ON SPECTRAL PROPERTIES OF MATRICES WITH POSITIVE CHARACTERISTIC VECTORS

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1. Introduction and terminology. Unless stated otherwise, all our matrices (denoted by capital letters) are square matrices of size $n \times n$ and composed of real numbers. A' denotes the transpose of A. The characteristic or eigenvectors of matrices are written as column vectors having n coordinates. If ξ is a vector, ξ' denotes its transpose.

A matrix $A = (a_{ij})$ is said to be positive (non-negative) if $a_{ij} > 0$ $(a_{ij} \geqq 0)$ for $i, j = 1, 2, \ldots, n$. We write $A < B$ $(A \leqq B)$ if and only if $B - A$ is positive (non-negative); similarly for column vectors.

Any positive characteristic vector ξ (if there be any) of a matrix A will be called a Perron vector of A, $A\xi = \omega\xi$, ω being the characteristic root or eigenvalue corresponding to ξ. If $\xi_1, \xi_2, \ldots, \xi_k$ are all the Perron vectors of A with corresponding characteristic roots $\omega_1, \omega_2, \ldots, \omega_k$, then $\omega = \max(\omega_1, \omega_2, \ldots, \omega_k)$ will be called the Perron root of A.

A matrix $A = (a_{ij})$ is said to be reducible if the set $N = \{1, 2, \ldots, n\}$ can be split into two disjoint sets N_1, N_2, $N_1 \cup N_2 = N$, $N_1 \cap N_2 = \emptyset$ such that $a_{ij} = 0$ whenever $i \in N_1, j \in N_2$. Equivalently, A is reducible if there exists a permutation matrix R such that

$$R'AR = \begin{bmatrix} B & C \\ O & D \end{bmatrix},$$

where B and D are square submatrices and O is a null rectangular matrix.

By a Frobenius matrix we shall mean a square irreducible matrix composed of real numbers, all of whose off-diagonal elements are non-negative. Such matrices arise naturally in many applications of matrix theory, e.g., in econometry. The following well-known theorem (**4**, p. 53, Theorem 1), reproved in § 3 as a particular case, was obtained by Frobenius as an extension of a remarkable result of Perron relating to positive matrices.

THEOREM. *A non-negative irreducible matrix has just one positive characteristic vector corresponding to a characteristic root which is positive, simple, and the moduli of all other characteristic roots do not exceed that root. (In case the matrix is positive, these moduli are less than that root.)*

A matrix A will be called polynomial-positive if there is a polynomial $c_0 I + c_1 A + c_2 A^2 + \ldots + c_m A^m$, with c_i real, such that $c_0 I + c_1 A + \ldots + c_m A^m > O$. Here, I denotes the identity matrix and O the null matrix.

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For a matrix $A = (a_{ij})$ we write r_i for the sum $a_{i1} + a_{i2} + \ldots + a_{in}$ of the elements in the ith row of A. Likewise, $c_j = a_{1j} + a_{2j} + \ldots + a_{nj}$. Define the row range of A by the difference of $\max(r_1, r_2, \ldots, r_n)$ and $\min(r_1, r_2, \ldots, r_n)$ (likewise, column range of A is defined).

In the following, we use the usual topologies of the real line and Euclidean (metric) spaces.

2. Perron vectors of matrices commuting with positive matrices. Let A be a matrix which commutes with a positive matrix P, i.e., $AP = PA$. In the following theorem we show that A has a Perron vector.

THEOREM 1. *If A commutes with a positive matrix P, then A has at least one positive characteristic vector.*

Proof. Let S be the set of all non-negative column vectors ξ, $\xi' = (x_1, x_2, \ldots, x_n)$ with $x_i \geqq 0$ and $\sum x_i = 1$. Then S is a closed bounded set. All Perron vectors of A belong to S. For each ξ of S define $\rho(\xi)$ to be the maximum possible value of ρ such that

$$\rho(\xi) \leqq A\xi. \tag{2.1}$$

Note that $\rho\xi = \min_i(a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n)x_i^{-1}$. If $x_i = 0$, interpret this quantity as $-\infty$ or 0 according as

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n < 0 \text{ or } \geqq 0.$$

Let T be the subset (of the extended real number system) consisting of all $\rho(\xi)$ as ξ varies over S. Note that T contains finite numbers, e.g., $\rho(\xi)$, where $\xi = (n^{-1}, n^{-1}, \ldots, n^{-1})$ is finite. From (2.1), using the fact that $\sum x_i = 1$, we obtain

$$\rho(\xi) \leqq c_1x_1 + c_2x_2 + \ldots + c_nx_n \leqq \max(c_1, c_2, \ldots, c_n), \tag{2.2}$$

where c_i is the ith column sum of A. Thus, T is bounded above. Denote the supremum of T by ω, ω a finite number.

