

magnitudes of the nuclei having estimates 4 and 5 and compared them with absolute magnitudes of corresponding galaxies (Fig. 2). For the nuclei of lower prominence we cannot derive the absolute magnitudes, as it is seen from the explanations in Table 1. But they are certainly to be placed in the lower part of Fig. 2. Thus we may be sure that the correlation between M_{gal} and M_r is very poor indeed.

Of course it is premature to go further into interpretation of these data. But we can hope that having more data about integral properties of nuclei (magnitudes, colours, spectra), we will be in a much better position to speak about their nature. And this will perhaps help us to understand better the nuclei which are capable to produce the phenomenon of a radio-galaxy.

7. THE GENERAL RELATIVISTIC INSTABILITY OF MASSIVE STARS

W. A. Fowler

(California Institute of Technology, Pasadena, Calif., U.S.A.)

1. Introduction

The pioneer theoretical investigations of Burbidge (1) and of Shklovsky (2) have shown that the observations on the extended radio sources imply the generation, storage and emission of prodigious amounts of energy, in round numbers of the order of $10^7 M_{\odot} c^2 \sim 10^{61}$ ergs or even more. On the very general grounds that the ultimate source of energy is the conversion of mass, it is thus clear that very large condensations of matter in some form or other are, or have been, associated with the radio sources. Burbidge (3) suggested supernovae explosions in large aggregates of stars as a possible mechanism for the original generation of the energy involved.

In the summer of 1962, after conversations with Geoffrey and Margaret Burbidge, Hoyle and I (4, 5) investigated what is perhaps the simplest of many possible models, namely that a mass of the order of $10^8 M_{\odot}$ or greater has condensed into a single star in which the energy generation takes place. On this point of view, using the standard theory of stellar structure, one immediately obtains optical luminosities of the order of 10^{46} ergs/sec and lifetimes for nuclear energy generation of the order of 10^5 to 10^6 years so that the overall energy release is 10^{59} ergs. These figures roughly match the observational data for the so-called quasi-stellar objects subsequently discovered by Schmidt (6). Hoyle and I were seeking an explanation of the energy requirements for extended radio sources and found that our model had a large optical luminosity. Problems in the stability of massive stars arise, as will be discussed in detail below. Questions of stability aside, it is apparent that nuclear energy generation by hydrogen burning in massive stars with $M \sim 10^8 M_{\odot}$ is adequate to match the energy requirements in the quasi-stellar objects.

However, the energy requirements for the extended radio sources involve nuclear burning in stars with $M \sim 10^{10} M_{\odot}$ or even more. This assumes that hydrogen burning with 0.7 per cent conversion efficiency goes to completion in about 15 per cent of the stellar mass, giving an overall efficiency of ~ 0.1 per cent and an energy output $\sim 10^7 M_{\odot} c^2$. The efficiency of conversion of thermal energy into that of the high energy electron and magnetic fields necessary to give the synchrotron radio emission may only be of the order of 1 per cent or even less. In this case nuclear burning in stellar masses approaching total galactic masses, $\sim 10^{12} M_{\odot}$, is required. Since there is no observational evidence for such wholesale nuclear conversion in the galaxies associated with the extended radio sources, Hoyle and I suggested gravitational collapse to the general relativistic limit as another possible source of energy. In principle all of the rest

mass can be converted to energy in gravitational collapse although this requires that $2GM/Rc^2$ approach unity. It was realized that this ultimate 100 per cent efficiency was probably not attainable during the collapse of an actual star because of the large redshifts in all forms of energy emission when $2GM/Rc^2 \sim 1$. Even so, the conversion of gravitational energy seemed more attractive to us than matter-antimatter annihilation which is also 100 per cent efficient in the limit. We were unable to suggest a satisfactory model for the assembly of the matter and antimatter under realistic conditions. It did not seem unrealistic to suggest that a massive star of one type of matter could condense from the gas and smaller stars of a large galaxy, most probably in the galactic nucleus.

