ON THE KAKEYA CONSTANT

F. CUNNINGHAM, JR., AND I. J. SCHOENBERG

We shall say that a plane set D has the *Kakeya property* if a unit segment can be turned continuously in D through 360° back to its original position. The famous solution of this problem by A. S. Besicovitch (**1**; **2**; **4**; **5**; **6**), to the effect that there are sets of arbitrarily small area having the Kayeka property, leaves open the problem obtained by adding the new condition that the set D be also *simply connected.* Since we do not know whether there is an attainable minimum, we define the Kakeya constant K to be the greatest lower bound of areas of simply connected sets having the Kakeya property. We shall refer to such sets as *Kakeya sets.*

It has long been known that $K \leqslant \frac{1}{8}\pi$, this being the area of a three-cusped hypocycloid inscribed in a circle of radius $\frac{3}{4}$. In 1952 R. J. Walker (**7**) determined by measurement the area of a certain set with a result that suggested that $K < \frac{1}{8}\pi$. In spite of its heuristic value Walker's note has not become well known. Independently of it, but using the same general idea, A. A. Blank (**3**) exhibited recently certain star-shaped polygons with the Kakeya property, having areas approaching $\frac{1}{8}\pi$, but not smaller than this value. Blank's examples suggested to each of us the possibility of finding Kakeya sets actually having areas smaller than $\frac{1}{8}\pi$, each such set giving an upper estimate for K. In the present note three different kinds of such sets are described. The first two (Part I) are due to Cunningham, the third (Part II) to Schoenberg. Each of these examples is self-contained and may be read independently. They also furnish progressively better estimates, the third example showing that

$$K \leqslant \frac{5 - 2\sqrt{2}}{24}\pi = (0.09048\ldots)\pi. \tag{1}$$

After completing this paper we were informed that Melvin Bloom has also found the estimate (1) by exactly the same construction as described in Part II. Since he obtained (1) several months earlier than Schoenberg, the priority belongs to Professor Bloom.

The second author wishes to record here his conjecture that the equality sign holds in the relation (1). In other words, he conjectures that the star-shaped domains $\mathfrak{A}_n$, described in Part II, play the role that the so-called Perron trees T_n play in Besicovitch's solution of the unrestricted Kakeya problem (**2**; **4**; **6**). Actually we do not even know whether $K > 0$.

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Part I

1. A modified hypocycloid. The most convenient analytic description of the three-cusped hypocycloid is parametric, using one complex variable as co-ordinate for the plane. We generate the hypocycloid Γ by rolling a circle of radius $\frac{1}{4}$ inside the fixed circle $|z| = \frac{3}{4}$. Taking the argument t of the centre of the moving circle as parameter, the motion of a point tracing Γ is

$$z = f(t) = \tfrac{1}{2}e^{it} + \tfrac{1}{4}e^{-2it}.$$

To see that the figure H consisting of Γ and its interior has the Kakeya property, consider for any t_0 the segment in H of the tangent to Γ at $f(t_0)$. If we set $t_1 = -\frac{1}{2}t_0$ and $t_2 = -\frac{1}{2}t_0 + \pi$, we find that

$$(1.1)\qquad \begin{aligned} f(t_1) - f(t_0) &= e^{-\frac{1}{2}it_0}\sin^2 \tfrac{3}{4}t_0,\\ f(t_2) - f(t_0) &= -e^{-\frac{1}{2}it_0}\cos^2 \tfrac{3}{4}t_0. \end{aligned}$$

Since $\arg f'(t_0) = -\frac{1}{2}t_0$, it is clear that $f(t_1)$ and $f(t_2)$ are the ends of the tangent segment, and from (1.1) it is also clear that $|f(t_1) - f(t_2)| = 1$.