Let $\xi_1, \xi_2, \ldots$ be a sequence of vectors in S such that $\rho(\xi_n) \to \omega$ as $n \to \infty$. As S is a closed bounded set, this sequence determines, by the Bolzano-Weierstrass Theorem, at least one limiting vector ξ, $\xi \in S$. Therefore, from the original sequence $\xi_1, \xi_2, \ldots$ we can pick a subsequence $\xi_{m_1}, \xi_{m_2}, \ldots$ which converges to ξ. Denote ξ_{m_n} by $\hat{\xi}_n$. Then

$$\lim_{n\to\infty} \hat{\xi}_n = \xi \quad \text{and} \quad \lim_{n\to\infty} \rho(\hat{\xi}_n) = \omega, \tag{2.3}$$

$$\rho(\hat{\xi}_n)\hat{\xi}_n \leqq A\hat{\xi}_n. \tag{2.4}$$

Letting $n \to \infty$ in (2.4) we easily obtain

$$\omega\xi \leqq A\xi \quad \text{or} \quad A\xi - \omega\xi \geqq 0, \tag{2.5}$$

where 0 denotes the null vector.

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We now show that equality must hold in (2.5). If $A\xi - \omega\xi = \eta$, where $\eta' = (p_1, p_2, \ldots, p_n)$ and at least one $p_i \neq 0$, then $P\eta$ is a positive column vector and $P\eta = PA\xi - \omega P\xi = A(P\xi) - \omega(P\xi)$. Thus, $\omega\zeta < A\zeta$, where ζ is a positive vector in S and $\zeta = c^{-1}P\xi$, c denoting the sum of the coordinates of the vector $P\xi$. Thus, for a suitably small $\epsilon > 0$, we would have $(\omega + \epsilon)\zeta < A\zeta$. But this would contradict the fact that ω is the supremum of S. Hence, equality must hold in (2.5) and ξ is a Perron vector of A, $A\xi = \omega\xi$. Note that

$$\begin{aligned}\omega &= \sup_S \min_i \,(a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n)x_i^{-1} \\ &= \max_S \min_i \,(a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n)x_i^{-1},\end{aligned}$$

and, clearly, ω is the Perron root of A. This max-min property characterizes the Perron root of A.

COROLLARY 1. *If A satisfies the hypothesis of Theorem 1 and if there is a non-negative vector ξ such that the vector $A\xi$ is non-negative, then the Perron root of A is non-negative.*

Since the set T contains a non-negative number, sup T is non-negative, and the corollary follows easily.

THEOREM 2. *If $A \leqq B$, $A \neq B$, and A and B commute with the positive matrices P and Q, respectively, then the Perron root of A is less than the Perron root of B.*

Proof. By Theorem 1, A and B have Perron roots α and β, respectively. Let ξ be a Perron vector of A corresponding to the characteristic root α, $A\xi = \alpha\xi$.

Remembering that $B - A$ has at least one positive element, we infer that $B\xi - \alpha\xi = (B - A)\xi$ is a non-negative vector with at least one positive coordinate. Thus, $\beta(Q\xi) - \alpha(Q\xi) = Q(B\xi - \alpha\xi)$ is a positive vector. Therefore there is a positive vector η in S (here S is as in Theorem 1) such that $B\eta > \alpha\eta$. Using the max-min property of Perron roots, we infer that $\alpha < \beta$.

THEOREM 3. *If A and A' both have positive characteristic vectors, then all the positive characteristic vectors of A correspond to the same characteristic root of A (namely, the Perron root of A); further, the Perron root of A is the Perron root of A'.*

Proof. Let $\xi_1, \xi_2, \ldots, \xi_k$ be all the Perron vectors of A corresponding to the characteristic roots $\omega_1, \omega_2, \ldots, \omega_k$; $A\xi_i = \omega_i\xi_i$, $i = 1, 2, \ldots, k$. Suppose that ω is the Perron root of A' and η is a corresponding Perron vector, $A'\eta = \omega\eta$. Then $\eta'A\xi_i = \omega_i\eta'\xi_i$, i.e., $\omega\eta'\xi_i = \omega_i\eta'\xi_i$. Since $\eta'\xi_i > 0$, we must have that $\omega = \omega_i$, $i = 1, 2, \ldots, k$.

THEOREM 4. *If A commutes with a positive matrix P, then all the Perron vectors of A correspond to the Perron root of A and the Perron root of A is the Perron root of A'.*

Proof. A' commutes with P'; thus, by Theorem 1, A' has a Perron vector. Now apply Theorem 3.

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It is possible for a matrix A of Theorem 4 to have several Perron vectors. See the numerical example in § 4.

3. Perron vector of polynomial positive matrices. Brauer (**1**; **2**) has studied power-positive matrices. These are very special types of polynomial-positive matrices which we study in this section. First we give a very simple generalization of a result due to Wielandt (**4**).

