WILSON SPACES AND STABLE SPLITTINGS OF BT^r

by C. A. McGIBBON

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Let $Q(X)$ denote $\varinjlim \Omega^n\Sigma^n(X)$ and let BT^r denote the classifying space of the r-torus. In [**8**], Segal showed that $Q(BT^1)$ is homotopy equivalent to a product $BU \times F$ where BU denotes the classifying space for stable complex vector bundles and F is a space with finite homotopy groups. This result has been a very useful one. For example, in [**5**] it was used to show that up to a stable homotopy equivalence there is only one loop structure on the 3-sphere at each odd prime p. (The subsequent work of Dwyer, Miller, and Wilkerson shows this result is even true unstably, at every prime p.) In [**6**] it was used to classify, up to homology, the stable self maps of the projective spaces $\mathbb{C}P^n$ and $\mathbb{H}P^n$. In [**5**] I asked if a splitting similar to Segal's might exist for $Q(BT^r)$ when $r \geq 2$. In particular, since the homotopy and homology groups of BU are torsion free it seemed natural to ask if $Q(BT^r)$, when $r > 1$, could likewise contain a retract with torsion free homology and homotopy groups and whose complement is rationally trivial. The purpose of this note is to show that the answer is no.

THEOREM 1. *For $r \geq 2$, the space $Q(BT^r)$ does not have the homotopy type, at any prime p, of a product $Z \times F$ where the homotopy groups and the (reduced) integral homology groups of Z are free $\mathbb{Z}_{(p)}$-modules, while those of F are finite.*

There are two main ingredients in the proof. The first is the Wilson spaces $B(n,p)$. These spaces were constructed by Wilson in his thesis [**9**] and later studied by Zabrodsky in his book [**10**]. For each prime p and each natural number n there exists a p-local H-space, $B(n,p)$, with the following properties:

(1) $\pi_q B(n,p) \approx \begin{cases} 0 & \text{if } q < n \\ \mathbb{Z}_{(p)} & \text{if } q = n. \end{cases}$

(2) Each of the higher homotopy groups and higher integral homology groups of $B(n,p)$ are free $\mathbb{Z}_{(p)}$-modules of finite rank.

(3) The space $B(n,p)$ is atomic; in other words, any self map of the space which induces an isomorphism on $\pi_n B(n,p)$ must be a homotopy equivalence.

In Theorem 6.2 of [**9**], Wilson shows that any p-local H-space of finite type over $\mathbb{Z}_{(p)}$, whose homotopy groups and homology groups are torsion free, is homotopy equivalent to some product of $B(n,p)$'s. A relevant example is

$$BU \simeq_p B(2,p) \times B(4,p) \times \ldots \times B(2p,p).$$

Here the first factor can be identified with the Eilenberg–MacLane space $K(\mathbb{Z}_{(p)},2)$ while the remaining $p-1$ factors give a p-local splitting of BSU. This particular splitting was first obtained by Peterson [**7**].

In view of Wilson's result, the proof of Theorem 1 amounts to showing that when $r \geq 2$, there is no product of $B(n,p)$'s which has the same rational homotopy type as $Q(BT^r)$ and which also occurs as a p-local retract of $Q(BT^r)$. Suppose for the moment that such a product *did* exist. Since the $B(n,p)$'s are atomic, it would follow that each retract of $Q(BT^r)$ would likewise decompose as a product of $B(n,p)$'s and rationally trivial spaces. In particular $Q(BT^2)$ is a retract of $Q(BT^r)$ when $r \geq 2$, so to prove the theorem it suffices to show that no such product exists at any prime when $r = 2$. To get started we need to know a little about the higher homotopy groups of a $B(n,p)$.

PROPOSITION 2. *Let $f_n(t)$ denote the Poincaré series for the graded $\mathbb{Z}_{(p)}$ module $\pi_* B(n,p)$ and let $v(r) = \dfrac{2p^r - 2}{p-1}$. Then*

$$f_n(t) = \begin{cases} t^n & \text{if } v(0) < n \leq v(1), \\ \dfrac{t^n}{1 - t^{2p-2}} & \text{if } v(1) < n \leq v(2), \\ \dfrac{t^n}{(1 - t^{2p-2})(1 - t^{2p^2-2})} & \text{if } v(2) < n \leq v(3), \\ \dfrac{t^n}{(1 - t^{2p-2})(1 - t^{2p^2-2}) \ldots (1 - t^{2p^k-2})} & \text{if } v(k) < n \leq v(k+1). \end{cases}$$

Proof. Fix p for the moment and let $B(n)$ denote $B(n,p)$. Wilson showed that

$$\Omega B(n+1) \simeq \begin{cases} B(n) & \text{if } n \neq v(r), \\ B(v) \times B(pv) & \text{if } n = v = v(r). \end{cases}$$

It follows that

$$\frac{f_{n+1}}{t} = f_n \quad \text{if } n \neq v(r),$$

while

$$\frac{f_{v+1}}{t} = f_v + f_{pv} \quad \text{if } n = v = v(r),$$
$$= f_v + t^{pv-(v+1)} f_{v+1}.$$

