The terms in (3·24) with $m \leq 3$ contribute

$$\ll \frac{\log T}{2^k}\sum_{j=0}^{k}\binom{k}{j}\sum_{n<6e^{2\pi\ell}} a_j(n)\sum_{m\leq 3} a_{k-j}(m)$$
$$= \frac{\log T}{2^k}\sum_{j=0}^{k}\binom{k}{j}\Bigg(\sum_{p\leq 6e^{2\pi\ell}} 1\Bigg)^j\Bigg(\sum_{p\leq 3} 1\Bigg)^{k-j} \ll e^{Ck}\log T.$$

The contribution of the terms with $m \geq 4$ is

$$\mathcal{E}_3(k) \ll X^{3kC\log\log X}\,\frac{\log T}{2^k}\sum_{j=0}^{k}\binom{k}{j}\sum_{n} a_j(n)\sum_{m} a_{k-j}(m)$$
$$= X^{3kC\log\log X}\,\pi(X^2)^k\log T \ll \frac{X^{2k}\log T}{(\log X)^k}X^{3kC\log\log X} \ll X^{4kC\log\log X}\log T.$$

Thus

$$\mathcal{E}_3(k) \ll X^{4kC\log\log X}\log T.$$

Combining our estimates for $\mathcal{E}_1(k)$, $\mathcal{E}_2(k)$, and $\mathcal{E}_3(k)$ in (3·19), we see that

$$S_1(k) \ll_\ell (\log T)X^{4Ck\log\log X}.$$

The estimation of $S_2(k)$ is very similar and leads to the same bound. Hence, by (3·18)

$$S(k) \ll_\ell (\log T)X^{4Ck\log\log X}.$$

Inserting this into (3·16), we find that

$$\sum_{0<\gamma\leq T} e(\ell\gamma)F_\Omega(\Re P(\gamma)-A) \ll_\ell X^{4CK\log\log X}\log T\int_0^\Omega G\Big(\frac{\omega}{\Omega}\Big)\sum_{1\leq k<K}\frac{(2\pi\omega)^k}{k!}\frac{d\omega}{\omega} + T.$$

As G is bounded over [0, 1], this is

$$\ll_\ell \Omega\, e^{2\pi\Omega}X^{4CK\log\log X}\log T + T.$$

By the choice of parameters X, Ω, and K in (3·7) and (3·11), it follows that for a fixed ℓ,

$$\sum_{0<\gamma\leq T} e(\ell\gamma)F_\Omega(\Re P(\gamma)-A) \ll e^{7(\log\log T)^2}T^{8C/(\log\log T)^{13}} + T \ll T = o(N(T)).$$

This establishes (3·9), and the same estimate clearly holds when A is replaced by B. This completes the proof that $\Gamma_{[a,b]}$ is uniformly distributed modulo one.

Since the number of γ's in $\Gamma_{[a,b]}$ with $0<\gamma\leq T$ that are not elements of $\Gamma^*_{[a,b]}$ (in other words, that are not simple) is at most $o(N(T))$, we see that $\Gamma^*_{[a,b]}$ is also uniformly distributed modulo one.

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4. *Proof of Theorem 1·3*

By the Erdös–Turán inequality (see [6, Chapter 1, Corollary 1·1]), if L is a positive integer and $[\alpha, \beta]$ is a subinterval of [0,1], then

$$\left| \sum_{\substack{0<\gamma\le T,\\ \{\gamma(\log T)/2\pi\}\in[\alpha,\beta]}} 1 \; - \; (\beta-\alpha)N(T) \right| \le \frac{N(T)}{L+1} \; + \; 3\sum_{\ell\le L}\frac{1}{\ell}\left| \sum_{0<\gamma\le T} e\Big(\ell\gamma\frac{\log T}{2\pi}\Big)\right|. \qquad (4\cdot1)$$

By Conjecture 1, for each integer $\ell > 0$,

$$\sum_{0<\gamma\le T} e\Big(\ell\gamma\frac{\log T}{2\pi}\Big) = \sum_{0<\gamma\le T} T^{i\ell\gamma} \ll T^{\frac12+\epsilon\ell}.$$

Hence, the right-hand side of (4·1) is

$$\ll \frac{N(T)}{L} + (\log L)\, T^{\frac12+\epsilon L}.$$

Taking $L=[1/2\epsilon]$ and assuming that $0<\epsilon<1/2$, we see that this is $\ll \epsilon N(T)$. Since ϵ can be arbitrarily small, this establishes (1·5).