THEOREM 5. *Every Frobenius matrix is polynomial-positive.*

Proof. Let A be a Frobenius matrix. Choose a suitably large constant c such that all the diagonal elements of the matrix $cI + A$ are positive. We shall now show that $(cI + A)^{n-1}$ is a positive matrix. It is sufficient to show that for any non-negative vector ξ, $(cI + A)^{n-1}\xi$ is a positive vector.

Clearly, if a coordinate of ξ is positive, then the corresponding coordinate of η, where $\eta = (cI + A)\xi$ is also positive. We now establish that the number of zero coordinates of η cannot equal the number of zero coordinates of ξ (provided it has any). If possible, let these be equal. Without loss of generality (by renumbering the coordinates of ξ and analogously in η, if necessary), we may suppose that $\xi' = (x_1, x_2, \ldots, x_r, 0, 0, \ldots, 0)$, $\eta' = (y_1, y_2, \ldots, y_r, 0, 0, \ldots, 0)$ with $x_i > 0$, $y_i > 0$, $i = 1, 2, \ldots, r$, and

$$\text{(3.1)} \qquad \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_r \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} = (cI + A) \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_r \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \\ \cdot \\ \cdot \\ \cdot \\ cx_r \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} + A \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_r \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}.$$

Setting

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where A_{12} and A_{21} are $r \times (n - r)$ and $(n - r) \times r$ non-negative matrices, respectively, we have, from (3.1), that

$$A_{21} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_r \end{bmatrix} = 0 \quad \text{with } x_i > 0$$

and, therefore, A_{21} is an $(n - r) \times r$ null matrix, contradicting the irreducibility of A. Thus, if ξ has some zero coordinates, η has fewer of them. As $(cI + A)^{n-1}\xi = (cI + A)\ (cI + A) \ldots (cI + A)\xi$, this vector has no zero coordinate (i.e., it is a positive vector). This proves the theorem.

Holladay and Varga (**5**) have shown that A^{2n-2} is positive if A is non-negative irreducible and if at least one diagonal element of A is positive.

THEOREM 6. *Every polynomial-positive matrix has a positive characteristic vector corresponding to its Perron root; there is no other characteristic vector corresponding to this root. The Perron root is a simple root of the characteristic equation of the matrix.*

Proof. Let A be a polynomial-positive matrix and let $P = f(A)$ be a positive matrix, where $f(x) = c_0 + c_1x + c_2x^2 + \ldots + c_mx^m$, c_i real.

As A commutes with the positive matrix P, by Theorem 1, A has a Perron vector (and thus a Perron root ω). Let ξ be a Perron vector corresponding to ω, $A\xi = \omega\xi$. If possible, let there be another (real) characteristic vector ξ_1 corresponding to ω, $A\xi_1 = \omega\xi_1$. Let $\xi_2 = c\xi + \xi_1$, where the constant c has been so chosen that ξ_2 has at least one zero coordinate and the rest of the coordinates non-negative. Then

$$A\xi_2 = A(c\xi + \xi_1) = \omega\xi_2.$$

Now $P\xi_2 = f(A)\xi_2 = f(\omega)\xi_2$. Therefore, if a coordinate of ξ_2 is 0, the corresponding coordinate of the vector $P\xi_2$ is also 0. But this is impossible, since $\xi_2 \geqq 0$, and therefore $P\xi_2 > 0$. Thus, A has only one characteristic vector corresponding to the Perron root ω.

We now show that ω is a simple root of the characteristic equation of A. If possible, let it be a multiple root. We shall use Schur's canonical form for matrices. We know that there is a unitary matrix U (i.e., with $U^*U = I$) such that U^*AU is an upper triangular matrix (with all its elements below the main diagonal equal to 0). We may assume that the first column vector of U is ξ, the unique Perron vector of A. We may also assume that the second column vector, say γ_1, is such that the first two diagonal elements of U^*AU are ω; this is possible as ω has been supposed to be a multiple root. It is easily deduced that

$$A\xi = \omega\xi, \tag{3.2}$$

$$A\gamma_1 = e_1\xi + \omega\gamma_1. \tag{3.3}$$

If $\gamma_1 = \gamma + i\gamma_2$ and $e_1 = e + ie_2$, where $i = \sqrt{-1}$, γ_1 and γ_2 are real vectors, and e_1 and e_2 real numbers, (3.3) yields

$$A\gamma = e\xi + \omega\gamma. \tag{3.4}$$

Note that $e \neq 0$, otherwise we would get two linearly independent vectors ξ and γ corresponding to the Perron root ω. (Linear independence of ξ and γ follows from the linear independence of ξ and γ_1 in U.)

