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ON ABSTRACT WIENER MEASURE*

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1. Introduction. In a recent paper, Sato [6] has shown that for every Gaussian measure μ on a real separable or reflexive Banach space $(X, \|\cdot\|)$ there exists a separable closed sub-space $\tilde{X}$ of X such that $\mu(\tilde{X}) = 1$ and $\mu_{\tilde{X}} \equiv \mu/\tilde{X}$ is the σ-extension of the canonical Gaussian cylinder measure $\mu_{\mathscr{H}}$ of a real separable Hilbert space $\mathscr{H}$ such that the norm $\|\cdot\|_{\tilde{X}} = \|\cdot\|/\tilde{X}$ is contiunous on $\mathscr{H}$ and $\mathscr{H}$ is dense in $\tilde{X}$. The main purpose of this note is to prove that $\|\cdot\|_{\tilde{X}}$ is measurable (and not merely continuous) on $\mathscr{H}$. From this and the Sato's result mentioned above, it follows that a Gaussian measure μ on a real separable or reflexive Banach space X has a restriction $\mu_{\tilde{X}}$ on a closed separable sub-space $\tilde{X}$ of X, which is an abstract Wiener measure. Gross [5] has shown that every measurable norm on a real separable Hilbert space $\mathscr{H}$ is admissible and continuous on $\mathscr{H}$. We show conversely that any continuous admissible norm on $\mathscr{H}$ is measurable. This result follows as a corollary to our main result mentioned above. We are indebted to Professor Sato and a referee who informed us about reference [2], where, among others, more general results than those included here are proved. Finally we thank Professor Feldman who supplied us with a pre-print of [2].#

We will assume that the reader is familiar with the notions of Gaussian measures and Gaussian cylinder measures on Banach spaces. We refer to Sections 1 and 2 of [6] for details. Throughout this paper, the underlined field for all Banach spaces X is the field R of real numbers, and X^* denotes the conjugate space of X.

2. Definitions and notation. In this section, we give basic definitions and notation that are used consistently.

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\# Since this paper was written, papers of Jain-Kallianpur (Proc. AMS 25 (1970), 890–895) and Kuelbs (ibid 27 (1971), 364–370) have appeared where, among others, result similar to those included here are obtained.

Let X be a Banach space. We denote by $\mathscr{U}_X(G)$, the σ-algebra of cylinder sets [6, p. 66] of X based on a finite dimensional sub-space G of X^*, and by $\mathscr{U}_X$, the algebra of all cylinder sets of X. We use the notation $\overline{\mathscr{U}}_X$ for the σ-algebra generated by $\mathscr{U}_X$.

Let $\|\cdot\|$ be a continuous norm on a separable Hilbert space $\mathscr{H}$ and X be the Banach space obtained by the completion of $\mathscr{H}$ in $\|\cdot\|$. Through the natural embedding, X^* can be considered as a sub-space of $\mathscr{H}^*$; therefore, the canonical Gaussian cylinder measure [6, p. 66] $\mu_{\mathscr{H}}$ on $(\mathscr{H}, \mathscr{U}(\mathscr{H}))$ induces a Gaussian cylinder measure [6, p. 66] μ_X on $(X, \mathscr{U}(X))$. If μ_X has the σ-extension to $(X, \overline{\mathscr{U}}(X))$, then μ_X is called *the σ-extension* of $\mu_{\mathscr{H}}$ and the norm $\|\cdot\|$ is called *admissible* on $\mathscr{H}$.

A semi-norm $\|\cdot\|$ is called *measurable* on a separable Hilbert space $\mathscr{H}$ if for every $\varepsilon > 0$, there exists a finite dimensional projection P_0 (depending on ε) such that for every finite dimensional projection P orthogonal to P_0, we have

$$\mu_{\mathscr{H}}\{x \in \mathscr{H} : \|Px\| > \varepsilon\} < \varepsilon,$$

where $\mu_{\mathscr{H}}$ is the canonical Gaussian cylinder measure on $\mathscr{H}$.