Feynman (7) first pointed out to us that general relativistic instabilities set in at a very early stage in the condensation of massive stars. Following Feynman's suggestion, Iben (8) carried out exact numerical integrations of the relativistic equations for a number of polytropes and confirmed Feynman's ideas. In my own work (Fowler (9, 10)) I have found that the first order post-Newtonian approximation is sufficient to illustrate the general physical principles involved and is particularly useful in investigations of the conditions under which nuclear reactions occur in massive stars.

Let me hasten to say that Chandrasekhar (11, 12, 13) has now given in very elegant form the exact treatment of the dynamical instability of massive stars. After some initial disagreements concerning numerical values, when we both performed our sums correctly (10, 12), agreement was reached on such matters as the radius for the onset of instability and so forth. Since the Dallas Conference in 1963 this field of study has become a very active one, and in particular, McVittie (14) Gratton (15) and Zel'dovich (16) have independently made significant contributions in the approach to the solution of the problem. At the California Institute of Technology, James Bardeen is carrying out numerical calculations on the dynamical collapse using the IBM 7094.

2. Binding Energy of a Massive Polytrope in Hydrostatic Equilibrium

The binding energy E_b of a star is equal but opposite in sign to the total energy E exclusive of the rest mass energy and, when the star has radius R , is given by

$$-E_b = E = (M - M_0)c^2 \quad (1)$$

where $M = M(R)$ is the mass of the star and M_0 is the total rest mass of its constituent particles. M is to be determined in principle by measuring the force exerted on a unit mass at a large distance ($\gg R$) from the star and then using Newton's inverse square law of gravitational attraction. On the other hand, M_0 can be measured by identifying and counting the constituent particles and multiplying by the appropriate rest mass.

One now employs the general relativistic equations for M and M_0 and the general relativistic equation for hydrostatic equilibrium throughout the star. Each expression in these equations is appropriately expanded in terms proportional to integral powers of the gravitational constant G and only the Newtonian term and the next higher order term are retained. In this way (9, 10) the post-Newtonian approximation for the total energy of a spherically symmetric, non-rotating star under hydrostatic equilibrium is found to be

$$E_{\text{eq}} = -6\pi \int \beta p r^2 dr + \frac{8\pi G}{c^2} \int p r M_r dr + \frac{6\pi G^2}{c^2} \int \rho M_r^2 dr \quad (2)$$

where r is the radial coordinate, p is the pressure, ρ is the mass-energy density expressed in mass per unit volume, M_r is the mass interior to r and β is the ratio of gas pressure to total pressure (gas plus radiation). In order to appreciate the order of the terms in equation (2) it should be noted that p is linear in G in the approximation under discussion so that the first

term on the right hand side of equation (2) is the classical Newtonian term and the last two are the post-Newtonian terms. In deriving equation (2) it was assumed that the stellar material is completely ionized into electrons and nuclei but that the temperature is below $T = 10^9$ degrees so that special relativistic effects for electrons and electron-positron pair formation can be neglected. Under these conditions the internal energy of particles and radiation per cm^3 is given by $3p(1 - \beta/2)$.

It is illuminating to express the classical term, which will be designated as $E_{\text{eq}}^{(1)}$, in terms of the appropriate average for β throughout the star. Thus

$$\begin{aligned} E_{\text{eq}}^{(1)} &= -6\pi(\beta) \int \rho r^2 dr = -\frac{1}{2}(\beta) \int 3p dV \\ &= -\frac{1}{2}(\beta)\Omega \\ &= -\frac{3(\beta)_n}{2(5-n)} \frac{GM^2}{R} \end{aligned} \tag{3}$$

Here the gravitational binding energy Ω , taken as a positive quantity, has been introduced. It is well known in classical hydrostatic equilibrium that $\Omega = \int 3p dV$ and that for a polytrope of index n , $\Omega = 3GM^2/(5-n)R$. In the approximation under consideration it is not necessary to distinguish between M and M_0 and so the superfluous subscript has not been retained. In the last form of equation (3) the dependence of (β) on the polytropic index is made explicit by appending the subscript n .

