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ARTICLE

On a conjecture of Conlon, Fox, and Wigderson

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Abstract

For graphs G and H, the Ramsey number $r(G, H)$ is the smallest positive integer N such that any red/blue edge colouring of the complete graph K_N contains either a red G or a blue H. A book B_n is a graph consisting of n triangles all sharing a common edge.

Recently, Conlon, Fox, and Wigderson conjectured that for any $0 < \alpha < 1$, the random lower bound $r(B_{\lceil \alpha n \rceil}, B_n) \ge (\sqrt{\alpha} + 1)^2 n + o(n)$ is not tight. In other words, there exists some constant $\beta > (\sqrt{\alpha} + 1)^2$ such that $r(B_{\lceil \alpha n \rceil}, B_n) \ge \beta n$ for all sufficiently large n. This conjecture holds for every $\alpha < 1/6$ by a result of Nikiforov and Rousseau from 2005, which says that in this range $r(B_{\lceil \alpha n \rceil}, B_n) = 2n + 3$ for all sufficiently large n.

We disprove the conjecture of Conlon, Fox, and Wigderson. Indeed, we show that the random lower bound is asymptotically tight for every $1/4 \le \alpha \le 1$. Moreover, we show that for any $1/6 \le \alpha \le 1/4$ and large n, $r(B_{\lceil \alpha n \rceil}, B_n) \le \left(\frac{3}{2} + 3\alpha\right) n + o(n)$, where the inequality is asymptotically tight when $\alpha = 1/6$ or $1/4$. We also give a lower bound of $r(B_{\lceil \alpha n \rceil}, B_n)$ for $1/6 \le \alpha < \frac{52-16\sqrt{3}}{121} \approx 0.2007$, showing that the random lower bound is not tight, i.e., the conjecture of Conlon, Fox, and Wigderson holds in this interval.

Keywords: Book; Ramsey number; refined regularity lemma

2020 MSC Codes: Primary: 05D10

1. Introduction

For graphs G and H, the Ramsey number $r(G, H)$ is the smallest positive integer N such that any red/blue edge colouring of the complete graph K_N contains either a red G or a blue H. A central problem in extremal graph theory is to determine the Ramsey number $r(G, H)$.

Let $B_n^{(k)}$ be the book graph consisting of n copies of K_{k+1}, all sharing a common K_k. We always call n the size of the book. When $k = 2$, we write B_n instead of $B_n^{(2)}$. Books have attracted a great deal of attention in graph Ramsey theory, see, for example, the recent breakthrough of Campos, Griffiths, Morris and Sahasrabudhe [1].

For the diagonal Ramsey number of books, Erdős, Faudree, Rousseau and Schelp [2] and independently Thomason [3] proved that $(2^k + o(1))n \le r(B_n^{(k)}, B_n^{(k)}) \le 4^k n$. In particular, Rousseau and Sheehan [4] showed that $r(B_n, B_n) = 4n + 2$ if $4n + 1$ is a prime power. Recently, a breakthrough result of Conlon [5] established that for each $k \ge 2$,

$$r(B_n^{(k)}, B_n^{(k)}) = 2^k n + o(n),$$

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which confirms a conjecture of Thomason [3] asymptotically and also gives an answer to a problem proposed by Erdős, Faudree, Rousseau and Schelp [2]. The error term $o(n)$ of the upper bound has been improved to $O\big(\frac{n}{(\log\log\log n)^{1/25}}\big)$ by Conlon, Fox, and Wigderson [6] using a different method, in particular avoiding the use of the full regularity lemma.

For the off-diagonal Ramsey number of books, a simple lower bound is as follows: for any $k, m, n \in \mathbb{N}$ with $m \le n$,

$$r(B_m^{(k)}, B_n^{(k)}) \ge k(n+k-1)+1. \tag{1}$$

Indeed, this follows from an observation of Chvátal and Harary [7] which states that if H is connected, then $r(G,H) \ge (\chi(G)-1)(|V(H)|-1)+1$. We say that H is G-good if this inequality is tight. Burr and Erdős [8] initiated the study of such Ramsey goodness problems; the reader is referred to the survey of Conlon, Fox, and Sudakov [9] for a detailed history of the area. Among these results, Nikiforov and Rousseau [10] obtained extremely general results. In particular, they proved the following theorem; see Fox, He and Wigderson [11] for a new proof avoiding the application of the regularity lemma.

Theorem 1.1 (Nikiforov and Rousseau [10]). *For every $k \ge 2$, there exists some $\alpha_0 \in (0,1)$ such that, for any $0 < \alpha \le \alpha_0$ and sufficiently large n,*

$$r(B_{\lceil \alpha n\rceil}^{(k)}, B_n^{(k)}) = k(n+k-1)+1.$$

Since we are concerned with the asymptotic behaviour of the Ramsey number, we always omit the ceiling and floor signs henceforth.

We know that α_0 in Theorem 1.1 is always small. In fact, if α is sufficiently far from 0, then the situation of the lower bound is much different. As pointed out by Conlon, Fox, and Wigderson [12], we can get *a random lower bound* for $r(B_{\alpha n}^{(k)}, B_n^{(k)})$ as follows (one can also see [13] for the special case of $k=2$). Indeed, for any $k \in \mathbb{N}$ and $0 < \alpha \le 1$, let $p = \frac{1}{\alpha^{1/k}+1}$ and $N = (p^{-k} - o(1))n$. Then we randomly and independently colour every edge of K_N blue with probability p and red with probability $1-p$. A standard application of the Chernoff bound yields that the probability containing a blue $B_n^{(k)}$ or a red $B_{\alpha n}^{(k)}$ is $o(1)$. Therefore, for any $k \in \mathbb{N}$ and $0 < \alpha \le 1$,

$$r(B_{\alpha n}^{(k)}, B_n^{(k)}) \ge (\alpha^{1/k}+1)^k n - o(n).$$

A simple calculation implies that for large k, if $\alpha > ((1+o(1))\frac{\log k}{k})^k$, then the above lower bound is much larger than that of (1), where the logarithm is to base e.

Furthermore, Conlon, Fox, and Wigderson [12] show that the random lower bound becomes asymptotically tight at this point.

Theorem 1.2 (Conlon, Fox, and Wigderson [12]). *For every $k \ge 2$, there exists some $\alpha_1 = \alpha_1(k) \in (0,1]$ such that, for any fixed $\alpha_1 \le \alpha \le 1$,*

$$r(B_{\alpha n}^{(k)}, B_n^{(k)}) = (\alpha^{1/k}+1)^k n + o(n).$$

Moreover, one may take $\alpha_1(k) = ((1+o(1))\frac{\log k}{k})^k$.