Choose a constant b such that $be > 0$ and a suitably large constant a such that the vector $\zeta = a\xi + b\gamma$ is positive; furthermore, we may assume that ζ belongs to the set S (of the proof of Theorem 1). Then we have that $A\zeta = A(a\xi + b\gamma) = a\omega\xi + b(e\xi + \omega\gamma) = \omega\zeta + be\xi$ by (3.2) and (3.4). Since $be > 0$ and $\xi > 0$, we now have that $\omega\zeta < A\zeta$. Hence, $\omega < \rho(\zeta)$ for $\zeta \in S$ (in the notation given in the proof of Theorem 1). But ω, being the Perron root of A, is equal to $\max_S \rho(\xi)$ as shown in Theorem 1. Thus, we have arrived at a contradiction. Consequently, ω is a simple root of this characteristic equation of A.

COROLLARY. *If $A = (a_{ij})$ is a Frobenius matrix, then A has a unique Perron vector with a simple Perron root, ω.*

Furthermore, if A is non-negative, then the moduli of other characteristic roots of A cannot exceed the Perron root (clearly positive) ω. If A is positive, these moduli are less than ω.

For, if $A\zeta = \lambda\zeta$, $\lambda \neq \omega$, $\zeta' = (z_1, z_2, \ldots, z_n)$, then ζ cannot be a Perron vector, and thus not all z_i are positive. Furthermore,

$$|\lambda|\ |z_i| = \left|\sum_j a_{ij}z_j\right| \leqq \sum_j a_{ij}|z_j|, \qquad i = 1, 2, \ldots, n.$$

Recalling the max-min property for the Perron root, we have that $|\lambda| \leqq \omega$. If $a_{ij} > 0$ and since not all z_i are positive, we obtain $|\lambda|\ |z_i| < \sum_j a_{ij}|z_j|$, and therefore $|\lambda| < \omega$. Thus, we have proved the theorem stated in § 1.

4. Common characteristic vectors of commuting matrices. In this section we give a few simple results relating to commuting matrices.

THEOREM 7. *If U and V are two matrices (composed of complex numbers) and $UV = VU$ and if μ and ν are any given characteristic roots of U and V, respectively, then there is a common characteristic vector ζ such that $U\zeta = \mu\zeta$ and $V\zeta = \nu\zeta$.*

Proof. Let η be a characteristic vector of U corresponding to the characteristic root μ; $U\eta = \mu\eta$. η, $V\eta$, $V^2\eta$, $\ldots$, $V^m\eta$, $\ldots$ cannot be linearly independent (in the field of complex numbers). Suppose that η, $V\eta$, $V^2\eta, \ldots,$ $V^m\eta$ are linearly dependent such that $\eta + c_1V\eta + c_2V^2\eta + \ldots + c_mV^m\eta = 0$, where the c's are constants and η, $V\eta$, $V^2\eta, \ldots, V^{m-1}\eta$ are linearly independent. Thus, $f(V)\eta = 0$, where

$$f(x) = c_0 + c_1x + c_2x^2 + \ldots + c_mx^m = (x - \nu)g(x), \text{ say.}$$

Then $(V - \nu)\zeta = 0$, where $\zeta = g(V)\eta$. Note that $\zeta \neq 0$ since η, $V\eta, \ldots,$ $V^{m-1}\eta$ are linearly independent. Therefore, $V\zeta = \nu\zeta$, and also $U\zeta = Ug(V)\eta = g(V)U\eta = \mu g(V)\eta = \mu\zeta$ since $UV^r = V^rU$, $r = 1, 2, \ldots$.

THEOREM 8. *If U and V commute and ζ is a characteristic vector of U corresponding to a simple characteristic root μ of U, then ζ is also a characteristic vector of V.*

Proof. The proof is very simple. We have that $U\zeta = \mu\zeta$, and thus $U(V\zeta) = VU\zeta = \mu(V\zeta)$. If $V\zeta \neq \nu\zeta$ for any constant ν, we would obtain two linearly independent characteristic vectors of U corresponding to the simple characteristic root μ of U; but this is impossible. Therefore, $V\zeta = \nu\zeta$ for some constant ν.

Since the Perron root of a positive matrix is simple, we obtain the following corollary.

COROLLARY. *If A is real and $AP = PA$ for some positive matrix P, then the Perron vector of P is a Perron vector of A.*

The matrix A of this corollary may have other Perron vectors. Consider the matrices

$$A = \begin{bmatrix} 4.0 & 1.0 & -3.0 \\ 0.4 & 2.2 & -0.6 \\ -1.6 & -0.8 & 4.4 \end{bmatrix}, \qquad P = \begin{bmatrix} 1 & 1 & 9 \\ 1 & 1 & 3 \\ 5 & 3 & 1 \end{bmatrix}, \qquad AP = PA.$$

A has a Perron vector ξ, where $\xi' = (1, 1, 1)$ which is not a Perron vector of P. Clearly, A has two Perron vectors corresponding to the Perron root 2.

As a companion to Theorem 8, we have the following theorem.