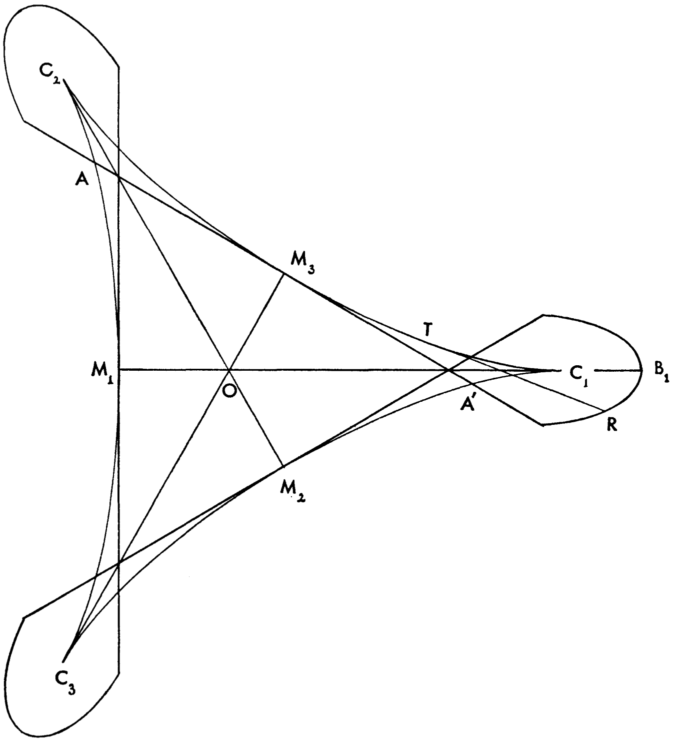

FIGURE 1

We shall construct a larger figure D (shown in Fig. 1) by moving a segment U of length $1 + \lambda > 1$, too long to turn in H. Let U be initially in the position

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$M_1 B_1$, where M_1 is $f(\pi) = -\frac{1}{4}$ and B_1 is $\frac{3}{4} + \lambda$. Move U so as to stay tangent to Γ, the point of tangency moving from the cusp C_1 along Γ to M_3, which is $f(\pi/3)$, the mid-point of $C_1 C_2$, while the end of U initially at M_1 moves along Γ from M_1 to A. Next slide U along the tangent line in which it lies until the lower end is on Γ at A', the upper end which was at A now projecting outside H. Continue the motion of the point of tangency from M_3 to the next cusp C_2, while moving the lower end of U along Γ from A' to M_2, which is $f(-\pi/3)$. The position of U is now symmetric to its original position, and it is clear how a succession of six such motions will turn U through 360° and bring it back to its original position. The figure D consists of all points encountered by U during the motion.

To compute the area of D, we add to the area of H, which is $\pi/8$, six times the extra area a required for the motion of U from its initial position until it is tangent to Γ at M_3. The area swept out by a moving tangent vector of length r, such as TR in Fig. 1, is

$$\frac{1}{2}\int_{\theta_1}^{\theta_2} r^2\, d\theta,$$

where θ is a variable angle giving its direction. Remembering that, for tangents to Γ, $d\theta = -\frac{1}{2}dt$, we get from (1.1)

$$a = \frac{1}{4}\int_0^{\pi/3} [(\sin^2 \tfrac{3}{4}t + \lambda)^2 - \sin^4 \tfrac{3}{4}t]dt = \frac{\pi}{12}\lambda^2 + \frac{\pi - 2}{12}\lambda.$$

The area of D is therefore $\frac{1}{8}\pi + 6a = \frac{1}{8}\pi + (\frac{1}{2}\pi - 1)\lambda + \frac{1}{2}\pi\lambda^2$.