The classical binding energy per unit mass-energy is obtained by dividing equation (3) by Mc^2 to obtain

$$\frac{E_{\text{eq}}^{(1)}}{Mc^2} = -\frac{3(\beta)_n}{4(5-n)} \left(\frac{R_g}{R}\right) \tag{4}$$

where $R_g = 2GM/c^2 = 3 \times 10^5 (M/M_\odot)$ cm is the limiting gravitational radius or *Schwarzschild limit* and the right hand side of equation (4) is the first and linear term in a power series in the dimensionless parameter $R_g/R = 2GM/Rc^2$. The post-Newtonian terms are, of course, quadratic in this parameter. For polytropes of index n , the post-Newtonian expression for the binding energy per unit mass can be reduced to

$$\frac{E_{\text{eq}}}{Mc^2} = -\frac{3(\beta)_n}{4(5-n)} \left(\frac{R_g}{R}\right) + \zeta_n \left(\frac{R_g}{R}\right)^2 + \dots \tag{5}$$

where

$$\zeta_n = \frac{3}{8(n+1)} \frac{R_n^2}{M_n^3} \left[\int_0^{R_n} \theta_n^{2n+1} \xi^4 d\xi + \frac{10}{n+2} \int_0^{R_n} \theta_n^{n+2} \xi^2 d\xi \right] \tag{6}$$

In equation (6), ξ is the dimensionless radial variable used by Chandrasekhar (17) in treating polytropes, R_n is the value of ξ at the surface of the polytrope, $\theta_n = \theta_n(\xi)$ is the Lane-Emden function for the polytrope and $M_n = -\xi^2 d\theta_n/d\xi$ at the polytropic surface. M_n is a dimensionless measure of the mass of the polytrope. It will be recalled that the run of the variables throughout the polytrope are given by $\rho = \rho_c \theta_n^m$ and $p = p_c \theta_n^{m+1}$ where the subscript c designates central values. For a nondegenerate gas $(T/\mu\beta) = (T/\mu\beta)_c \theta_n$.

Equation (6) can be evaluated analytically for $n = 0, 1$, and 3 and the results are

$$\zeta_0 = \frac{57}{280} = 0.2036, \zeta_1 = \frac{1}{\pi} = 0.3183 \text{ and } \zeta_3 = \frac{3}{16} \left(\frac{3}{\pi}\right)^{\frac{1}{2}} R_3 = 1.264 \tag{7}$$

where $R_3 = 6.897$ has been used.

3. The Critical Radius, Temperature and Density for the Onset of Dynamical Instability

The coefficient ζ_n is positive and thus the internal energy required for hydrostatic equilibrium eventually becomes positive, the binding energy is negative and the system is unbound rather than bound. The energy goes through a minimum or the binding energy through a maximum at a critical radius given by

$$\frac{R_c}{R_g} = \frac{8(5-n)}{3} \frac{\zeta_n}{(\beta)_n} \approx \frac{4(5-n)}{9} \frac{\zeta}{(\Gamma_1 - 4/3)_n} \quad (8)$$

This ratio is $19/7(\beta)_0 = 2.714/(\beta)_0$ for $n = 0$, $32/3\pi(\beta)_1 = 3.395/(\beta)_1$ for $n = 1$ and $(3/\pi)^{1/2} R_3/\beta_3 = 6.740/\beta_3$ for $n = 3$. For $n = 3$, β is a constant throughout the polytrope and averaging is unnecessary. In the last form of equation (8) the relation $\Gamma_1 - 4/3 \approx \beta/6$ has been employed as a fair approximation in massive stars. Γ_1 is the first of the adiabatic coefficients defined in (17). The results for R_c/R_g are identical to those obtained in (11, 12, 13).