From Theorem 1.1 and Theorem 1.2, we know that for every $k \ge 2$, there exist $\alpha_0, \alpha_1 \in (0,1]$ such that the following holds:

(i) if $0 < \alpha \le \alpha_0$, then $r(B_{\alpha n}^{(k)}, B_n^{(k)}) = k(n+k-1)+1$;

(ii) if $\alpha_1 \le \alpha \le 1$, then $r(B_{\alpha n}^{(k)}, B_n^{(k)}) = (\alpha^{1/k}+1)^k n + o(n)$.

However, the values of α_0 and α_1 are not very clear in general. Moreover, although Theorem 1.2 shows that the random lower bound becomes asymptotically tight when $k \to \infty$, it does not say anything non-trivial in the simplest case $k=2$ except when $\alpha = 1$.

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Nikiforov and Rousseau [14] proved that

$$\text{for any fixed } \alpha < 1/6 \text{ and all large } n,\ r(B_{\alpha n}, B_n) = 2n + 3, \tag{2}$$

which implies the Ramsey goodness of large books. Moreover, the constant 1/6 is asymptotically best possible, i.e., for any $\alpha > 1/6$ and all large n,

$$r(B_{\alpha n}, B_n) > 2n + 3.$$

Recently, the second author together with Chen and You [13] proved that for all large $m \leq n$, $r(B_m, B_n) \leq 2(m + n) + o(n)$. However, the behaviour of $r(B_{\alpha n}, B_n)$ is still not well-understood for $1/6 \leq \alpha < 1$.

More recently, Conlon, Fox, and Wigderson [12, Conjecture 6.1] proposed the following conjecture, which would imply that the random lower bound $r(B_{\alpha n}, B_n) \geq (\sqrt{\alpha} + 1)^2 n - o(n)$ is not tight for any $\alpha < 1$.

Conjecture 1.3 (Conlon, Fox, and Wigderson [12]). *For every $\alpha < 1$, the random lower bound for $r(B_{\alpha n}, B_n)$ is not tight. In other words, there exists some $\beta > (\sqrt{\alpha} + 1)^2$ such that $r(B_{\alpha n}, B_n) \geq \beta n$ for all n sufficiently large.*

Note that Conjecture 1.3 holds for any $\alpha \leq 1/6$, by (2). However, in this paper, we disprove Conjecture 1.3 by showing that the random lower bound is asymptotically tight for any $1/4 \leq \alpha \leq 1$.

Theorem 1.4. *For any fixed $1/4 \leq \alpha \leq 1$ and sufficiently large n,*

$$r(B_{\alpha n}, B_n) = (\sqrt{\alpha} + 1)^2 n + o(n).$$

Moreover, we give an upper bound of $r(B_{\alpha n}, B_n)$ for $1/6 \leq \alpha \leq 1/4$.

Theorem 1.5. *For any fixed $1/6 \leq \alpha \leq 1/4$ and sufficiently large n,*

$$r(B_{\alpha n}, B_n) \leq (3/2 + 3\alpha)\, n + o(n).$$

The inequality is asymptotically tight when $\alpha = 1/6$ or $1/4$.

We also have the following lower bound.

Theorem 1.6. *Let $1/6 \leq \alpha \leq \frac{52-16\sqrt{3}}{121} \approx 0.2007$ be fixed, and let $p = \frac{1-\sqrt{\alpha(3-2\alpha)}}{1-2\alpha}$. Then for all sufficiently large n,*

$$r(B_{\alpha n}, B_n) \geq \frac{3n}{1 + 2p^2} - o(n).$$

The inequality is asymptotically tight when $\alpha = 1/6$.

Remark 1.7. Since $\frac{3}{1+2p^2} > (\sqrt{\alpha} + 1)^2$ for any $1/6 \leq \alpha < \frac{52-16\sqrt{3}}{121}$, we have that Conjecture 1.3 holds in this interval.

1.1. Notation

For a book B_n, we refer to the common edge as the base of the book B_n. For a graph $G = (V, E)$ with vertex set V and edge set E, we write bk_G for the size of the largest book in a graph G. Let uv denote an edge of G. For $X \subseteq V$, $e(X)$ is the number of edges in X. For two disjoint subsets $X, Y \subseteq V$, $e_G(X, Y)$ denotes the number of edges between X and Y. The neighbourhood of a vertex v in $U \subseteq V$ is denoted by $N_G(v, U)$, and $\deg_G(v, U) = |N_G(v, U)|$ and the degree of v in G is $\deg_G(v) = |N_G(v, V)|$. Let $X \sqcup Y$ denote the disjoint union of X and Y. Let $[n] = \{1, 2, \ldots, n\}$, and $[m, n] = \{m, m + 1, \ldots, n\}$. We always delete the subscriptions if there is no confusion from the context.

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The rest of the paper is organised as follows. In Section 2, we will collect several useful lemmas. In Sections 3 and 4, we shall present the proofs of Theorems 1.4, 1.5 and 1.6. Finally, we will have some discussion in Section 5.

2. Preliminaries

The proofs rely on the regularity method [15], which is a powerful tool in extremal graph theory. There are many important applications of the regularity lemma, and we refer the reader to the nice surveys [16–18] and many other recent references.

Let $G=(V, E)$ be a graph. For two vertex sets $A, B \subseteq V(G)$, we call $d(A, B)=\frac{e(A,B)}{|A||B|}$ the density of the pair (A, B). Let $\varepsilon>0$, a pair (A, B) is said to be *ε-regular* if $|d(A, B)-d(X, Y)| \le \varepsilon$ for every $X \subseteq A$, $Y \subseteq B$ with $|X| \ge \varepsilon|A|$ and $|Y| \ge \varepsilon|B|$. Also, a subset A is said to be ε-regular if the pair (A, A) is ε-regular.

Given a graph G, an equitable ε-regular partition $V(G)=\sqcup_{i=1}^{k} V_i$ of G is a partition of $V(G)$ such that (i) $||V_i|-|V_j|| \le 1$ for all distinct i and j; (ii) each V_i is ε-regular; and (iii) for every $1 \le i \le k$, there are at most εk values $1 \le j \le k$ such that the pair (V_i, V_j) is not ε-regular.

When establishing the asymptotic order of $r(B_n^{(k)}, B_n^{(k)})$, Conlon [5, Lemma 3] applied a refined version of the regularity lemma which guarantees a regular subset in each part of the partition for any graph. We will use the following refined regularity lemma due to Conlon, Fox, and Wigderson [6, Lemma 2.1], which is a strengthening of that due to Conlon [5] and the usual version of Szemerédi's regularity lemma [15].