The figure D is of course larger than necessary, since a segment of length $1 + \lambda$ can turn in it. We can reduce the figure by a contraction in the ratio of $1 : (1 + \lambda)^{-1}$ and still have the Kakeya property. After this reduction, the final area of the figure will be

$$A(\lambda) = [\tfrac{1}{8}\pi + (\tfrac{1}{2}\pi - 1)\lambda + \tfrac{1}{2}\pi\lambda^2](1 + \lambda)^{-2}.$$

The construction was for arbitrary $\lambda > 0$. We now choose λ so as to minimize $A(\lambda)$. Elementary calculus shows that $A(\lambda)$ is minimal when

$$\lambda = (4 - \pi)/(4 + 2\pi),$$

the minimum value being

$$(2\pi - 2)/(\pi + 8) = (0.1224\ldots)\pi < \tfrac{1}{8}\pi.$$

2. A modified star-shaped polygon. Let n be an odd integer $\geqslant 3$. Take n evenly spaced points on a circle of centre O, indexed $A_0, A_1, \ldots, A_n = A_0$ in such a way that each counterclockwise angle $A_i O A_{i+1} = \pi(1 + n)/n$; $i = 0, 1, \ldots, n - 1$ (see Fig. 2). The radius of the circle is to be such that $A_i A_{i+1} = 1$. Let S_n be the closed star-shaped region whose boundary is a $2n$-gon with $A_1, \ldots, A_n$ among its vertices, and whose sides are segments of the lines joining consecutive pairs A_i, A_{i+1}. Denote by B_i the (interior) vertex opposite A_i.

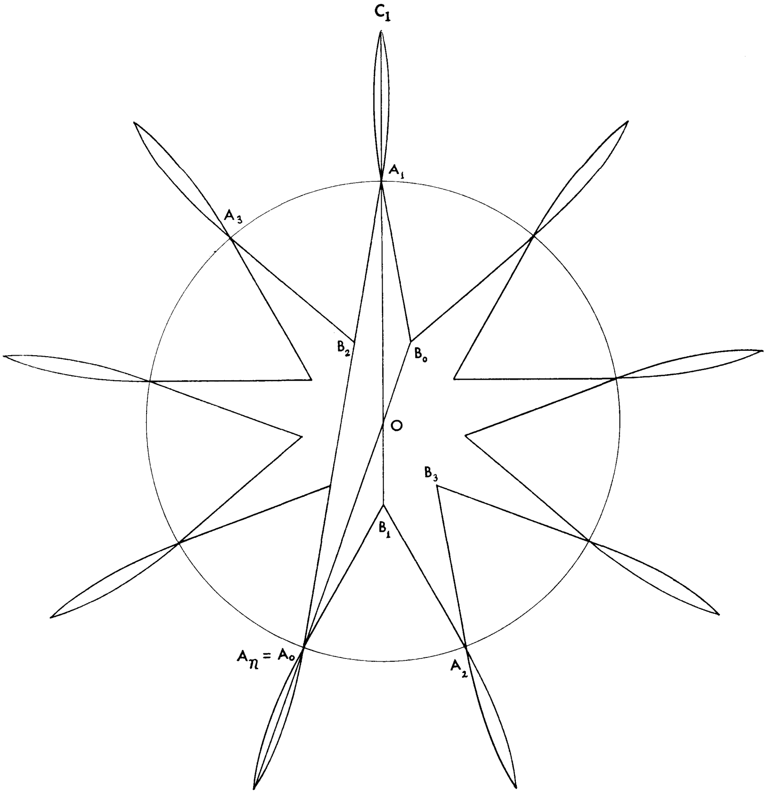

FIGURE 2

Since S_n does not have the Kakeya property, we shall enlarge it slightly. Let a unit segment U be placed in the position $A_0 A_1$ and then moved in the following way. The end of U initially at A_0 is moved along the boundary of S_n from A_0 to B_1 and thence to A_2, while U is constrained to pass through A_1 during the motion. The other end of U starts at A_1, describes an arc of a conchoid $A_1 C_1$ outside the circle, and returns to A_1 along another such arc, symmetric to the first. The over-all result of the motion is to transport U from position $A_0 A_1$ to position $A_2 A_1$. The motion is now imitated with A_0, A_1, A_2 replaced by A_1, A_2, A_3 respectively (the roles of the ends of U being interchanged), and so on, until $2n$ such motions have brought U back to its original position. The set D_n required for the motion is the union of S_n and n lobes bounded by conchoid arcs.