THEOREM 9. *If A is real and normal (i.e., $AA' = A'A$) and λ is a real characteristic root of A, then there is a real common characteristic vector η such that $A\eta = \lambda\eta$ and $A'\eta = \delta\eta$.*

Proof. By Theorem 7, there is a common characteristic vector η of A and A' such that $A\eta = \lambda\eta$, $A'\eta = \delta\eta$. If η is real, the theorem holds.

Suppose that $\eta = \eta_1 + i\eta_2$, where η_1 and η_2 are real vectors. Then $A\eta_1 = \lambda\eta_1$, $A\eta_2 = \lambda\eta_2$, and $\eta_2'A'\eta = \delta\eta_2'\eta$, i.e., $\lambda\eta_2'\eta = \delta\eta_2'\eta$. If $\lambda \neq \delta$ we must have that $\eta_2'\eta = 0$; whence $\eta_2'\eta_2 = 0$, i.e., $\eta_2 = 0$; thus η is real and $A\eta = \lambda\eta$, $A'\eta = \delta\eta$. If $\lambda = \delta$, then, clearly, one of η_1, η_2 is not 0, and therefore we get a common characteristic vector for A and A'.

5. Determination of the Perron root and vector of a Frobenius matrix. Brauer (**1**; **3**) has given an efficient procedure for obtaining the Perron root and vector of a positive matrix and of a non-negative matrix, to any desired degree of accuracy. Here, we shall use his procedure to obtain the Perron root and vector of a Frobenius matrix, to any degree of accuracy.

Let d be the row range (definition in § 1) of a Frobenius matrix $A = (a_{ij})$; for non-triviality, we take $d > 0$. Let r be the smallest row sum in A. We say that a row of a matrix belongs to (or is in) the lower class L or L' according as its row sum lies in the interval $[r, r + d/4)$ or $[r + d/4, r + d/2)$. Similarly, a row belongs to the upper class U' or U according as its row sum lies in the interval $[r + d/2, \quad r + 3d/4)$ or $[r + 3d/4, \quad r + d]$. Let

$$g = (4r + 4 + 3d)/(4r + 4c + 2d) > 1,$$

where c is the smallest non-negative number such that $a_{ii} + c \geqq 0$ for all i.

Multiply those rows of A which belong to the lower classes L and L' and divide the corresponding columns by g. This is Brauer's procedure. Obviously, this is a similarity transformation of A into a similar matrix A_1, $A_1 = D^{-1}AD$, where D is a diagonal matrix having some diagonal elements equal to g and the rest to one. By the procedure, row sums of rows belonging to the lower classes are never diminished while the row sums of rows in the upper classes are never increased since $g > 1$.

Let r_i and $r_i^{(1)}$ be the sums of the elements in the ith rows of A and A_1, respectively. If the ith row of A belongs to a lower class, then the ith row of A_1 cannot belong to the class U since

$$r_i < r + d/2$$

and

$$r_i^{(1)} \leqq a_{ii} + (r_i - a_{ii})g = gr_i - a_{ii}(g - 1) < c(g - 1) + g(r + d/2) = r + 3d/4.$$

Thus, the procedure may keep a row of A belonging to L' in the same class or send it to the class U' (but not to U or L). Similarly, a row belonging to L may be kept in L or may be sent to L' or U' (but not to U). Next, if the ith row of A belongs to an upper class, then the ith row of A_1 cannot belong to L since

$$r_i \geqq r + d/2$$

and

$$r_i^{(1)} \geqq a_{ii} + (r_i - a_{ii})g^{-1} = a_{ii}(1 - g^{-1}) + r_i g^{-1} \geqq -c(1 - g^{-1}) + (r + d/2)g^{-1} > r + d/4.$$

Thus, the procedure may keep a row of A belonging to U' in the same class or send it to L' (but not to L or U). Similarly, a row of A belonging to U may be kept in U or sent to U' or L' (but not to L).

If no row of A_1 belongs to L or to U, then the row range of A_1 is less than or equal to $3d/4$ and Brauer's procedure has reduced the row range of A by a factor at most $3/4$.

If some row of A_1 belongs to L and another to U, we apply Brauer's procedure to A_1, using the same g given above. We obtain a similar matrix A_2. If the row range of A_2 is less than or equal to $3d/4$, we terminate the procedure; if not, using the same g we continue the procedure and obtain similar matrices, $A_1, A_2, A_3, \ldots$.

We shall now show that Brauer's procedure terminates after a finite number of applications, giving us a matrix A_k with row range less than or equal to $3d/4$.

If possible, let the procedure continue ad infinitum. This means that there are some (one or more) rows in $A, A_1, A_2, \ldots$ which belong to L and some other (one or more) rows which belong to U in all these matrices. For the purpose of proving our result we may suppose (if necessary, by initially permuting the rows and analogously permuting the columns) that the first t rows of $A, A_1, A_2, \ldots$ permanently belong to the lower classes and the last u rows of these matrices permanently belong to the upper classes. Besides these, there might be some rows which change classes (between L' and U' only) infinitely often; let the number of such rows be v. Thus, $u \geqq 1$, $t \geqq 1$, $v \geqq 0$, $t + u + v = n$.