Lemma 2.1 (Conlon, Fox, and Wigderson [6]).*For every $\varepsilon>0$ and $M_0 \in \mathbb{N}$, there is some $M=M(\varepsilon, M_0)>M_0$ such that for every graph G, there is an equitable ε-regular partition $V(G)=\sqcup_{i=1}^{k} V_i$ where $M_0 \le k \le M$.*

We will use the following version of the counting lemma, proved by Conlon [5, Lemma 5]. (A similar counting lemma was proved by Nikiforov and Rousseau [14, Corollary 11], but they required all of the clusters to be different.) For the general local counting lemma, see Rödl and Schacht [18, Theorem 18].

Lemma 2.2 (Conlon [5]). *For any $\delta>0$, there is $\varepsilon>0$ such that if U_1, U_2 (not necessarily distinct), $W_1, \ldots, W_l$ are vertex sets with (U_1, U_2) ε-regular of density at least δ and (U_i, W_j) ε-regular of density d_{ij} for all $i \in [2]$ and $j \in [l]$, then there exists an edge u_1u_2 where $u_1 \in U_1$ and $u_2 \in U_2$ such that u_1 and u_2 have at least $\sum_{j=1}^{l}(d_{1j}d_{2j}-\delta)|W_j|$ common neighbours in $\sqcup_{j=1}^{l} W_j$.*

We will also use the following well-known Turán's bound.

Lemma 2.3 (Turán [19]). *For any graph G of order n with average degree d, the independence number $\alpha(G)$ is at least $\frac{n}{1+d}$.*

3. Proofs of Theorem 1.4 and Theorem 1.5

In the following, for a red/blue edge colouring of K_N, we always use R/B to denote the subgraph induced by all red/blue edges. For every i, j, $(i \neq j)$, let d_{ij} be the red density of the pair (V_i, V_j), i.e., $d_{ij}=\frac{e_R(V_i,V_j)}{|V_i||V_j|}$.

We will use the following definition introduced by Conlon, Fox, and Wigderson [12].

Definition 3.1. Fix parameters $\ell \in \mathbb{N}$ and $\varepsilon, \gamma \in (0, 1)$ and suppose that we are given a red/blue colouring of $E(K_N)$. Then an ℓ-tuple of pairwise disjoint vertex sets $V_1, \ldots, V_\ell \subseteq V(K_N)$ is called an $(\ell, \varepsilon, \gamma)$-red-blocked configuration if the following properties are satisfied:

1. Each V_i is ε-regular with itself,
2. Each V_i has internal red density at least γ, and
3. For all $i \neq j$, the pair (V_i, V_j) is ε-regular and has blue density at least γ.

Similarly, we say that $V_1, \ldots, V_\ell$ is an $(\ell, \varepsilon, \gamma)$-blue-blocked configuration if properties (1–3) hold, but the roles of red and blue interchanged.

We first have a specific structure as in the following lemma, where the proof relies on the refined regularity lemma due to Conlon, Fox, and Wigderson [6] together with the idea from Nikiforov and Rousseau [14].

Lemma 3.2. *Let $1/6 \leq \alpha \leq 1$, $\varepsilon, \gamma \in (0, 1)$ where ε is sufficiently small in terms of γ, and n is a large integer. Consider a red/blue edge colouring of K_N and an equitable ε-regular partition of $V(R)$ guaranteed by Lemma 2.1 with $N = (x + y\alpha)n + o(n)$, where $x + y = xy$, $1 \leq x \leq y \leq 2x$. If R is B_n-free and B is $B_{\alpha n}$-free, then the following two properties hold:*

(1) there exists no $(3, \varepsilon, \gamma)$-red-blocked configuration;

(2) there exists no $(2, \varepsilon, \gamma)$-blue-blocked configuration.

Proof. Let $N = (x + y\alpha + \eta)n$ where $\eta > 0$ is sufficiently small. Consider a red/blue edge colouring of K_N on vertex set $[N]$. We assume that $\gamma > 0$ is taken sufficiently small in terms of η. Set $\delta = \gamma^2/2$. Note that ε is sufficiently small in terms of δ, since ε is sufficiently small in terms of γ.

By Lemma 2.1, there is an equitable ε-regular partition $[N] = \sqcup_{i=1}^{k} V_i$ for the red graph R, i.e., (i) $||V_i| - |V_j|| \leq 1$ for all distinct i and j; (ii) each part V_i is ε-regular; and (iii) for every $1 \leq i \leq k$, there are at most εk values $1 \leq j \leq k$ such that the pair (V_i, V_j) is not ε-regular. Because the colours are complementary, the same conclusion holds for the blue graph. For convenience, we may assume $|V_i| = N/k =: t$ for all $i \in [k]$. By the assumption that R is B_n-free and B is $B_{\alpha n}$-free, we have

$$bk_R < \frac{1}{x + y\alpha + \eta} N = \frac{1}{x + y\alpha + \eta} kt \leq \left(\frac{1}{x + y\alpha} - \gamma \right) kt, \tag{3}$$

$$bk_B < \frac{\alpha}{x + y\alpha + \eta} N = \frac{\alpha}{x + y\alpha + \eta} kt \leq \alpha \left(\frac{1}{x + y\alpha} - \gamma \right) kt. \tag{4}$$

Now we shall prove that there exists no $(3, \varepsilon, \gamma)$-red-blocked configuration. On the contrary, we may assume that V_1, V_2, V_3 is a $(3, \varepsilon, \gamma)$-red-blocked configuration. Let M be the set of all $s \in [k] \setminus [3]$ such that every pair (V_i, V_s) for $i \in [3]$ is ε-regular; clearly, $|M| \geq (k-3) - 3\varepsilon k \geq (1 - 4\varepsilon)k$.

We compute the maximum size of the red books with bases in $E(V_i)$ for $i \in [3]$. Since the red density of the pair (V_i, V_s) is d_{is}, we apply Lemma 2.2 to obtain that

$$bk_R \geq \sum_{s \in M} (d_{is}^2 - \delta)t. \tag{5}$$

On the other hand, applying Lemma 2.2, we obtain that the maximum size S of the blue books with bases in $E(V_1, V_2)$ satisfies

$$bk_B \geq S \geq \sum_{s \in M} ((1 - d_{1s})(1 - d_{2s}) - \delta)t. \tag{6}$$

Considering in turn (V_1, V_3) and (V_2, V_3), we obtain exactly in the same way,

$$bk_B \geq \sum_{s \in M} ((1 - d_{1s})(1 - d_{3s}) - \delta)t, \tag{7}$$

$$bk_B \geq \sum_{s \in M} ((1 - d_{2s})(1 - d_{3s}) - \delta)t. \tag{8}$$

Let

$$d_s = \sum_{i=1}^{3} d_{is}, \text{ and } d_0 = \frac{1}{|M|}\sum_{s\in M} d_s.$$