The area of D_n can be computed exactly by elementary means, but we are only interested in the limit as $n \to \infty$. We can obtain Kakeya sets of area as close to this limit as we please by taking n large enough. From the figure, read off the following angles: $\angle A_0 OB_1 = \pi/n$, $OA_0 B_1 = \pi/2n$, $\angle A_0 B_1 O = \pi - (3\pi/2n)$. Since $A_0 A_1 = 1$, the radius of the circle tends to $\frac{1}{2}$, and the law of sines applied to triangle $A_0 OB_1$ gives that OB_1 tends to $\frac{1}{6}$. The limit

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of the area of S_n can now be read off as the sum of a circle of radius $\frac{1}{6}$ (representing the central portion) and n sectors of radius $\frac{1}{3}$ and angle π/n (representing the points of the star). Thus the area of S_n tends to $\pi/12$.

For the lobe at A_0 we use polar co-ordinates with A_0 as origin. The line segment $B_0 A_1$ will have a polar equation

$$r = a \csc\left(\frac{3\pi}{2n} - \theta\right)$$

with θ variable from 0 to $\pi/2n$, and the value of $a = \sin(\pi/n)$ determined by the polar co-ordinates of A_1, which are $\theta = \pi/2n$, $r = 1$. As one end P of the moving segment U goes from B_1 to A_1 along this line, the other end describes half the boundary of the lobe. Multiplying the integral for the area of this half-lobe by $2n$, we get for the area of all lobes

$$n \int_0^{\pi/2n} \left[1 - \sin\frac{\pi}{n} \csc\left(\frac{3\pi}{2n} - \theta\right)\right]^2 d\theta$$
$$= \frac{1}{2}\pi \int_0^1 \left[1 - \sin\frac{\pi}{n} \csc\left(\frac{3\pi}{2n} - \frac{\pi x}{2n}\right)\right]^2 dx.$$

where the second integral comes from the first by the substitution $2n\theta = \pi x$. Now as $n \to \infty$, the integrand in the second integral converges to $[1 - 2(3 - x)^{-1}]^2$ uniformly on $0 \leqslant x \leqslant 1$. The limit integral is easily found to equal $\pi(\frac{5}{6} - 2 \log \frac{3}{2})$, giving for the total area of D_n in the limit

$$\pi\left(\frac{11}{12} - 2 \log\frac{3}{2}\right) = (0.1057\ldots)\pi < \frac{\pi}{8}.$$

Part II.

3. The main result. Here we wish to prove that

$$K \leqslant \frac{5 - 2\sqrt{2}}{24}\pi = (0.09048\ldots)\pi. \tag{3.1}$$

The proof is based on the following construction: Let n be an *odd* integer $\geqslant 3$. Take n evenly spaced points on a circle of radius unity and centre O, indexed $A_0, A_1, \ldots, A_n = A_0$, in such a way that each counterclockwise angle $A_i O A_{i+1} = \pi(n - 1)/n$, $(i = 0, \ldots, n - 1)$ (Fig. 3). We draw the n radii OA_i and the n circular arcs

$$\Gamma_i = A_i A_{i+1} \qquad (i = 0, \ldots, n - 1),$$

such that Γ_i is tangent to the radii OA_i and OA_{i+1} at the points A_i and A_{i+1}, respectively. These n arcs form a star-shaped figure and we denote by $\mathfrak{A}_n$ the least simply connected domain containing it. (For large n (of the order of 10^4) the domain $\mathfrak{A}_n$ looks like the curve traced out on the floor by the tip of a Foucault pendulum swinging for 12 hours.)