Now multiply through by t and solve for f_{v+1} to obtain

$$f_{v+1} = \frac{t f_v}{1 - t^{pv-v}}.$$

The result then follows by induction on n. □

COROLLARY 3. *Assume that X is a 2-connected space with finite type over $\mathbb{Z}_{(p)}$ and that it has the rational homotopy type of a product of $B(n,p)$. If $C_n = \mathrm{rank}_{\mathbb{Q}} \pi_n X \otimes \mathbb{Q}$, then for any m the sequence $C_m, C_{m+2p-2}, C_{m+4p-4}, C_{m+6p-6}, \ldots$ is nondecreasing.*

It might be instructive to note that, at each prime, $Q(BT^2)$ *does* have the rational homotopy type of a certain product of $B(n,p)$. Since these spaces are H-spaces, this claim amounts to showing that their rational homotopy groups are isomorphic. The Poincaré series for $\pi_* Q(BT^2) \otimes \mathbb{Q}$ is easily seen to equal the Poincaré series for $\tilde{H}_*(BT^2, \mathbb{Q})$, which is, of course $(1-t^2)^{-2} - 1$. The claim is then a consequence of the following power series identity, which is straightforward to verify:

$$\frac{1}{(1-t^2)^2} - 1 = \sum_{k=1}^{p} (k+1) f_{2k} + (p-1) \sum_{k=p+1}^{p^2+p} f_{2k}.$$

Thus to prove Theorem 1 we have to consider more than just the stable rational homotopy type of BT^2. We need to take a closer look at its stable p-local homotopy type.

To this end we use the results of the Manchester topology group on p-local splittings of ΣBT^2 ([**1**], [**2**]). They used the modular representation of $M(2,p)$, the semigroup of all 2×2 matrices over $\mathbb{Z}/p$, to obtain a homotopy equivalence

$$\Sigma(BT^2)_{(p)}\simeq\bigvee_{\rho} d_\rho Y_\rho.$$

This decomposition is an interesting one. In it, the wedge summands Y_ρ are indexed by isomorphism classes of irreducible modular representations of $M(2,p)$. As ρ varies, the Y_ρ run through p^2-1 distinct infinite dimensional homotopy types. The mod p cohomology of each Y_ρ is indecomposable as a module over the Steenrod algebra and so each Y_ρ is in fact stably irreducible. The coefficient d_ρ indicates the number of copies of Y_ρ that occur as summands. It is also the dimension, over $\mathbb{Z}/p$, of the module affording the representation ρ.

Assume now that Theorem 1 is false for BT^2. Then, at some prime p, there must be a product of $B(n,p)$ which occurs as a retract of $Q(BT^2)$ and which also has the same rational homotopy type as $Q(BT^2)$. Since Y_ρ is a p-local retract of ΣBT^2, it is clear that $\Omega Q(Y_\rho)$ is a p-local retract of $Q(BT^2)$. Since the $B(n,p)$ are atomic, it follows that each $\Omega Q(Y_\rho)$ must also have the rational homotopy type of a product of $B(n,p)$.

Let us now take a closer look at some of these wedge summands, bearing in mind that if $p(t)$ is the Poincaré series for the reduced rational homology of Y_ρ then $(1/t)p(t)$ is the Poincaré series for the rational homotopy groups of $\Omega Q(Y_\rho)$. At the prime 2, ΣBT^2 breaks up into five pieces

$$\Sigma BT^2\simeq 2Y_\alpha\vee Y_\delta\vee 2Y_\sigma.$$

Here $Y_\alpha\simeq\Sigma\mathbb{C}P^\infty$. The summand Y_σ corresponds to the Steinberg representation and, by [**3**], has Poincaré series $t^9/(1-t^2)(1-t^6)$. The remaining piece Y_δ corresponds to the determinant representation. Its Poincaré series is then seen to be

$$\frac{t^5}{(1-t^2)^2}-\frac{2t^9}{(1-t^2)(1-t^6)}\doteq t^5+2t^7+t^9+\text{higher terms.}$$

In view of Corollary 3, it is then evident that $\Omega Q(Y_\delta)$ does *not* have the rational homotopy type of a product of $B(n,p)$.

The determinant representation δ also provides a counterexample at odd primes. In [**2**], it was shown that, at $p\geq 3$, the Poincaré series for the rational homology of the summand Y_δ is

$$p(t)=t^{2p+3}+t^{6p-1}+t^{8p-3}+\text{higher terms.}$$

Once again this provides a sequence of ranks which fails to meet the conclusion of Corollary 3. Thus at each prime p there is at least one retract of $Q(BT^2)$ that does not have the rational homotopy type of a product of $B(n,p)$ and so the theorem follows.

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Mathematics Department
Wayne State University
Detroit, Michigan, 48202, U.S.A.
Current Address:
Department of Pure Maths.
16 Mill Lane
Cambridge, CB2 1SB
England