Let the element in the i, j position of A_k be denoted by $a_{ij}^{(k)}$, $k = 1, 2, 3, \ldots$. Consider the $t \times (u + v)$ submatrices situated at the upper right-hand corner regions of all these matrices; Brauer's procedure never reduces the elements of these submatrices.

For any j with $n - u + 1 \leqq j \leqq n$, we easily obtain $a_{1j}^{(k)} = a_{1j}g^k$, $k = 1, 2, \ldots$. If $a_{1j} \neq 0$, then $a_{1j} > 0$ and $a_{1j}^{(k)} \to \infty$ as $k \to \infty$. Since the first row sum of each of our matrices is less than $r + d/2$, we infer that $a_{1j} = 0$. For any j with $t + 1 \leqq j \leqq n - u$, a moment's consideration shows that the sequence $a_{1j}, a_{1j}^{(1)}, a_{1j}^{(2)}, \ldots$ contains as an infinite subsequence, the sequence $a_{1j}, a_{1j}g, a_{1j}g^2, \ldots$; thus, we again infer that $a_{1j} = 0$, $t + 1 \leqq j \leqq n - u$. Thus, all the elements in the first row of all the submatrices are 0. In a like manner we see that $a_{ij} = 0$, $1 \leqq i \leqq t$, $t + 1 \leqq j \leqq n$.

Hence, a $t \times (n - t)$ submatrix of A is a null matrix, contradicting the irreducibility of A. Consequently, Brauer's procedure must terminate, giving us a similar matrix B_1 with row range less than or equal to $3d/4$.

If the least row sum of B_1 is $r^{(1)}$ and row range is $d^{(1)}$, apply Brauer's procedure to B_1, using $g^{(1)} = (4r^{(1)} + 4c + 3d^{(1)})/(4r^{(1)} + 4c + 2d^{(1)})$. We obtain a similar matrix B_2 with row range $3d^{(1)}/4 \leqq (\frac{3}{4})^2 d$. Similarly, starting with B_2 and with a corresponding $g^{(2)}$ we obtain a similar matrix B_3 with row range less than or equal to $(\frac{3}{4})^3 d$, and so on. Hence, it follows that there is a matrix $B^{(m)}$ (with smallest row sum $r^{(m)}$, greatest row sum $r'^{(m)}$, say) similar to A such that the row range of $B^{(m)} = r'^{(m)} - r^{(m)} \leqq 1/m$; $B^{(m)} = D_m^{-1}AD_m$, where D_m is a diagonal matrix with positive diagonal elements. We easily obtain

$$B^{(m)} = E_m^{-1}AE_m, \tag{5.1}$$

where E_m is a diagonal matrix with diagonal elements $x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)}$, $\sum_i x_i^{(m)} = 1$, and with row range of $B^{(m)} \leqq 1/m$. The above discussion may be summarized in the following theorem.

THEOREM 10. *Given any Frobenius matrix A, any $\epsilon > 0$, there is a matrix B similar to A such that the row range of $B < \epsilon$ and $B = E^{-1}AE$, where E is a diagonal matrix with positive diagonal elements. B can be obtained by Brauer's procedure (as explained above).*

Let $\xi^{(m)}$ be the vector with coordinates $x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)}$, $x_i^{(m)} > 0$, $\sum_i x_i^{(m)} = 1$. Then from (5.1) it follows (on recalling the fourth paragraph of this section) that

$$r\xi^{(m)} \leqq r^{(m)}\xi^{(m)} \leqq A\xi^{(m)} \leqq r'^{(m)}\xi^{(m)} \leqq r'\xi^{(m)}, \tag{5.2}$$

$$r'^{(m)} - r^{(m)} \leqq 1/m. \tag{5.3}$$

Consider the bounded monotone increasing sequence $r^{(1)}, r^{(2)}, \ldots$; it tends to a limit, ω, say. As the associated sequence of vectors $\xi^{(1)}, \xi^{(2)}, \ldots$ belong to the closed bounded set

$$S = \{\xi \colon \xi' = (x_1, x_2, \ldots, x_n),\ x_i > 0 \text{ for all } i,\ \textstyle\sum x_i = 1\},$$

we can pick a subsequence $\xi^{(m_1)}, \xi^{(m_2)}, \ldots$, converging to a non-negative vector ξ of S. Write $\xi^{(m_k)} = \xi_k$, $r^{(m_k)} = r_{(k)}$, $r'^{(m_k)} = r_{(k)}'$. Then $\lim_{k\to\infty} \xi_k = \xi$ and, as $\lim_{k\to\infty}(r_{(k)}' - r_{(k)}) = 0$, $\lim_{k\to\infty} r_{(k)}' = \omega$. From (5.2) we now obtain

$$r_{(k)}\xi_k \leqq A\xi_k \leqq r_{(k)}'\xi_k. \tag{5.4}$$

Letting $k \to \infty$, we deduce that $A\xi = \omega\xi$. Since A is a Frobenius matrix, by Theorem 5, there is a polynomial in A, $f(A)$, say, such that $f(A) > 0$, and thus $f(\omega)\xi = f(A)\xi > 0$; whence $\xi > 0$.