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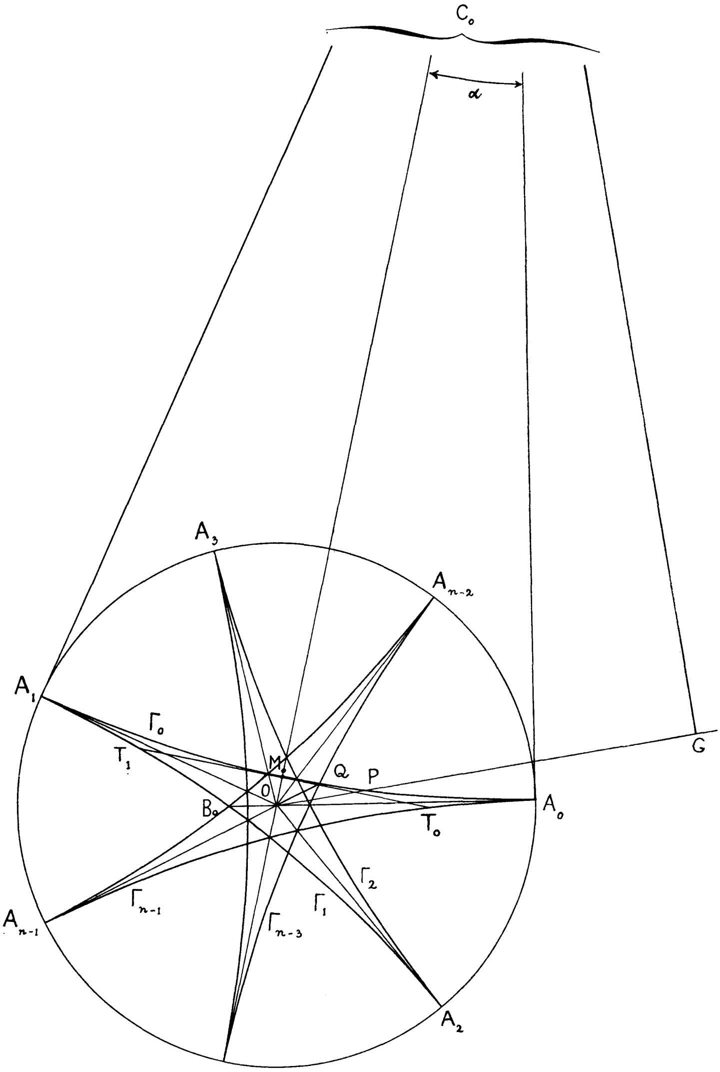

FIGURE 3

Let the variable point P start from A_0 and trace out continuously the arcs $\Gamma_0, \Gamma_1, \ldots, \Gamma_{n-1}$, until it returns to A_0. We draw the tangent, at P, to the arc Γ_i on which it lies and denote by t the segment which is the intersection of this tangent with the domain $\mathfrak{A}_n$. Evidently t turns through an angle $= \pi/n$ as P moves from A_0 to A_1 and similarly on every arc Γ_i. Thus t performs a continuous rotation of total angle π as P traces all of $\Gamma = \bigcup \Gamma_i$. This motion of t will also be the motion of the segment U as required by Kakeya's problem; by this we mean that we require that in all positions of P we have that $U \subset t$.

This last requirement evidently needs some information on the way the length of t varies as P describes Γ. Let

$$(P) = \text{length of } t.$$

We state three lemmas, to be established in the next section, which will readily imply the inequality (3.1).

LEMMA 1. *As P describes the arc $\Gamma_0 = A_0 A_1$, (P) reaches its maximum, namely $A_0 B_0$, when $P = A_0$; from there (P) decreases until P reaches the mid-point M_0 of Γ_0, when $(P) = (M_0) = T_0 T_1$; from there it increases to $(A_1) = (A_0)$.*

This lemma shows that

$$\min_{P \epsilon \Gamma} (P) = (M_0). \tag{3.2}$$

The behaviour of this minimum as $n \to \infty$ is described by

LEMMA 2. *As $n \to \infty$*

$$\lim (M_0) = 2(2 - \sqrt{2}) = 1.17158\ldots. \tag{3.3}$$

Finally, we need

LEMMA 3. *If we denote by $|\mathfrak{A}_n|$ the area of the domain $\mathfrak{A}_n$, then*

$$\lim_{n\to\infty} |\mathfrak{A}_n| = \frac{23 - 16\sqrt{2}}{3}\,\pi = (0.12421\ldots)\pi. \tag{3.4}$$