Adding (6), (7), and (8) each multiplied by $y/3$ together with (5) for each $i \in [3]$ multiplied by $x/3$, noting $2x \geq y$, and applying Cauchy's inequality to the double sum we obtain

$$\begin{aligned}
y\cdot bk_B + x\cdot bk_R \geq & \frac{y}{3}\sum_{s\in M}\left(3-2d_s+\sum_{1\leq i<j\leq 3} d_{is}d_{js} - 3\delta\right)t + \frac{x}{3}\sum_{s\in M}\left(\sum_{i=1}^{3} d_{is}^2 - 3\delta\right)t \\
\geq & \sum_{s\in M}\left(\frac{y}{3}\left(3-2d_s+\frac{d_s^2}{2}-\frac{1}{2}\sum_{i=1}^{3} d_{is}^2\right)+\frac{x}{3}\sum_{i=1}^{3} d_{is}^2\right)t-(x+y)\delta kt \\
\geq & \sum_{s\in M}\left(\frac{y}{3}\left(3-2d_s+\frac{d_s^2}{2}\right)+\left(\frac{x}{3}-\frac{y}{6}\right)\cdot\frac{1}{3}d_s^2\right)t-(x+y)\delta kt \\
= & \sum_{s\in M}\left(\frac{y}{3}\left(3-2d_s\right)+\frac{x+y}{9}d_s^2\right)t-(x+y)\delta kt \\
\geq & |M|\left(\frac{y}{3}\left(3-2d_0\right)+\frac{x+y}{9}d_0^2\right)t-(x+y)\delta kt.
\end{aligned}$$

Therefore, from (3) and (4), we obtain that

$$|M|\left(\frac{y}{3}\left(3-2d_0\right)+\frac{x+y}{9}d_0^2\right)t-(x+y)\delta kt < \left(y\alpha\left(\frac{1}{x+y\alpha}-\gamma\right)+x\left(\frac{1}{x+y\alpha}-\gamma\right)\right)kt.$$

Since $|M| \geq (1-4\varepsilon)k$, we have

$$(1-4\varepsilon)\left(\frac{x+y}{9}d_0^2-\frac{2y}{3}d_0+y\right)-(x+y)\delta < 1-(x+y\alpha)\gamma.$$

Thus, by noting $\delta = \gamma^2/2$, ε is sufficiently small in terms of γ, we have

$$\frac{x+y}{9}d_0^2-\frac{2y}{3}d_0+y-1 < \frac{1-(x+y\alpha)\gamma+(x+y)\delta}{1-4\varepsilon}-1 = \frac{-(x+y\alpha)\gamma+(x+y)\gamma^2/2+4\varepsilon}{1-4\varepsilon}.$$

This leads to a contradiction since the right-hand side is negative for sufficiently small γ and the left-hand side is non-negative for $\frac{x+y}{9} > 0$ and the discriminant of the quadratic form $\Delta = (\frac{2y}{3})^2 - \frac{4(x+y)(y-1)}{9} = \frac{4}{9}(-xy+x+y) = 0$ since $x+y = xy$.

It remains to prove that there exists no $(2,\varepsilon,\gamma)$-blue-blocked configuration. On the contrary, suppose that V_1, V_2 is a $(2,\varepsilon,\gamma)$-blue-blocked configuration without loss of generality. Let M be the set of all $s \in [k]\setminus[2]$ such that all pairs (V_1, V_s) and (V_2, V_s) are ε-regular; clearly $|M| \geq (k-2)-2\varepsilon k \geq (1-3\varepsilon)k$.

By a similar argument as aforementioned, we obtain that for $i \in [2]$,

$$bk_B \geq \sum_{s\in M}((1-d_{is})^2-\delta)t. \tag{9}$$

Similarly,

$$bk_R \geq \sum_{s\in M}(d_{1s}d_{2s}-\delta)t. \tag{10}$$

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Let $d_s = \sum_{i=1}^{2} d_{is}$, and $d_0 = \frac{1}{|M|}\sum_{s\in M} d_s$. Adding (10) multiplied by x together with (9) for each $i \in [2]$ multiplied by $y/2$, noting $x \le y$, and applying Cauchy's inequality to the double sum we obtain

$$\begin{aligned} y\cdot bk_B + x\cdot bk_R \ge& \frac{y}{2}\sum_{s\in M}\left(2-2d_s+\sum_{i=1}^{2} d_{is}^2 - 2\delta\right)t + x\sum_{s\in M}\left(d_{1s}d_{2s}-\delta\right)t \\ \ge& \sum_{s\in M}\left(\frac{y}{2}(2-2d_s)+\frac{y}{2}\sum_{i=1}^{2} d_{is}^2 + \frac{x}{2}\left(d_s^2-\sum_{i=1}^{2} d_{is}^2\right)\right)t-(x+y)\delta kt \\ \ge& \sum_{s\in M}\left(\frac{y}{2}(2-2d_s)+\frac{y+x}{2}\cdot\frac{d_s^2}{2}\right)t-(x+y)\delta kt \\ \ge& |M|\left(\frac{y}{2}(2-2d_0)+\frac{y+x}{4}d_0^2\right)t-(x+y)\delta kt. \end{aligned}$$

Therefore, from (3) and (4), we obtain that

$$|M|\left(\frac{y}{2}(2-2d_0)+\frac{y+x}{4}d_0^2\right)t-(x+y)\delta kt < \left(y\alpha\left(\frac{1}{x+y\alpha}-\gamma\right)+x\left(\frac{1}{x+y\alpha}-\gamma\right)\right)kt.$$

Since $|M| \ge (1-3\varepsilon)k$, we have

$$(1-3\varepsilon)\left(\frac{y+x}{4}d_0^2 - yd_0 + y\right)-(x+y)\delta < 1-(x+y\alpha)\gamma.$$

Thus, by noting $\delta = \gamma^2/2$, ε is sufficiently small in terms of γ, we have that

$$\frac{y+x}{4}d_0^2 - yd_0 + y - 1 < \frac{1-(x+y\alpha)\gamma+(x+y)\delta}{1-3\varepsilon}-1 = \frac{-(x+y\alpha)\gamma+(x+y)\gamma^2/2+3\varepsilon}{1-3\varepsilon}.$$

This leads to a contradiction since the right-hand side is negative for sufficiently small γ and the left-hand side is non-negative for $\frac{y+x}{4} > 0$ and the discriminant of the quadratic form $\Delta = y^2-(y+x)(y-1) = -xy+x+y = 0$. This completes the proof of Lemma 3.2. □

Now, we are ready to give proofs for Theorem 1.4 and Theorem 1.5.