The above gives a constructive proof for the existence of a Perron vector and root of a Frobenius matrix. This proof is different from Brauer's proof of the analogous result for positive matrices and is more direct.

6. Bounds for the characteristic roots corresponding to Perron vectors. In this section we study some upper and lower bounds for the characteristic root λ, corresponding to a Perron vector of a real matrix $A = (a_{ij})$.

THEOREM 11. *If A has a non-negative characteristic vector corresponding to a characteristic root λ, then $c \leqq \lambda \leqq c'$, where c and c' denote the minimum and maximum of the column sums of A.*

Proof. Let $x_1, x_2, \ldots, x_n$ (with $x_i \geqq 0$, $\sum x_i = 1$) be the coordinates of the non-negative vector corresponding to the characteristic root λ. Then

$$\lambda x_i = a_{i1} x_1 + a_{i2}x_2 + \ldots + a_{in}x_n, \qquad i = 1, 2, \ldots, n.$$

Adding these we have that $\lambda = c_1x_1 + c_2x_2 + \ldots + c_nx_n$, c_i denoting the ith column sum of A and the theorem follows easily.

COROLLARY. *If A is polynomial-positive and ω is its Perron root, then $r \leqq \omega \leqq r'$, where r and r' are the minimum and maximum of the row sums of A.*

Since A' is polynomial-positive, it has a Perron vector by Theorem 6. Now apply Theorem 11 to A'.

THEOREM 12. *If A commutes with a positive matrix $P = (p_{ij})$ and ω and λ denote the Perron roots of A and P, respectively, then*

$$\lambda \max_i \min_j (a_{ij}/p_{ij}) \leqq \omega \leqq \lambda \min_i \max_j (a_{ij}/p_{ij}).$$

Proof. Since λ is a simple characteristic root of P, the Perron vector ξ, with $\xi' = (x_1, x_2, \ldots, x_n)$, $x_i > 0$, of P is also a Perron vector of A for its Perron root ω, by the corollary to Theorem 8. Thus

$$\omega x_i = a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n, \qquad i = 1, 2, \ldots, n,$$
$$\lambda x_i = p_{i1}x_1 + p_{i2}x_2 + \ldots + p_{in}x_n, \qquad i = 1, 2, \ldots, n;$$

whence

$$\begin{aligned}\omega/\lambda &= (a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n)/(p_{i1}x_1 + p_{i2}x_2 + \ldots + p_{in}x_n)\\ &\leqq \max(a_{i1}/p_{i1}, a_{i2}/p_{i2}, \ldots, a_{in}/p_{in}).\end{aligned}$$

Thus $\omega/\lambda \leqq \min_i \max_j a_{ij}/p_{ij}$; likewise for the left-hand inequality.

THEOREM 13. *If all the off-diagonal elements of A are positive and m denotes the smallest of them, then the Perron root ω of A satisfies the inequality*

$$\omega \geqq \max_i \tfrac{1}{2}\{r + a_{ii} - m + [(r - a_{ii} - m)^2 + 4m(r_i - a_{ii})]^{\frac{1}{2}}\},$$

where r_i is the ith row sum of A, $r = \min(r_1, r_2, \ldots, r_n)$.

Proof. A' being a Frobenius matrix has a Perron vector and a Perron root which is equal to ω, by Theorem 3. By Theorem 11 (applied to A') we have that $r_i \leqq \omega$, $i = 1, 2, \ldots, n$.

Let $x_1, x_2, \ldots, x_n$, $x_i > 0$, be the coordinates of the Perron vector of A corresponding to ω. Let x_s and x_t be the smallest and largest of the x_i's.

We have that $\omega x_i = \sum_j a_{ij}x_j$, $i = 1, 2, \ldots, n$, and therefore $0 = a_{i1}x_1 + a_{i2}x_2 + \ldots + (a_{ii} - \omega)x_i + a_{ii+1}x_{ii+1} + \ldots + a_{in}x_n$. As $x_i > 0$ and $a_{ij} > 0$, $i \neq j$, we see that $a_{ii} < \omega$ for all i.