5. *Proof of Theorem 1·4*

By the Erdös–Turán inequality again, if L is a positive integer and $[\alpha, \beta]$ is a subinterval of [0,1], then

$$\left| \sum_{\substack{0<\gamma\le T,\gamma\in\Gamma_{[a,b]}\\ \{\gamma(\log T)/2\pi\}\in[\alpha,\beta]}} 1 \; - \; (\beta-\alpha)N_{[a,b]}(T) \right|$$
$$\le \frac{N_{[a,b]}(T)}{L+1} \; + \; 3\sum_{\ell\le L}\frac{1}{\ell}\left| \sum_{0<\gamma\le T,\gamma\in\Gamma_{[a,b]}} e\Big(\ell\gamma\frac{\log T}{2\pi}\Big)\right|. \qquad (5\cdot1)$$

Thus, to prove (1·6), we need to estimate

$$\sum_{0<\gamma\le T,\gamma\in\Gamma_{[a,b]}} e\Big(\ell\gamma\frac{\log T}{2\pi}\Big) = \sum_{0<\gamma\le T} T^{i\ell\gamma}\; \mathbb{1}_{[a,b]}\left(\frac{\log\big(\big|\zeta^{(m_\gamma)}(\frac12+i\gamma)\big|/(\log\gamma)^{m_\gamma}\big)}{\sqrt{\frac12\log\log\gamma}}\right) \qquad (5\cdot2)$$

for positive integers ℓ. We do this, for the most part, by following the procedure of estimating the corresponding sum in (3·1) in the previous section. To start with, the same analysis that led to (3·2) leads to

$$\sum_{0<\gamma\le T} T^{i\ell\gamma}\; \mathbb{1}_{[a,b]}\left(\frac{\log\big(\big|\zeta^{(m_\gamma)}(\frac12+i\gamma)\big|/(\log\gamma)^{m_\gamma}\big)}{\sqrt{\frac12\log\log\gamma}}\right)$$
$$= \sum_{0<\gamma\le T} T^{i\ell\gamma}\; \mathbb{1}_{[a,b]}\left(\frac{\Re P(\gamma)}{\sqrt{\frac12\log\log T}}\right) + O\left(N(T)\frac{(\log\log\log T)^2}{\sqrt{\log\log T}}\right). \qquad (5\cdot3)$$

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Similarly, the analysis that led to (3·8), with the same choices of the parameters A, B, X, and Ω, and Dirichlet polynomial P, shows that

$$\begin{aligned}&\sum_{0<\gamma\le T} T^{i\ell\gamma}\mathbb{1}_{[A,B]}(\Re P(\gamma))\\&\quad=\frac{1}{2}\sum_{0<\gamma\le T} T^{i\ell\gamma}F_\Omega(\Re P(\gamma)-A)-\frac{1}{2}\sum_{0<\gamma\le T} T^{i\ell\gamma}F_\Omega(\Re P(\gamma)-B)+O\Big(\frac{N(T)}{\Omega}\Big).\end{aligned} \tag{5·4}$$

And, similarly to (3·12), we find that

$$\begin{aligned}\sum_{0<\gamma\le T} T^{i\ell\gamma}F_\Omega(\Re P(\gamma)-A)&=F_\Omega(A)\sum_{0<\gamma\le T}T^{i\ell\gamma}\\&\quad+\sum_{0<\gamma\le T}T^{i\ell\gamma}\Im\int_0^\Omega G\Big(\frac{\omega}{\Omega}\Big)e^{-2\pi iA\omega}\sum_{1\le k<K}\frac{(2\pi i\omega)^k}{k!}(\Re P(\gamma))^k\frac{d\omega}{\omega}\\&\quad+O\Big(\sum_{0<\gamma\le T}|\Re P(\gamma)|^K\int_0^\Omega G\Big(\frac{\omega}{\Omega}\Big)\frac{(2\pi\omega)^K}{K!}\frac{d\omega}{\omega}\Big),\end{aligned} \tag{5·5}$$

where $K=2\big[(\log\log T)^6\big]$.