Proof sketches of Theorem 1.4 and Theorem 1.5. Consider a red/blue edge colouring of K_N for some suitable N. On the contrary, we suppose that $bk_R < n$ and $bk_B < \alpha n$. Firstly, we apply the refined regularity lemma due to Conlon, Fox, and Wigderson [6] to obtain an equitable ε-regular partition of $V(R)$ which guarantees the regularity of each cluster with itself. Secondly, from the assumption that $bk_R < n$ and $bk_B < \alpha n$, Lemma 3.2 implies that there are no $(3, \varepsilon, \gamma)$-red-blocked and $(2, \varepsilon, \gamma)$-blue-blocked configurations, which are the bases for subsequent calculations of the corresponding book sizes. According to the densities of clusters, we partition these clusters into red clusters and blue clusters. For Theorem 1.4, applying the counting lemma due to Conlon [5] to a single red/blue cluster, and combining with Turán's bound on the subgraph H defined on the red clusters, we finally obtain $2bk_R + bk_B \ge 2n + \alpha n$, which leads to a contradiction. For Theorem 1.5, based on the lower bounds of the book sizes of R/B, we need to consider the total blue/red densities between red-cluster sets and blue-cluster sets. The situation is more complicated and the computation is much more technical.

The reason we have some improvements is that the specific structure, i.e., there exists no $(3, \varepsilon, \gamma)$-red-blocked and $(2, \varepsilon, \gamma)$-blue-blocked configurations, is more in line with the essence for 2-books than the structure that no $(2, \varepsilon, \gamma)$-red-blocked and $(2, \varepsilon, \gamma)$-blue-blocked configurations used by Conlon, Fox, and Wigderson [12]. In particular, the refined regularity lemma by Conlon, Fox, and Wigderson [6] is a key ingredient of the proofs.

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Proof of Theorem 1.4. Let $1/4 \leq \alpha \leq 1$ be fixed, and $\eta > 0$ is sufficiently small and n is sufficiently large. Let $N = (x + y\alpha + \eta)n$ where $x + y = xy$, $1 \leq x \leq y \leq 2x$. Consider a red/blue edge colouring of K_N on vertex set $[N]$. Let $\gamma > 0$ be sufficiently small in terms of η. Set $\delta = \gamma^2/2$. We assume that $\varepsilon > 0$ is sufficiently small in terms of δ and γ.

By Lemma 2.1, there is an equitable ε-regular partition $[N] = \sqcup_{i=1}^{k} V_i$ for the red graph R, i.e., (i) $||V_i| - |V_j|| \leq 1$ for all distinct i and j; (ii) each part V_i is ε-regular; and (iii) for every $1 \leq i \leq k$, there are at most εk values $1 \leq j \leq k$ such that the pair (V_i, V_j) is not ε-regular. Because the colours are complementary, the same conclusion holds for the blue graph. For convenience, we may assume $|V_i| = N/k =: t$ for all $i \in [k]$. It suffices to show that for all sufficiently large n, there exists a red B_n or a blue $B_{\alpha n}$. On the contrary,

$$bk_R < \frac{1}{x + y\alpha + \eta} N = \frac{1}{x + y\alpha + \eta} kt \leq \left(\frac{1}{x + y\alpha} - \gamma \right) kt, \tag{11}$$

$$bk_B < \frac{\alpha}{x + y\alpha + \eta} N = \frac{\alpha}{x + y\alpha + \eta} kt \leq \alpha \left(\frac{1}{x + y\alpha} - \gamma \right) kt. \tag{12}$$

We call a cluster V_i *red* if at least half of its internal edges are red and *blue* otherwise. Clearly, every cluster is either red or blue.

Now we assume that $V_1, \ldots, V_l$ are blue clusters without loss of generality and set $l = \lambda k$, $0 \leq \lambda \leq 1$. To finish the proof, we will show that (11) or (12) is not true.

We first fix a blue cluster V_i and compute the maximum size of the blue books whose bases lie in $E(V_i)$. Let M_i be the set of all $s \in [k] \setminus \{i\}$ such that (V_i, V_s) is an ε-regular pair, and let $M_{i1} = M_i \cap [l]$. Clearly, $l \geq |M_{i1}| \geq l - 1 - \varepsilon k \geq (\lambda - 2\varepsilon)k$ since $|M_i| \geq (1 - \varepsilon)k$.

By Lemma 3.2 (2), for every $s \in M_{i1}$, the red density d_{is} of the pair (V_i, V_s) satisfies $d_{is} \leq \gamma$, so the blue density of (V_i, V_s) is at least $1 - \gamma$. Therefore, by Lemma 2.2 and noting $\delta = \gamma^2/2$, the maximum size S of the blue books whose bases are in $E(V_i)$ satisfies

$$S \geq \sum_{s \in M_{i1}} \left((1 - \gamma)^2 - \delta \right) t \geq (1 - 2\gamma)|M_{i1}|t.$$

Then, by noting $|M_{i1}| \geq (\lambda - 2\varepsilon)k$, we find that

$$bk_B \geq S \geq (1 - 2\gamma)(\lambda - 2\varepsilon)kt. \tag{13}$$

Next we consider the maximum size of the red books whose bases are contained in a red cluster. Let us define the graph H as follows. The vertex set of H is $[l + 1, k]$ and two vertices $i, j \in [l + 1, k]$ are joined if and only if the red density d_{ij} of the ε-regular pair (V_i, V_j) satisfies $d_{ij} > 1 - \gamma$.

By Lemma 3.2 (1), the complement of H is triangle-free, i.e., the independence number of H is at most two, hence Lemma 2.3 implies that the average degree of H is at least $(k - l)/2 - 1$. So there exists a vertex $i \in [l + 1, k]$ such that the degree of i in H is at least $(k - l)/2 - 1$. Let N_i be the set of all $s \in [k] \setminus \{i\}$ such that (V_i, V_s) is an ε-regular pair, and let $N_{i1} = N_i \cap N_H(i)$. Clearly, $k - l \geq |N_{i1}| \geq (k - l)/2 - 1 - \varepsilon k$ since $|N_i| \geq (1 - \varepsilon)k$.

By Lemma 2.2 and $\delta = \gamma^2/2$, the maximum size S of the red books whose bases lie in $E(V_i)$ satisfies

$$S \geq \sum_{s \in N_{i1}} \left((1 - \gamma)^2 - \delta \right) t \geq (1 - 2\gamma)|N_{i1}|t.$$

Then, by noting $|N_{i1}| \geq (k - l)/2 - 1 - \varepsilon k \geq \left(\frac{1-\lambda}{2} - 2\varepsilon \right) k$, we find that

$$bk_R \geq S \geq (1 - 2\gamma) \left(\frac{1 - \lambda}{2} - 2\varepsilon \right) kt. \tag{14}$$

Adding (13) and (14) multiplied by 2, we obtain

$$2bk_R + bk_B \geq (1 - 2\gamma)\left(1 - \lambda - 4\varepsilon\right) kt + (1 - 2\gamma)(\lambda - 2\varepsilon)kt \geq (1 - 2\gamma - 6\varepsilon)kt$$