We have that $\omega x_s = \sum_j a_{sj}x_j$, i.e.,

$$\begin{aligned}\omega &= a_{s1}x_1/x_s + a_{s2}x_2/x_s + \ldots + a_{st}x_t/x_s + \ldots + a_{sn}x_n/x_s\\ &\geqq \textstyle\sum_j a_{sj} - a_{st} + a_{st}x_t/x_s\\ &= r_s - a_{st} + a_{st}x_t/x_s\\ &\geqq r - a_{st} + a_{st}x_t/x_s,\end{aligned}$$

and therefore $(\omega - r + a_{st})/a_{st} \geqq x_t/x_s$; whence, $(\omega - r + m)/m \geqq x_t/x_s$, i.e., $x_s/x_t \geqq m/(\omega - r + m)$ and thus

$$x_i/x_j \geqq m/(\omega - r + m), \qquad i, j = 1, 2, \ldots, n. \tag{6.1}$$

For each i, we have that $\omega = \sum_j a_{ij}x_j/x_i \geqq (r_i - a_{ii})m/(\omega - r + m) + a_{ii}$, and hence, for all i,

$$\omega^2 + \omega(m - r - a_{ii}) + (ra_{ii} - r_i m) \geqq 0. \tag{6.2}$$

The smaller root of the equation $x^2 + x(m - r - a_{ii}) + (ra_{ii} - r_i m) = 0$ is

$$\begin{aligned}\tfrac{1}{2}\{(r + a_{ii} - m) &- [(r - a_{ii} - m)^2 + 4m(r_i - a_{ii})]^{\frac{1}{2}}\}\\ &\leqq \tfrac{1}{2}\{(r + a_{ii} - m) - [(r - a_{ii} - m)^2 + 4m(r - a_{ii})]^{\frac{1}{2}}\}\\ &= \tfrac{1}{2}\{(r + a_{ii} - m) - |r - a_{ii} + m|\}\\ &= a_{ii} - m \text{ or } r - m\\ &< \omega.\end{aligned}$$

It therefore follows from (6.2) that ω cannot be less than the roots of the above equation.

THEOREM 14. *If all the off-diagonal elements of A are positive and A_k denotes the principal submatrix obtained by suppressing its kth row and column, then*

$$\omega^{(k)} \leqq \omega - \omega m m_k/(\omega - r + m),$$

where ω and $\omega^{(k)}$ are the Perron roots of A and A_k, respectively, m is the least off-diagonal element of A, r is the least row sum of A, and $m_k = \min_i a_{ik} (k \neq i)$.

Proof. It is sufficient to prove the theorem for the submatrix A_1. Since A_1 is a Frobenius matrix, it has a Perron vector and root $\omega^{(1)}$. We use the notation of Theorem 13. Since $\omega x_i = \sum_j a_{ij}x_j$, $i = 1, 2, \ldots, n$, we have, for $i = 2, 3, \ldots, n$, that

$$\begin{aligned} a_{i2}x_2 + a_{i3}x_3 + \ldots + a_{in}x_n &= (\omega - a_{i1}x_1/x_i)x_i \\ &\leqq (\omega - a_{i1}m/(\omega - r + m))x_i \quad \text{by (6.1)} \\ &\leqq (\omega - m_1 m/(\omega - r + m))x_i. \end{aligned}$$

Using the max-min property of Perron roots (as applied to A_1) we have that $\omega^{(1)} \leqq \omega - m_1 m/(\omega - r + m)$.

THEOREM 15. *If all the off-diagonal elements of $A = (a_{ij})$ are non-negative and λ is a characteristic root of A to which there corresponds a positive characteristic vector, then*

$$\lambda \geqq \sum_i a_{ii}/n + 2 \sum_{\substack{i,j;\\ i<j}} (a_{ij}a_{ji})^{\frac{1}{2}}/n.$$

Proof. Let $x_1, x_2, \ldots, x_n$, with $x_i > 0$ for all i, be the coordinates of the Perron vector. Then $\lambda = x_i^{-1}\sum_j a_{ij}x_j$, $i = 1, 2 \ldots, n$. Adding these, we obtain

$$\begin{aligned} n\lambda &= \sum_i a_{ii} + \sum_{\substack{i,j;\\ i<j}} (a_{ij}x_j/x_i + a_{ji}x_i/x_j) \\ &= \sum_i a_{ii} + \sum_{\substack{i,j;\\ i<j}} \{[(a_{ij}x_j/x_i)^{\frac{1}{2}} - (a_{ji}x_i/x_j)^{\frac{1}{2}}]^2 + 2(a_{ij}a_{ji})^{\frac{1}{2}}\} \\ &\geqq \sum_i a_{ii} + 2\sum_{\substack{i,j;\\ i<j}} (a_{ij}a_{ji})^{\frac{1}{2}}. \end{aligned}$$

COROLLARY. *If $A = (a_{ij})$ is a Frobenius matrix and ω is its Perron root, then*

$$\omega \geqq \sum_i (a_{ii}/n) + 2 \sum_{\substack{i,j;\\ i<j}} (a_{ij}a_{ji})^{\frac{1}{2}}/n.$$

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