The early onset of instability can now be traced to the fact that R_c is inversely proportional to $(\beta)_n$ which is small for massive stars. Fowler and Hoyle (18) have shown in massive stars that

$$\beta \approx \frac{1}{\mu} \left[\frac{3}{4\pi} (n+1)^3 \frac{\mathfrak{R}^4}{aG^3} \right]^{1/2} \left(\frac{M_n}{M} \right)^{1/2} \rho_n^{(n-3)/4} \quad (9)$$

where μ is the mean molecular weight and the other symbols have the customary meanings. For a polytrope of index $n = 3$, β is constant throughout the polytrope and is given by

$$\beta \approx \frac{4.3}{\mu} \left(\frac{M_\odot}{M} \right)^{1/2} \quad (10)$$

This expression holds roughly for the average value throughout any polytrope and for hydrogen with $\mu = \frac{1}{2}$ yields $\beta \approx 10^{-3}$ in a polytrope with mass $M = 10^8 M_\odot$. The upshot is that R_c is several thousand times R_g for such a mass, the actual factor being sensitive to the polytropic index. *It is interesting to note that (5), (8) and (10) yield $E_{eq} \approx 2 M_\odot c^2 \sim 4 \times 10^{54}$ ergs at the minimum for all large masses.*

The onset of instability below the critical radius

$$R_c \sim 2.3 \times 10^5 (M/M_\odot)^{3/2} \text{ cm} \quad (n = 3, \mu = \frac{1}{2}) \quad (11)$$

can be understood in the following way. Consider an adiabatic compression at a point below the critical radius. Hydrostatic equilibrium after the perturbation requires more internal energy and pressure than before and since this is not made available in the adiabatic compression, further collapse ensues. Consider an adiabatic expansion. Now hydrostatic equilibrium requires less internal energy and pressure than given adiabatically so expansion continues. Clearly the radius at which E reaches a minimum is critical in this regard. At larger radii the decrease in the equilibrium energy as R decreases gives the well known classical stability. When an actual star reaches the critical radius it will lose energy by radiation and the general relativistic instability will lead to collapse rather than expansion unless some internal energy resource can be called upon.

Can nuclear energy supply the energy necessary to halt the general relativistic collapse and perhaps even reverse the motion by supplying more than that required by equation (5) for hydrostatic equilibrium? This is a problem still under attack but this much can be made clear. The central temperature and the central density at criticality can be shown (9, 10) to be relatively insensitive to the polytropic structure in contrast to the outer radius and are given by

$$T_c = 2.5 \times 10^{13} (M_\odot/M) \text{ degrees} \quad (12)$$

$$\rho_c = 2.0 \times 10^{18} (M_\odot/M)^{7/2} \text{ gm cm}^{-3} \quad (13)$$

It will be noted that the critical values are only $T_c = 2.5 \times 10^5$ degrees and $\rho_c = 2.0 \times 10^{-10}$ gm cm⁻³ for $M = 10^8 M_\odot$. The density is very small indeed but it will be recalled that the central density at the Schwarzschild radius for a polytrope of index 3 is only ~ 100 gm cm⁻³. The main point is that general relativistic considerations come into play in massive stars long before central temperatures and densities necessary for nuclear reactions to take place are reached. For hydrogen burning, $T \sim 8 \times 10^7$ degrees at $\rho \sim 10^{-2}$ gm cm⁻³ are required.

4. General Relativistic Gravitational Collapse

Nuclear energy or any form of energy must thus be generated during the collapse stage and the time scale for collapse becomes highly relevant in connection with generation rates per unit time. The hydrodynamic equation for the acceleration in the post-Newtonian approximation can be written

$$\frac{dv}{dt} = \frac{d^2r}{dt^2} \approx -\frac{1}{\rho} \frac{dp}{dr} - \frac{GM_r}{r^2} \left(1 + \frac{4GM_r}{rc^2} \right) \tag{14}$$

where M_r is the mass interior to r . The numerical coefficient of the post-Newtonian term is approximately correct only for the polytrope with index $n = 0$ (constant ρ) and then only in hydrostatic equilibrium. However, equation (14) is sufficiently accurate for our present purposes.