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To prove Theorem 1.4, we take $x=\sqrt{\alpha}+1$, $y=1+\frac{1}{\sqrt{\alpha}}$. Since ε is sufficiently small in terms of γ and $\frac{2+\alpha}{x+y\alpha}\le 1$ for $\frac{1}{4}\le\alpha\le 1$, we have

$$2bk_R+bk_B\ge\left(\frac{2+\alpha}{x+y\alpha}-2\gamma-\alpha\gamma\right)kt.$$

It follows that either $2bk_R\ge(\frac{2}{x+y\alpha}-2\gamma)kt$, or $bk_B\ge(\frac{\alpha}{x+y\alpha}-\alpha\gamma)kt$, which contradicts (11) or (12), respectively. The proof of Theorem 1.4 is complete. □

Proof of Theorem 1.5. Let $1/6\le\alpha\le 1/4$ be fixed, and $\eta>0$ is sufficiently small and n is sufficiently large. Let $N=(x+y\alpha+\eta)n$, where $x=3/2$ and $y=3$, so $x+y=xy$. Consider a red/blue edge colouring of K_N on vertex set $[N]$. Let $\gamma>0$ be sufficiently small in terms of η. Set $\delta=\gamma^2/2$, and $\varepsilon>0$ is taken sufficiently small in terms of δ and γ.

Similarly, by Lemma 2.1, there is an equitable ε-regular partition $[N]=\sqcup_{i=1}^{k}V_i$ for the red graph R. For convenience, we may assume $|V_i|=N/k=:t$ for all $i\in[k]$. It suffices to show that for all sufficiently large n, there exists a red B_n or a blue $B_{\alpha n}$. On the contrary,

$$bk_R<\frac{1}{x+y\alpha+\eta}N=\frac{1}{x+y\alpha+\eta}kt\le\left(\frac{1}{x+y\alpha}-\gamma\right)kt, \tag{15}$$
$$bk_B<\frac{\alpha}{x+y\alpha+\eta}N=\frac{\alpha}{x+y\alpha+\eta}kt\le\alpha\left(\frac{1}{x+y\alpha}-20\gamma\right)kt. \tag{16}$$

We call a cluster V_i *red* if at least half of its internal edges are red and *blue* otherwise. Clearly, every cluster is either red or blue. Now we assume that $V_1,\dots,V_l$ are blue clusters without loss of generality and set $l=\lambda k$, $0\le\lambda\le 1$. We first consider a blue cluster V_i and compute the maximum size of the blue books whose bases lie in $E(V_i)$. Let M_i be the set of all $s\in[k]\setminus\{i\}$ such that (V_i,V_s) is an ε-regular pair, and let

$$M_{i1}=M_i\cap[l],\ \text{ and }\ M_{i2}=M_i\cap[l+1,k].$$

Clearly, $l\ge|M_{i1}|\ge l-1-\varepsilon k\ge(\lambda-2\varepsilon)k$ since $|M_i|\ge(1-\varepsilon)k$.

By Lemma 3.2 (2), for every $s\in M_{i1}$, the red density d_{is} of the pair (V_i,V_s) satisfies $d_{is}\le\gamma$, so the blue density of (V_i,V_s) is at least $1-\gamma$. Therefore, by Lemma 2.2 and noting $\delta=\gamma^2/2$, the maximum size S of the blue books whose bases are in $E(V_i)$ satisfies

$$\begin{aligned}S&\ge\sum_{s\in M_{i1}}\left((1-\gamma)^2-\delta\right)t+\sum_{s\in M_{i2}}((1-d_{is})^2-\delta)t\\&\ge(1-2\gamma)|M_{i1}|t+\sum_{s\in M_{i2}}(1-d_{is})^2t-\delta|M_{i2}|t.\end{aligned}$$

Then, by noting ε is sufficiently small in terms of δ, $|M_{i1}|\ge(\lambda-2\varepsilon)k$ and $|M_{i2}|\le k$, and applying Cauchy's inequality, we obtain

$$\begin{aligned}bk_B\ge S\ge&(1-2\gamma)(\lambda-2\varepsilon)kt+\sum_{s\in M_{i2}}(1-d_{is})^2t-\delta kt\\\ge&(1-2\gamma)\lambda kt+\frac{1}{|M_{i2}|}\left(\sum_{s\in M_{i2}}(1-d_{is})\right)^2t-2\delta kt.\end{aligned}$$

Recall (16) and $\delta=\gamma^2/2$, we obtain that

$$\frac{1}{|M_{i2}|}\left(\sum_{s\in M_{i2}}(1-d_{is})\right)^2 t<\left(\frac{\alpha}{x+y\alpha}-20\gamma-(1-2\gamma)\lambda+2\delta\right)kt\leq\left(\frac{\alpha}{x+y\alpha}-\lambda-15\gamma\right)kt. \tag{17}$$

We may assume $\lambda\leq\frac{\alpha}{x+y\alpha}<1/2$, otherwise the right-hand side of (17) is negative, which is not possible since the left-hand side is non-negative. Since $|M_{i2}|\leq k-l=(1-\lambda)k$, we have

$$\sum_{s\in M_{i2}}(1-d_{is})<k\sqrt{\left(\frac{\alpha}{x+y\alpha}-\lambda-15\gamma\right)(1-\lambda)}.$$

Summing over all $i\in[l]$ and noting that $l=\lambda k<k/2$, we obtain that the total blue densities of all regular pairs (V_i,V_s) where $i\in[l]$ and $s\in[l+1,k]$ satisfies that

$$\sum_{i=1}^{l}\sum_{s\in M_{i2}}(1-d_{is})<\lambda k^2\sqrt{\left(\frac{\alpha}{x+y\alpha}-\lambda-15\gamma\right)(1-\lambda)}. \tag{18}$$

Next we consider the maximum size of the red books whose bases are contained in a red cluster. Let us define the graph H as follows. The vertices of H are the numbers $[l+1,k]$ and two vertices i,j are joined if and only if the red density d_{ij} of the ε-regular pair (V_i,V_j) satisfies $d_{ij}>1-\gamma$.