By (3·5), $F_\Omega(A)\ll 1$ and, by Conjecture 1,

$$\sum_{0<\gamma\le T}T^{i\ell\gamma}\ll T^{\frac12+\epsilon\ell}$$

for any $\epsilon>0$. Thus, the first term on the right-hand side of (5·5) is $O(T^{\frac12+\epsilon\ell})$. The third term is estimated in the same way as the third term in (3·12) and is likewise $O(T)$. Hence

$$\begin{aligned}&\sum_{0<\gamma\le T}T^{i\ell\gamma}F_\Omega(\Re P(\gamma)-A)\\&=\int_0^\Omega G\Big(\frac{\omega}{\Omega}\Big)\sum_{1\le k<K}\Im\big(e^{-2\pi iA\omega}\,i^k\big)\frac{(2\pi\omega)^k}{k!}\sum_{0<\gamma\le T}T^{i\ell\gamma}(\Re P(\gamma))^k\frac{d\omega}{\omega}+O(T)+O(T^{\frac12+\epsilon\ell}).\end{aligned} \tag{5·6}$$

The remaining term here is handled in the same way as the corresponding term in (3·16), except that we use Conjecture 1 rather than Lemma 3·1 to estimate the sums over γ. We carry this out now.

Similarly to the analysis of the sum $S(k)$ in (3·17) that gave (3·18), we find that

$$\begin{aligned}\sum_{0<\gamma\le T}T^{i\ell\gamma}(\Re P(\gamma))^k&=\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\sum_{0<\gamma\le T}T^{i\ell\gamma}\sum_n\frac{a_j(n)}{n^{1/2+i\gamma}}\sum_m\frac{a_{k-j}(m)}{m^{1/2-i\gamma}}\\&=\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\sum_n\frac{a_j(n)}{\sqrt{n}}\sum_m\frac{a_{k-j}(m)}{\sqrt{m}}\sum_{0<\gamma\le T}\Big(\frac{mT^\ell}{n}\Big)^{i\gamma},\end{aligned}$$

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where, as before, $a_r(p_1 \ldots p_r)$ denotes the number of permutations of the primes $p_1, \ldots, p_r$, which might or might not be distinct. By Conjecture 1, for each m and n

$$\sum_{0<\gamma\le T} \left(\frac{mT^\ell}{n}\right)^{i\gamma} \ll T^{1+\ell\epsilon-\ell/2}\left(\frac{m}{n}\right)^{-\frac{1}{2}+\epsilon} + T^{\frac{1}{2}+\ell\epsilon}\left(\frac{m}{n}\right)^{\epsilon}.$$

Thus

$$\begin{aligned}&\sum_{0<\gamma\le T} T^{i\ell\gamma}(\Re P(\gamma))^k \\ &\ll \frac{T^{1+\ell\epsilon-\frac{\ell}{2}}}{2^k}\sum_{j=0}^{k}\binom{k}{j}\sum_n \frac{a_j(n)}{n^\epsilon}\sum_m \frac{a_{k-j}(m)}{m^{1-\epsilon}} + \frac{T^{\frac{1}{2}+\ell\epsilon}}{2^k}\sum_{j=0}^{k}\binom{k}{j}\sum_n \frac{a_j(n)}{n^{1/2+\epsilon}}\sum_m \frac{a_{k-j}(m)}{m^{1/2-\epsilon}} \\ &= \mathcal{T}_1(k)+\mathcal{T}_2(k),\end{aligned} \tag{5·7}$$