In classical free fall the pressure gradient in a star is set equal to zero and the acceleration is just that due to the gravitational forces. The increase in kinetic energy of fall can be readily computed from the change in the gravitational potential energy. Starting from rest at a radius large compared to R , the velocity of free fall at R is

$$v_{ff} \approx c \left(\frac{2GM}{Rc^2} \right)^{\frac{1}{2}} \tag{15}$$

and the characteristic e-folding time in R or $T_8 = T/10^8$ is

$$\begin{aligned} \tau_{ff} &\approx \frac{R}{c} \left(\frac{Rc^2}{2GM} \right)^{\frac{1}{2}} = 10^3 \left(\frac{R}{R_\odot} \right)^{3/2} \left(\frac{M_\odot}{M} \right)^{\frac{1}{2}} \text{ sec} \\ &\approx \frac{160}{(T_8/c)^{3/2}} \left(\frac{M}{M_\odot} \right)^{\frac{1}{2}} \text{ sec} \\ &\sim 2 \times 10^4 \text{ sec for } M = 10^8 M_\odot, (T_8)_c = 0.8 \text{ (H-burning)} \end{aligned} \tag{16}$$

It can be argued that the *gravitational collapse* is not free fall but arises from the post-Newtonian terms in the general relativistic expressions for the pressure gradient. The pressure gradient would just be balanced by the classical terms if general relativity were not taken into account and hence to order of magnitude the acceleration is equal to the post-Newtonian term. The kinetic energy per unit mass becomes equal to $\frac{1}{2}c^2 (2GM/Rc^2)^2$ not just $\frac{1}{2}c^2 (2GM/Rc^2)$ and so

$$v_{gc} \approx c \left(\frac{2GM}{Rc^2} \right)$$

Note that $v_{gc} \approx v_{ff} \approx c$ in the limit $2GM/Rc^2 = 1$.

The e-folding time is

$$\begin{aligned} \tau_{gc} &\approx \frac{R}{c} \left(\frac{Rc^2}{2GM} \right) = 5 \times 10^5 \left(\frac{R}{R_\odot} \right)^2 \left(\frac{M_\odot}{M} \right) \text{ sec} \\ &\approx \frac{4 \times 10^4}{(T_8/c)^2} \text{ sec independent of } M \\ &\sim 10^5 \text{ sec} \sim 1 \text{ day}, (T_8)_c = 0.8 \end{aligned} \tag{17}$$

We are reminded of the quotation from *The Lucky Chance* by Aphra Behn (1640–89): ‘Faith, sir, we are here to day, and gone to morrow.’ In spherically symmetric general relativistic collapse the time scale for the release of nuclear energy is very short and for $M > 10^7 M_\odot$ the collapse is probably not stopped. However, for $M < 10^7 M_\odot$ the nuclear resources would seem to be adequate to stop and reverse the collapse. Oscillations of the star then become possible if adequate modes of energy transmission to and emission from the surface are available. It can be shown that ordinary thermal mechanisms are grossly inadequate. Shock wave phenomena leading to the generation of high energy particles presumably come into play and may well lead to the excitation of the H II and radio-emitting regions surrounding the quasi-stellar objects. Detailed calculations are under way in Pasadena on these problems.

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DISCUSSION

H. Bondi. I want to stress and amplify a few of the points Fowler made.

1. The scale-independence of the relativistic equations is their most important characteristic, while nuclear reactions are strongly scale-dependent. There are accordingly some advantages in first considering the purely gravitational problem.

2. H. A. Buchdahl (*Phys. Rev.* **116**, 1027, 1959) showed that according to general relativity, the gravitational contraction of a mass M cannot yield more energy than Mc^2 , an important satisfactory result in view of the unbounded amount available according to Newtonian theory. If the energy emerges, say, in electromagnetic waves, whose emission leaves baryon number unchanged, then at a late stage of contraction the object has still its original number of baryons but they are virtually massless.