A proof of (3.1) is now easy. Let $\epsilon > 0$. By Lemma 3 we know that

$$|\mathfrak{A}_n| < \frac{23 - 16\sqrt{2}}{3}\,\pi + \eta \qquad (\eta > 0, \text{given}),$$

provided that $n > N(\eta)$. Let us write $l = (M_0)$ for the right side of (3.2). By Lemma 1 we know that we can turn within $\mathfrak{A}_n$ a segment of length l. We now contract $\mathfrak{A}_n$ in the linear ratio $l:1$, obtaining a domain $\mathfrak{A}_n^*$ within which we can surely turn a unit segment U and whose area satisfies the inequality

$$|\mathfrak{A}_n^*| = |\mathfrak{A}_n| l^{-2} < \frac{23 - 16\sqrt{2}}{3l^2}\,\pi + \eta l^{-2}. \tag{3.5}$$

By Lemma 2, $l \to 4 - 2\sqrt{2}$. Noting that

$$\frac{23 - 16\sqrt{2}}{3(4 - 2\sqrt{2})^2} = \frac{5 - 2\sqrt{2}}{24},$$

we conclude from (3.5) that

$$|\mathfrak{A}_n^*| < \frac{5 - 2\sqrt{2}}{24}\,\pi + \epsilon,$$

provided that η is appropriately chosen and that n is sufficiently large. Since

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ϵ is arbitrary, the last inequality evidently implies the inequality (3.1) and our proof is complete.

For further orientation we also state.

LEMMA 4. *As* $n \to \infty$,

$$\lim (A_0) = 2(2 - \sqrt{2}). \tag{3.6}$$

On comparing (3.3) with (3.6) we see that the oscillation of (P)

$$\max_{P\epsilon\Gamma} (P) - \min_{P\epsilon\Gamma} (P) \to 0, \qquad \text{as } n \to \infty.$$

We therefore see that if n is sufficiently large, the domain $\mathfrak{A}_n^*$ has interior tangents t^* which all differ in length from unity by as little as we wish. This remark is meant to give some weight to the conjecture that the equality holds in (3.1). Actually we do not even know if $K > 0$.

4. A proof of Lemma 1. In Figure 4, we denote by C_i the centre of the arc Γ_i. We also set $C_0 A_0 = C_0 A_1 = A_1 C_1 = A_0 C_{n-1} = a$, $\angle A_0 C_0 A_1 = 2\alpha < \pi/2$. Let $l = T_0 T_1$ and more specifically

$$l_0 = PT_0, \qquad l_1 = PT_1, \qquad \phi_0 = \angle PT_0 N_0, \qquad \phi_1 = \angle PT_1 N_1,$$

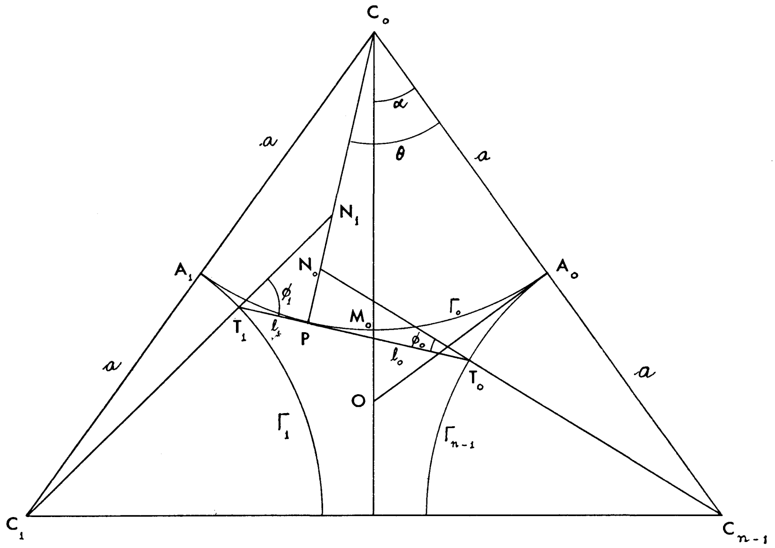

FIGURE 4

where N_0 and N_1 are the intersections of the radius $C_0 P$ with the lines $C_{n-1} T_0$ and $C_1 T_1$, respectively. Finally, let the position of P be determined by the angle