A Gaussian measure μ on $(X, \overline{\mathscr{U}}(X))$, where $(X, \|\cdot\|)$ is a separable Banach space, is called an *abstract wiener measure* if μ is the σ-extension of the canonical Gaussian cylinder measure $\mu_{\mathscr{H}}$ of a separable Hilbert space such that the norm $\|\cdot\|$ is measurable on $\mathscr{H}$ and $\mathscr{H}$ is dense in X in the norm $\|\cdot\|$.

3. Gaussian measure and abstract Wiener measure.

The results of this note are contained in the latter half part of Theorem 3.1 and Corollary 3.1. We begin with two preliminary lemmas.

LEMMA 3.1. *Let $\{\xi_j : j = 1, \cdots, n\}$ be a (Gaussian system) with mean zero defined on the probability space $(\Omega, \mathscr{B}, P)$. Let ξ_j, $j = 1, \cdots, k$, $(1 \leq k \leq n)$, be linearly independent and non-degenerate and $\xi_i = \sum_{j=1}^{k} a_{ij}\xi_j$, $i = k+1, \cdots, n$, a_{ij}'s real. Let ϕ be defined by $\phi(\omega) = (\xi_1(\omega), \cdots, \xi_k(\omega), \sum_{j=1}^{k} a_{k+1\,j}\xi_j(\omega), \cdots, \sum_{j=1}^{k} a_{nj}\xi_j(\omega))$, $\omega \in \Omega$. Then for any Borel measurable convex subset E of R^n symmetric about the origin and for any $\underline{a} = (a_1, \cdots, a_n) \in \phi(\Omega)$, we have*

$$P\{\omega : \underline{\xi}_n(\omega) \in E\} \geq P\{\omega : \underline{\xi}_n(\omega) \in \underline{a} + E\}, \tag{3.1}$$

where $\underline{\xi}_n(\omega) = (\xi_1(\omega), \cdots, \xi_n(\omega))$.

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Proof. Without loss of generality we can assume that $\xi_1, \cdots, \xi_k$ are defined on the canonical probability space $(R^k, \mathscr{B}(R^k), P)$ with $\xi_j(x_1, \cdots, x_k) = x_j$, $j = 1, \cdots, k$, where $\mathscr{B}(R^k)$ is the class of Borel subsets of the Euclidean space R^k. In this setting the map ϕ: $R^k \to R^n$ becomes $\phi(x_1, \cdots, x_k) = (x_1, \cdots x_k, \sum_{j=1}^{k} a_{k+1\,j} x_j, \cdots, \sum_{j=1}^{k} a_{n\,j} x_j)$. It is clear that ϕ is linear and one to one. Since $E \subseteq R^n$ is convex, symmetric about zero and $\phi(R^k)$ is a subspace of R^n, it follows that $F = E \cap \phi(R^k)$ is also convex and symmetric about zero. Now

$$P\{\underline{\xi}_n \in E\} = P\{\underline{\xi}_n \in F\} = P\{\underline{\xi}_k \in \phi^{-1}(F)\}, \tag{3.2}$$

where $\underline{\xi}_k = (\xi_1, \cdots, \xi_k)$. Since ϕ^{-1}: $\phi(R^k) \to R^k$ is linear, it follows that $\phi^{-1}(F)$ is convex and symmetric about zero. Using Corollary 2 of [1, p. 172], we have

$$P\{\underline{\xi}_k \in \phi^{-1}(F)\} \geq P\{\underline{\xi}_k \in \underline{b} + \phi^{-1}(F)\}, \tag{3.3}$$

where $\underline{b} = \phi^{-1}(\underline{a})$. It is easy to verify that $\phi^{-1}(\underline{a}) + \phi^{-1}(F) = \phi^{-1}(\underline{a} + F)$ and $\phi(R^k) \cap (\underline{a} + E) = \underline{a} + \phi(R^k) \cap E = \underline{a} + F$. Using these facts, (3.2) and (3.3), we conclude inequality (3.1) as follows:

$$P\{\underline{\xi}_n \in E\} \geq P\{\underline{\xi}_k \in \phi^{-1}(\underline{a} + F)\} = P\{\underline{\xi}_n \in \underline{a} + F\} = P\{\underline{\xi}_n \in \underline{a} + E\}.$$

It must be noted that inequality (3.1) is trivially true when all ξ_j's are degenerate at zero. Therefore the conclusion of Lemma 3.1 holds in both cases, namely when all ξ_j's are degenerate at zero or at least one of the ξ_j's is nondegenerate.