say. Now

$$\begin{aligned}\mathcal{T}_1(k) &\ll \frac{T^{1+\ell\epsilon-\frac{\ell}{2}}}{2^k}\sum_{j=0}^{k}\binom{k}{j}\sum_n a_j(n)\sum_m a_{k-j}(m) \\ &\ll \frac{T^{1+\ell\epsilon-\frac{\ell}{2}}}{2^k}\sum_{j=0}^{k}\binom{k}{j}\left(\sum_{p\le X^2} 1\right)^j\left(\sum_{p\le X^2} 1\right)^{k-j} \ll T^{\frac{1}{2}+\ell\epsilon}\pi(X^2)^k,\end{aligned}$$

since $\ell \ge 1$. Similarly,

$$\mathcal{T}_2(k) \ll \frac{T^{\frac{1}{2}+\ell\epsilon}}{2^k}\sum_{j=0}^{k}\binom{k}{j}\sum_n a_j(n)\sum_m a_{k-j}(m) = T^{\frac{1}{2}+\ell\epsilon}\pi(X^2)^k.$$

Combining our estimates for $\mathcal{T}_1(k)$ and $\mathcal{T}_2(k)$ in (5·7), we see that

$$\sum_{0<\gamma\le T} T^{i\ell\gamma}(\Re P(\gamma))^k \ll T^{\frac{1}{2}+\ell\epsilon}\,\pi(X^2)^k \ll T^{\frac{1}{2}+\ell\epsilon}X^{2k}.$$

Using this in (5·6), we find that

$$\begin{aligned}\sum_{0<\gamma\le T} T^{i\ell\gamma}F_\Omega(\Re P(\gamma)-A) &\ll T^{\frac{1}{2}+\ell\epsilon}\,X^{2K}\int_0^\Omega G\left(\frac{\omega}{\Omega}\right)\sum_{1\le k<K}\frac{(2\pi\omega)^k}{k!}\frac{d\omega}{\omega} + O(T) + O(T^{\frac{1}{2}+\epsilon\ell}) \\ &\ll \Omega e^{2\pi\Omega}\,T^{\frac{1}{2}+\ell\epsilon}\,X^{2K} + O(T).\end{aligned}$$

By (3·7) and (3·11) this is

$$\ll e^{7(\log\log T)^2}\,T^{\frac{1}{2}+\epsilon\ell}\,T^{\frac{5}{(\log\log T)^{14}}} + T \ll T,$$

provided $\ell \le L$ and ϵ is small enough relative to L. Inserting this (and the same estimate when A is replaced by B) into (5·4), we find that

$$\sum_{0<\gamma\le T} T^{i\ell\gamma}\mathbb{1}_{[A,B]}(\Re P(\gamma)) \ll T + \frac{N(T)}{\Omega} \ll \frac{N(T)}{(\log\log T)^2}.$$

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By (5·2) and (5·3) we then obtain

$$\sum_{0<\gamma\le T,\gamma\in\Gamma_{[a,b]}} e\Big(\ell\gamma\frac{\log T}{2\pi}\Big) \ll N(T)\frac{(\log\log\log T)^2}{\sqrt{\log\log T}}$$

for $\ell \le L$. Using this bound in (5·1), we see that

$$\left|\sum_{\substack{0<\gamma\le T,\gamma\in\Gamma_{[a,b]}\\ \{\gamma(\log T)/2\pi\}\in[\alpha,\beta]}} 1 \;-\; (\beta-\alpha)N_{[a,b]}(T)\right| \ll \frac{N_{[a,b]}(T)}{L+1} \;+\; N(T)(\log L)\frac{(\log\log\log T)^2}{\sqrt{\log\log T}}.$$

By our hypotheses on a and b, $N_{[a,b]}(T) \gg N(T)$. Hence, since L may be arbitrarily large, we find that this equals $o(N_{[a,b]}(T))$. This proves (1·6). The analogous inequality when $\Gamma_{[a,b]}$ is replaced by $\Gamma^*_{[a,b]}$ follows from this on noting that the number of γ in $\Gamma_{[a,b]}$ with $0<\gamma\le T$ that are not elements of $\Gamma^*_{[a,b]}$ is at most $o(N_{[a,b]}(T))$.

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