$$\theta = \angle A_0 C_0 P$$

so that $T_0 T_1 = l(\theta) = l$ becomes a function of θ. We find that

(4.1) $$dl/d\theta = l_1 \tan \phi_1 - l_0 \tan \phi_0 = PN_1 - PN_0 = C_0 N_0 - C_0 N_1.$$

By trigonometry, $C_0 N_0 = f(\theta)$, where

$$f(\theta) = a(1 - 2\cos\theta)(1 + \sec\theta)^{\frac{1}{2}} + 2a\cos\theta.$$

We claim that $f(\theta)$ is an increasing function of θ in the range $0 \leqslant \theta < \pi/2$. This becomes clear if we set $x = \cos\theta$ and observe that

$$a^{-1}f(\theta) = (1 + x^{-1})^{\frac{1}{2}} + (-2)\{1 + (1 + x^{-1})^{\frac{1}{2}}\}^{-1}$$

is a *decreasing* function of x in the range $0 < x \leqslant 1$, in fact even in the range $0 < x < \infty$, because both terms on the right side are decreasing functions. From (4.1) and the symmetry of the figure we find that

$$dl/d\theta = f(\theta) - f(2\alpha - \theta).$$

Now, if $0 < \theta < \alpha$, then $\theta < 2\alpha - \theta$; hence $f(\theta) < f(2\alpha - \theta)$ and $dl/d\theta < 0$. On the contrary, $\alpha < \theta < 2\alpha$ implies that $\theta > 2\alpha - \theta, f(\theta) > f(2\alpha - \theta)$; hence $dl/d\theta > 0$. Evidently l has a strict minimum for $\theta = \alpha$.

5. A proof of Lemma 2. We refer again to Figure 4, where we now assume that the parameters a and α are connected by the relation

(5.1) $$a \tan\alpha = OA_0 = 1.$$

Moreover, we now assume in Figure 4 that $\theta = \alpha$, hence that P is at M_0. We let $\alpha \to 0$ and wish to show that $T_0 T_1 \to 2(2 - \sqrt{2})$.

Since (Section 4)

$$C_0 N_0 = f(\alpha) = a(1 - 2\cos\alpha)(1 + \sec\alpha)^{\frac{1}{2}} + 2a\cos\alpha,$$

we conclude that $C_0 N_0/a \to 2 - \sqrt{2}$. But then $C_{n-1} N_0/a \to 2 - (2 - \sqrt{2}) = \sqrt{2}$. Setting $\beta = \angle C_0 C_{n-1} N_0$, the law of sines gives

$$\frac{\beta}{\alpha} \to \frac{2 - \sqrt{2}}{\sqrt{2}} = \sqrt{2} - 1.$$

Now $\angle T_0 N_0 P = \alpha + \beta$, and by (5.1) we obtain

$$\begin{aligned} T_0T_1 &= 2T_0 P = 2N_0 P \tan(\alpha + \beta) = 2a(1 - a^{-1}f(\alpha))\tan(\alpha + \beta) \\ &= 2(1 - a^{-1}f(\alpha))\frac{\tan(\alpha + \beta)}{\tan\alpha} \to 2(1 - (2 - \sqrt{2}))(1 + (\sqrt{2} - 1)) \\ &= 2(2 - \sqrt{2}), \end{aligned}$$

which was to be proved.