By Lemma 3.2 (1), the complement of H is triangle-free, hence Lemma 2.3 implies that the average degree of H is at least $(k-l)/2-1$. Thus, recall $l\leq k/2$, if k is sufficiently large, then

$$e(H)\geq\frac{k-l}{2}\left(\frac{k-l}{2}-1\right)\geq\left(\frac{1}{4}-\varepsilon\right)(k-l)^2. \tag{19}$$

For any $i\in[l+1,k]$, let N_i be the set of all $s\in[k]\setminus\{i\}$ such that (V_i,V_s) is ε-regular, and let

$$N_{i1}=N_i\cap N_H(i),\ \text{ and }\ N_{i2}=N_i\cap[l].$$

Since $|N_i|\geq(1-\varepsilon)k$, we have $\deg_H(i)\geq|N_{i1}|\geq\deg_H(i)-\varepsilon k$. Therefore, since ε is sufficiently small in terms of δ, the maximum size S_i of the red books whose bases lie in $E(V_i)$ satisfies

$$\begin{aligned}S_i\geq&\sum_{s\in N_{i1}}\left((1-\gamma)^2-\delta\right)t+\sum_{s\in N_{i2}}(d_{is}^2-\delta)t\\ \geq&(1-2\gamma)|N_{i1}|t+\sum_{s\in N_{i2}}d_{is}^2t-\delta|N_{i2}|t\\ \geq&(1-2\gamma)\deg_H(i)t+\sum_{s\in N_{i2}}d_{is}^2t-2\delta kt.\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{i=l+1}^{k}S_i\geq&\sum_{i=l+1}^{k}(1-2\gamma)\deg_H(i)t+\sum_{i=l+1}^{k}\sum_{s\in N_{i2}}d_{is}^2t-2\delta k(k-l)t\\ \geq&(1-2\gamma)2e(H)t+\sum_{i=l+1}^{k}\sum_{s\in N_{i2}}d_{is}^2t-2\delta k(k-l)t.\end{aligned}$$

Hence by (19) and ε is sufficiently small in terms of δ, we obtain that

$$\begin{aligned}\sum_{i=l+1}^{k} S_i \geq &(1-2\gamma)(1/2-2\varepsilon)(k-l)^2 t + \sum_{i=l+1}^{k}\sum_{s\in N_{i2}} d_{is}^2 t - 2\delta k(k-l)t\\ \geq &\frac{1}{2}(1-2\gamma)(k-l)^2 t + \sum_{i=l+1}^{k}\sum_{s\in N_{i2}} d_{is}^2 t - 4\delta k(k-l)t.\end{aligned}$$

Then by noting that $|N_{i2}| \leq l = \lambda k$ and applying Cauchy's inequality to the double sum, we have

$$\begin{aligned}bk_R \geq \frac{1}{k-l}\sum_{i=l+1}^{k} S_i \geq &\frac{1}{2}(1-2\gamma)(k-l)t + \frac{1}{k-l}\sum_{i=l+1}^{k}\sum_{s\in N_{i2}} d_{is}^2 t - 4\delta kt\\ \geq &\frac{1}{2}(1-2\gamma)(1-\lambda)kt + \frac{1}{(k-l)l}\sum_{i=l+1}^{k}\left(\sum_{s\in N_{i2}} d_{is}\right)^2 t - 4\delta kt\\ \geq &\frac{1}{2}(1-2\gamma)(1-\lambda)kt + \frac{1}{(k-l)^2 l}\left(\sum_{i=l+1}^{k}\sum_{s\in N_{i2}} d_{is}\right)^2 t - 4\delta kt.\end{aligned}$$

So we obtain

$$bk_R \geq \frac{1}{2}(1-2\gamma)(1-\lambda)kt - 4\delta kt. \tag{20}$$

Recall (15), we obtain that

$$\begin{aligned}\frac{1}{(k-l)^2 l}\left(\sum_{i=l+1}^{k}\sum_{s\in N_{i2}} d_{is}\right)^2 t &< \left(\frac{1}{x+y\alpha} - \gamma\right)kt - \frac{1}{2}(1-2\gamma)(1-\lambda)kt + 4\delta kt\\ &\leq \left(\frac{1}{x+y\alpha} - \frac{1-\lambda}{2} + 4\delta\right)kt.\end{aligned}$$

Recall $l = \lambda k$, we obtain that the total red densities of all regular pairs (V_i, V_s) where $i \in [l+1, k]$ and $s \in [l]$ satisfies that

$$\sum_{i=l+1}^{k}\sum_{s\in N_{i2}} d_{is} < (1-\lambda)k^2\sqrt{\left(\frac{1}{x+y\alpha} - \frac{1-\lambda}{2} + 4\delta\right)\lambda}. \tag{21}$$

Therefore, adding (18) and (21), we obtain

$$\begin{aligned}\sum_{i=1}^{l}\sum_{s\in M_{i2}} 1 = \sum_{i=1}^{l}\sum_{s\in M_{i2}} (1-d_{is}) + \sum_{i=1}^{l}\sum_{s\in M_{i2}} d_{is} = \sum_{i=1}^{l}\sum_{s\in M_{i2}} (1-d_{is}) + \sum_{i=l+1}^{k}\sum_{s\in N_{i2}} d_{is}\\ < \lambda k^2\sqrt{\left(\frac{\alpha}{x+y\alpha} - \lambda - 15\gamma\right)(1-\lambda)} + (1-\lambda)k^2\sqrt{\left(\frac{1}{x+y\alpha} - \frac{1-\lambda}{2} + 4\delta\right)\lambda}.\end{aligned}$$

Note that $\sum_{i=1}^{l}\sum_{s\in M_{i2}} 1 \geq (k-l)l - \varepsilon k^2 = ((1-\lambda)\lambda - \varepsilon)k^2$, we have

$$((1-\lambda)\lambda - \varepsilon)k^2 < \lambda k^2\sqrt{\left(\frac{\alpha}{x+y\alpha} - \lambda - 15\gamma\right)(1-\lambda)} + (1-\lambda)k^2\sqrt{\left(\frac{1}{x+y\alpha} - \frac{1-\lambda}{2} + 4\delta\right)\lambda}.$$

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Suppose $\lambda \le \eta/10$, then we are done from (20) that

$$bk_R \ge \frac{1}{2}(1-2\gamma)(1-\lambda)kt - 4\delta kt \ge \left(\frac{1}{2}-\gamma-\frac{\lambda}{2}-4\delta\right)\left(\frac{3}{2}+3\alpha+\eta\right)n \ge n$$

by noting $\alpha \ge 1/6$, and γ and δ are sufficiently small in terms of η. Therefore, we may assume $\lambda > \eta/10$.

Since γ is sufficiently small in terms of λ, α, x and y, and $\delta = \gamma^2/2$, and ε is sufficiently small in terms of γ, we obtain

$$(1-\lambda)\lambda < \lambda\sqrt{\left(\frac{\alpha}{x+y\alpha}-\lambda\right)(1-\lambda)} + (1-\lambda)\sqrt{\left(\frac{1}{x+y\alpha}-\frac{1-\lambda}{2}\right)\lambda},$$

and consequently,

$$\sqrt{(1-\lambda)\lambda} < \sqrt{\left(\frac{\alpha}{x+y\alpha}-\lambda\right)\lambda} + \sqrt{\left(\frac{1}{x+y\alpha}-\frac{1-\lambda}{2}\right)(1-\lambda)}. \tag{22}$$

Since (22) makes sense, we obtain $1-\frac{2}{x+y\alpha} \le \lambda \le \frac{\alpha}{x+y\alpha}$. Recall that $1/6 \le \alpha \le 1/4$, $x = 3/2$ and $y = 3$, so $\frac{\alpha+11/6}{x+y\alpha} \le 1$, implying $\frac{\alpha}{x+y\alpha} \le \frac{1}{12} + \frac{11}{12}(1-\frac{2}{x+y\alpha}) \le \frac{1}{12} + \frac{11}{12}\lambda = \frac{1}{12}(1-\lambda)+\lambda$. Thus $\frac{\alpha}{x+y\alpha} - \lambda \le \frac{1}{12}(1-\lambda)$, and so $\sqrt{(\frac{\alpha}{x+y\alpha}-\lambda)\lambda} \le \frac{1}{2\sqrt{3}}\sqrt{(1-\lambda)\lambda}$.