LEMMA 3.2. *Let μ be a Gaussian measure on a separable Banach space $(X, \|\cdot\|_X)$; then, for every $\varepsilon > 0$,*

$$\mu\{x \in X\colon \|x\|_X < \varepsilon\} > 0. \tag{3.4}$$

Proof. Since in a separable Banach space the σ-algebra generated by norm open sets coincides with $\mathscr{U}_X$, equation (3.4) makes sense. Using separability of X, we choose a sequence $\{x_n\colon n = 1, 2, \cdots\}$ of elements of X such that $X = \bigcup_{j=1}^{\infty} \Delta(x_j, \varepsilon/2)$, where $\Delta(x_j, \varepsilon/2) = \{x \in X\colon \|x - x_j\|_X \leq \varepsilon/2\}$. Since $1 = \mu(X) \leq \sum_{j=1}^{\infty} \mu\{\Delta(x_j, \varepsilon/2)\}$, it follows that there exists some n_0 such that

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$$\mu\{\Delta(x_{n_0},\ \varepsilon/2)\} > 0. \tag{3.5}$$

Using the separability of X once more, we can find a sequence $\{\xi_n: n = 1, 2, \cdots\}$ of elements of X^* such that $\|x\|_X = \sup_j |\xi_j(x)|$, for every $x \in X$. Now

$$\begin{aligned}\mu\{x \in X:\ \|x\|_X < \varepsilon\} &= \mu\{x \in X:\ \sup_j |\xi_j(x)| < \varepsilon\} \\ &\geq \lim_{k\to\infty} \mu\{x \in X:\ |\xi_j(x)| \leq \varepsilon/2,\ j = 1, \cdots, k\}. \end{aligned} \tag{3.6}$$

By Lemma 3.1, we have

$$\begin{aligned}&\mu\{x \in X:\ |\xi_j(x)| \leq \varepsilon/2,\ j = 1, \cdots, k\} \\ &\geq \mu\{x \in X:\ |\xi_j(x) - \xi_j(x_{n_0})| \leq \varepsilon/2,\ j = 1, \cdots, k\},\end{aligned} \tag{3.7}$$

for every positive integer k. From (3.5), (3.6) and (3.7), we obtain

$$\begin{aligned}\mu\{x \in X:\ \|x\|_X < \varepsilon\} &\geq \lim_{k\to\infty} \mu\{x \in X:\ |\xi_j(x) - \xi_j(x_{n_0})| \leq \varepsilon/2,\ j = 1, \cdots, k\} \\ &= \mu\{x \in X:\ \sup_j |\xi_j(x) - \xi_j(x_{n_0})| \leq \varepsilon/2\} \\ &= \mu\{x \in X:\ \|x - x_{n_0}\|_X \leq \varepsilon/2\} \\ &= \mu\{\Delta(x_{n_0},\ \varepsilon/2)\} > 0.\end{aligned}$$

The proof is complete.

Now we state and prove the first result of this note. As indicated in the beginning of this section, the only new result in the following theorem is to prove the measurability of a certain norm. The rest of the theorem is due to Sato [6].