3. In a paper (shortly to be published in *Proc. R. Soc.*) I have set up the exact equations of contraction allowing for the emission of electromagnetic radiation, and have a number of results. In particular I have studied the case of slow adiabatic contraction (yielding a $p \sim \rho^{4/3}$ law in the

Newtonian case). The pressure-density dependence is steeper than this (and keeps on getting steeper) in the relativistic case, showing, in agreement with Fowler, that in contraction it gets progressively more difficult for materials to be stiff enough to resist gravitational collapse.

4. In another paper (soon to be published in *Proc. R. Soc.*) I have investigated static spherically symmetric models. Here the approach to the Schwarzschild singularity (gravitational potential $u = \frac{1}{2}$) is of special interest, and I have found limits for various cases. If the density is nowhere negative the limit is $u = 6\sqrt{2} - 8 = 0.485$, if the density exceeds three times the pressure and nowhere increases outwards, the limit is $u = 0.319$, etc.

W. A. Fowler. In regard to the energy available from gravitational collapse, Buchdahl is correct in principle. In practice it is necessary to describe a mechanism for the emission of the energy. Prof. Gell-Mann and I investigated the first order rate of emission of gravitational waves by a massive star which fissions under rotation and found that only 0.01 to $0.1 Mc^2$ could be emitted before the binary components merged in the ultimate Schwarzschild limit. This is only 10 times the well known yield from nuclear processes, namely 0.001 to $0.01 Mc^2$.

F. Zwicky. 1. I like to remark that the general theory of relativity on which Prof. Fowler bases his calculation is not necessarily correct. I am particularly concerned because (a) after long search I have never found one of the dense compact galaxies to act as a gravitational lens, and (b) there are no clusters of clusters of galaxies and the dispersion in velocities among the clusters is so small that a breakdown of Newton's law at distances of some tens of millions of parsecs is the most likely explanation.

2. What is still more serious, however, that no family of cosmic objects is known into which Prof. Fowler's superstars could be incorporated organically. A much more likely explanation for the powerful radio sources with compact stellar-like nuclei can be found in the view that these objects are 'pathological' individuals within the now well established family of the compact and extremely compact galaxies.

3. Finally it should be remarked that the term quasi-stellar radio source is a misnomer but only the radio sources involved are not stellar but only their visual nuclei are stellar. One should therefore talk about radio sources with compact visual nuclei and reserve the designation quasi-stellar radio sources to those objects which emit the radio waves themselves from a point-like region.

L. Gratton. I wish to report very shortly on the work which has been going on at our group in Roma. This is much on the same line as Prof. Fowler's and the results are similar.

Essentially a number of polytropic relativistic models have been integrated numerically, assuming the equation of state

$$\epsilon = 3P + QP^q$$

where ϵ is the energy per unit volume, P the pressure, Q and q are constants.

The cases considered are $q = 5/3$ and $q = 4/3$; Q is the parameter. The first case reproduces the well-known properties of neutron stars and need no further mention. The second case corresponds to the neglect of material pressure relative to radiation pressure and applies to stars of very large masses.

The interesting point is that if we follow the evolution of a mass of $10^8 M_{\odot}$ up to the point at which the central density is of the order of 0.3 gr cm^{-3} , through gravitational contraction from infinity (neglecting instability), it is found that the total energy radiated in space is of 4×10^{61} ergs. At this point the central temperature is of $6 \times 10^{80} \text{ K}$, so that nuclear process cannot be of importance; the radius is of $1.2 \times 10^{15} \text{ cm}$. With an assumed lifetime of 10^5 yr , the surface temperature is $300\,000^\circ \text{ K}$ and the absolute visual magnitude — 26, in remarkable agreement with the observed data for the quasi-stellar radio sources.