6. Proofs of Lemmas 3 and 4. We wish to determine $\lim|\mathfrak{A}_n|$ as $n \to \infty$. We refer to Figure 3 and start by determining the vertex A_i such that

$\angle A_0 OA_i = 2\pi/n$. We easily see that this must be the vertex A_{n-2} and that the arcs Γ_{n-3} and Γ_0 are symmetric to each other with respect to the bisector of the angle $A_0 OA_{n-2}$. Let Q be their intersection:

$$Q = \Gamma_0 \cap \Gamma_{n-3}.$$

We draw the segment OQ and observe that in view of the n-fold rotational symmetry of the domain $\mathfrak{A}_n$ we have

$$|\mathfrak{A}_n| = 2n \cdot \text{Area } OA_0 Q, \tag{6.1}$$

where the area $OA_0 Q$ is bounded by the segments OA_0, OQ, and the portion A_0Q of the arc Γ_0. We proceed to determine this area by means of integration in polar co-ordinates with origin at O.

Let P be a point on the arc $A_0 Q$ and let

$$\rho = OP, \qquad \theta = \angle A_0 OP.$$

From the diagram

$$\angle A_0 OC_0 = \tfrac{1}{2}(\angle A_0 OA_1) = \frac{1}{2}\frac{n-1}{2}\frac{2\pi}{n} = \frac{\pi}{2} - \alpha, \tag{6.2}$$

where we set

$$\alpha = \angle OC_0 A_0 = \pi/2n. \tag{6.3}$$

Further useful facts are

$$\angle A_0 OQ = \pi/n = 2\alpha, \qquad OC_0 = 1/\sin \alpha. \tag{6.4}$$

We project C_0 onto OP into G and find by (6.2) that $\angle POC_0 = \frac{1}{2}\pi - \alpha - \theta$ and therefore by (6.4) that

$$OG = \cos(\tfrac{1}{2}\pi - \alpha - \theta)/\sin \alpha = \sin(\alpha + \theta)/\sin \alpha.$$

But then we see that $\rho = OP$ is the smaller root of the quadratic equation

$$x^2 - 2\frac{\sin(\alpha + \theta)}{\sin \alpha} x + 1 = 0. \tag{6.5}$$

Solving for ρ, we obtain

$$\rho(\theta) = \frac{\sin(\alpha + \theta)}{\sin \alpha} - \left\{\left(\frac{\sin(\alpha + \theta)}{\sin \alpha}\right)^2 - 1\right\}^{\frac{1}{2}}. \tag{6.6}$$

In view of the first relation (6.4), we obtain

$$\text{Area } OA_0 Q = \frac{1}{2}\int_0^{2\alpha} (\rho(\theta))^2 d\theta$$

and (6.1) shows that we need the limit of the expression

$$|\mathfrak{A}_n| = n\int_0^{2\alpha} (\rho(\theta))^2 d\theta \tag{6.7}$$

as $n \to \infty$. We change variables by setting $2n\theta = \pi x$ so that (6.7) becomes

$$|\mathfrak{A}_n| = \frac{\pi}{2} \int_0^2 \rho^2 \left(\frac{\pi x}{2n}\right) dx. \tag{6.8}$$

However, as $n \to \infty$,

$$\frac{\sin(\alpha + \theta)}{\sin \alpha} = \sin\left(\frac{\pi}{2n} + \frac{\pi x}{2n}\right) \Big/ \sin\frac{\pi}{2n} \to x + 1,$$

uniformly in x, so that (6.6) and (6.8) yield

$$\lim |\mathfrak{A}_n| = \frac{\pi}{2} \int_0^2 \{x + 1 - \sqrt{[(x+1)^2 - 1]}\}^2 dx.$$

This is an elementary integral which is easily evaluated and seen to equal the right-hand side of (3.4).

A proof of Lemma 4 is immediate: Evidently $OQ = \rho(2\alpha)$ and from (6.6) we find that $\lim OQ = 3 - 2\sqrt{2}$ as $n \to \infty$ and therefore $\alpha \to 0$. But then $\lim(A_0) = 1 + \lim OQ = 4 - 2\sqrt{2}$ and Lemma 4 is established.

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Bryn Mawr College,
The University of Pennsylvania, and
Mathematics Research Center,
U.S. Army, University of Wisconsin