THEOREM 3.1. *Let μ be a Gaussian measure on a separable or reflexive Banach space $(X, \|\cdot\|_X)$. Then there exists a separable closed subspace $\tilde{X}$ of X such that $\mu(\tilde{X}) = 1$ and $\mu_{\tilde{X}} \equiv \mu/\tilde{X}$ is an abstract Wiener measure; i.e., there exists a separable Hilbert space $\mathscr{H}$ such that $\mu_{\tilde{X}}$ is the σ-extension of the canonical Gaussian cylinder measure $\mu_{\mathscr{H}}$ of $\mathscr{H}$, $\|\cdot\|_{\tilde{X}} \equiv \|\cdot\|_{X/\tilde{X}}$ is measurable on $\mathscr{H}$ and $\mathscr{H}$ is dense in $\tilde{X}$ in the norm $\|\cdot\|_{\tilde{X}}$.*

Proof. Let $\tilde{X}$ and $\mathscr{H}$ be the same as defined on pages 70 and 71 of [6] respectively. In view of Theorem 2 of [6], we only need to show that $\|\cdot\|_X$ is measurable on $\mathscr{H}$.

It is clear from [6, p. 71] that $\tilde{X}^*$ can be identified with a subset of $\mathscr{H}^*$, and moreover, $\tilde{X}^*$ is dense in $\mathscr{H}^*$. By Corollary 1 of [5, p. 38], the identity map on $\tilde{X}^*$ regarded as densely defined map of $\mathscr{H}^*$ into random variables over the probability space $(\tilde{X}, \overline{\mathscr{U}}_{\tilde{X}}, \mu_{\tilde{X}})$ extends to a representative F of the canonical normal distribution [4, p. 372] over $\mathscr{H}$ in a unique manner. Furthermore, the corresponding canonical Gaussian cylinder measure $\mu_{\mathscr{H}}$ satisfies (1.2) of [6] for any $\xi_1, \cdots, \xi_n \in X^*$, and any Borel set D of R^n.

Using separability of the space $\tilde{X}$, we can choose a sequence $\{\xi_n: n = 1, 2, \cdots\}$ of elements of $\tilde{X}^*$ such that $\|x\|_{\tilde{X}} = \sup_j |\xi_j(x)|$, for every $x \in \tilde{X}$. Since $\tilde{X}^* \subseteq \mathscr{H}^*$, it follows that the restriction $\phi_j \equiv \xi_j/\mathscr{H}$ belongs to $\mathscr{H}^*$, for each j. Define the sequence of pseudonorms $\{\|\cdot\|_j': j = 1, 2, \cdots\}$ on $\mathscr{H}$ by $\|x\|_j' = |\phi_j(x)|$. Since $\phi_j \in \mathscr{H}^*$, the function $f_j(x) = |\phi_j(x)|$ is continuous tame function [5, p. 32] on $\mathscr{H}$, for each $j = 1, 2, \cdots$. It follows from the definition of F that the random variable $\tilde{f}_j$ corresponding to f_j is $|\xi_j|$ which is defined on the probability space $(\tilde{X}, \overline{\mathscr{U}}_{\tilde{X}}, \mu_{\tilde{X}})$. Since ξ_j is Gaussian random variable, it follows that, for every $\varepsilon > 0$,

$$\mu_{\tilde{X}}\{x \in \tilde{X}: |\xi_j(x)| < \varepsilon\} > 0. \tag{3.8}$$

Applying Theorem 1 of [3, p. 406], Corollary 4.5 of [4, p. 383] and (3.8), we have that $\|\cdot\|_j'$ is measurable pseudonorm for each j. Let $\|x\|_n = \max_{1 \leq j \leq n} \|x\|_j'$; then $\|\cdot\|_n$ is a pseudonorm on $\mathscr{H}$ and $\|x\|_n \leq \|x\|_1' + \cdots + \|x\|_n'$. Since finite sum of measurable pseudonorms is a measurable pseudonorm, it follows that $\|\cdot\|_1' + \cdots + \|\cdot\|_n'$ and hence $\|\cdot\|_n$ is a measurable pseudonorm; moreover, the random variable $\|\tilde{x}\|_n$ corresponding to $\|x\|_n$ is $\max_{1 \leq j \leq n} |\xi_j(x)|$.