Moreover, since $\frac{1}{2} - \frac{1}{x+y\alpha} \ge 0$, we obtain that

$$\frac{1}{x+y\alpha} - \frac{1}{2} \le \left(2\left(1-\frac{1}{2\sqrt{3}}\right)^2 - 1\right)\left(\frac{1}{2}-\frac{1}{x+y\alpha}\right) \le \left(\left(1-\frac{1}{2\sqrt{3}}\right)^2 - \frac{1}{2}\right)\lambda.$$

Thus $\frac{1}{x+y\alpha} - \frac{1-\lambda}{2} \le (1-\frac{1}{2\sqrt{3}})^2\lambda$, and so $\sqrt{(\frac{1}{x+y\alpha}-\frac{1-\lambda}{2})(1-\lambda)} \le (1-\frac{1}{2\sqrt{3}})\sqrt{(1-\lambda)\lambda}$.

Therefore, adding these two terms, the right-hand side of (22) is at most $\sqrt{(1-\lambda)\lambda}$, which leads to a contradiction. The proof of Theorem 1.5 is complete. □

4. Proof of Theorem 1.6

Let $\frac{1}{6} \le \alpha \le \frac{52-16\sqrt{3}}{121}$ be fixed, and $p = \frac{1-\sqrt{\alpha(3-2\alpha)}}{1-2\alpha}$, and $N = (\frac{3}{1+2p^2}-\eta)n$, where $\eta > 0$ is sufficiently small and n is sufficiently large. We shall show that for sufficiently large N there exists a partially random red/blue colouring of the edges of K_N for which

$$bk_R < n, \quad \text{and} \quad bk_B < \alpha n.$$

For convenience, assume that N is divisible by 3. Partition $[N]$ into three sets A_1, A_2, A_3, each with $N/3$ vertices, and colour the graphs induced by A_1, A_2, A_3 in red. Then edges of the form uv where $u \in A_i$, $v \in A_j$ $(1 \le i < j \le 3)$ are independently coloured red with probability p and blue with probability $1-p$. For $u, v \in A_i$, the size of the red book with base uv is a random variable with expected value

$$\frac{N}{3} - 2 + \frac{2N}{3}p^2 \le \frac{N}{3}(1+2p^2) = \frac{1}{3}\left(\frac{3}{1+2p^2}-\eta\right)(1+2p^2)n = \left(1-\frac{(1+2p^2)\eta}{3}\right)n. \tag{23}$$

Now suppose $u \in A_i$ and $v \in A_j$ where $i \ne j$. If uv is a blue edge, the size of the blue book with base uv is a random variable with expected value

$$\frac{N}{3}(1-p)^2 = \frac{1}{3}\left(\frac{3}{1+2p^2}-\eta\right)(1-p)^2 n = \left(\frac{(1-p)^2}{1+2p^2} - \frac{(1-p)^2\eta}{3}\right)n = \left(1-\frac{(1+2p^2)\eta}{3}\right)\alpha n,$$

by noting that $\alpha = \frac{(1-p)^2}{1+2p^2}$.

By noting (23), if uv is a red edge, then the size of the red book with base uv is a random variable with expected value

$$\frac{N}{3}p^2 + \left(\frac{2N}{3} - 2\right)p \le \frac{N}{3} - 2 + \frac{2N}{3}p^2 \le \left(1 - \frac{(1+2p^2)\eta}{3}\right)n.$$

We will use the following version of the Chernoff bound [20, Corollary 2.4 and Theorem 2.8]: let $X_1, \ldots, X_t$ be independent random variables taking values in $\{0,1\}$ and let $X = \sum_{i=1}^{t} X_i$. If $x \ge c\mathbb{E}(X)$ for some $c > 1$, then $\Pr(X \ge x) \le e^{-c'x}$, where $c' = \ln c - 1 + \frac{1}{c}$.

Plugging in $c = 1/(1 - \frac{(1+2p^2)\eta}{3})$, since $y = \ln x - 1 + \frac{1}{x}$ is increasing when $x \ge 1$, we find that $c' > 0$. Since $\Pr(X_1 \ge n) \le e^{-c'n}$ and $\Pr(X_2 \ge \alpha n) \le e^{-c'\alpha n}$, where X_1 denotes the size of a red book and X_2 denotes the size of a blue book, applying a union bound over all edges, we obtain that the probability that there is a red book B_n or a blue book $B_{\alpha n}$ tends to 0 as $n \to \infty$. Thus for large enough N the desired red/blue colouring of the edges of K_N exists.

5. Concluding remarks

From the result of Nikiforov and Rousseau [14], we know the exact value of $r(B_{\alpha n}, B_n)$ for $0 < \alpha < 1/6$; and from Theorem 1.4, we know the asymptotic behaviour of $r(B_{\alpha n}, B_n)$ for $1/4 \le \alpha \le 1$, i.e., the random lower bound $r(B_{\alpha n}, B_n) \ge (\sqrt{\alpha} + 1)^2 n + o(n)$ is asymptotically tight for $1/4 \le \alpha \le 1$. Moreover, the asymptotic behaviour of $r(B_n, B_n)$ is already known from a more general result, see [5,6]. For the remaining cases, when $1/6 \le \alpha \le 1/4$, we only know that $r(B_{\alpha n}, B_n) \le (\frac{3}{2} + 3\alpha)n + o(n)$ from Theorem 1.5. We do not know whether this upper bound is asymptotically tight or not for any $1/6 < \alpha < 1/4$. Note that for any $1/6 < \alpha < 1/4$,

$$3/2 + 3\alpha > (\sqrt{\alpha} + 1)^2,$$

therefore, if $r(B_{\alpha n}, B_n) = (\frac{3}{2} + 3\alpha)n + o(n)$ holds in this interval, then it means that Conjecture 1.3 proposed by Conlon, Fox, and Wigderson [12] indeed holds in this interval. In particular, we already know that for any $\frac{1}{6} \le \alpha < \frac{52 - 16\sqrt{3}}{121} \approx 0.2007$, Conjecture 1.3 holds from Theorem 1.6.

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