Since $\lim_{n \to \infty} \max_{1 \leq j \leq n} |\xi_j(x)| = \|x\|_{\tilde{X}}$ for every $x \in \tilde{X}$, it follows that the sequence $\{\|\tilde{x}\|_n: n = 1, 2, \cdots\}$ of random variables on $(\tilde{X}, \overline{\mathscr{U}}_{\tilde{X}}, \mu_{\tilde{X}})$ converges to the random variable $\|x\|_{\tilde{X}}$ in probability. Since $\tilde{X}$ is separable and $\mu_{\tilde{X}}$ is Gaussian measure, it follows, by Lemma 3.2, that, for every $\varepsilon > 0$, $\mu_{\tilde{X}}\{x \in \tilde{X}: \|x\|_{\tilde{X}} > \varepsilon\} > 0$. Thus we have a sequence $\{\|x\|_n: n = 1, 2, \cdots\}$ of nondecreasing measurable pseudonorms on $\mathscr{H}$ such that $\|x\|_{\tilde{X}}$, the limit in probability of the sequence of random variables $\{\|\tilde{x}\|_n: n = 1, 2, \cdots\}$ exists and has the property that $\mu_{\tilde{X}}\{x \in \tilde{X}: \|x\|_{\tilde{X}} > \varepsilon\} > 0$. The measurability of $\|x\|_{\tilde{X}}$ now follows from Corollary 4.4 of [4, p. 383].

The following Corollary gives the converse of the Gross's result mentioned in the introduction; specifically, we show that every continuous admissible norm on a separable Hilbert space is measurable.

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COROLLARY 3.1. *Let $\|\cdot\|$ be a continuous admissible norm on a separable Hilbert space H; then $\|\cdot\|$ is measurable on H.*

Proof. Let X be the Banach space obtained by completing H with respect to $\|\cdot\|$, and let μ_X be the Gaussian measure on $(X, \bar{\mathscr{U}}_X)$ which is the σ-extension of the canonical Gaussian cylinder measure μ_H on H. Let $\mathscr{H}$, $\tilde{X}$ and $\mu_{\mathscr{H}}$ be the same as in Theorem 3.1.

First we show $X = \tilde{X}$. In view of the definition of $\tilde{X}$ (see Lemma 2 of [6]) it is enough to show that if $f \in X^*$ and $f \neq 0$ then $v(f) = \int_X f^2(x) d\mu_X(x) \neq 0$. Let $f \in X^*$ with $f \neq 0$; since $\|\cdot\|$ is continuous on H and H is dense in X, it follows that $\hat{f}$, the restriction of f to H, belongs to H^* and $\hat{f} \neq 0$ on H. Using the fact that μ_H, the canonical Gaussian cylinder measure on H, is countably additive on $\mathscr{U}_H(G)$, where G is any finite dimensional subspace, and the definition of μ_X, we have $v(f) = \int_X f^2(x) d\mu_X(x) = \int_H f^2(h) d\mu_H(h) = \|\hat{f}\|^2_{H^*} \neq 0$, since $\hat{f} \neq 0$ on H, where $\|\cdot\|_{H^*}$ denote the norm in H^*.

Using the same argument as in the end of the previous paragraph we have that if $f, g \in \tilde{X}^*$ then $\int_X f(x)g(x)d\mu_X(x) = \langle \hat{f}, \hat{g} \rangle_{H^*}$, where $\hat{f} = f/H$, $\hat{g} = g/H$ and $\langle \cdot \rangle_{H^*}$ is the inner product in H^*. By Lemma 3 of [6], we have that $\|\cdot\|_{H^*}$ is continuous on $\tilde{X}^*$. This fact and the fact that H is dense in $\tilde{X}$ allow us to use a similar argument as in Lemma 5 of [6] to conclude that $\tilde{X}^*$ is dense in H^* under the obvious identification. By repeating the proof of Theorem 3.1 from second paragraph on replacing $\mathscr{H}$ by H and $\mathscr{H}^*$ by H^*, it follows that $\|\cdot\|$ is measurable on H.

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