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CONVERGENCE OF AVERAGED OCCUPATION TIMES

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1. **Introduction.** Let $X=\{X_t, t\geq 0\}$ be a stationary Markov process with values in a measurable space $(S, \mathscr{B})$, transition function p, and initial distribution concentrated at a point $x\in S$. The occupation times of a set $A\in\mathscr{B}$ are defined for $t\geq 0$ by

$$\mu(t) = \int_0^t 1_A(X_s)\,ds$$

where 1_A is the indicator function of A. The expected occupation times are given by

$$E\{\mu(t)\} = \int_0^t p(s, x, A)\,ds.$$

If $\lim_{t\to\infty} E\{\mu(t)\}=\infty$ and $u(t)\sim E\{\mu(t)\}$ as $t\to\infty$, then it is natural to study the asymptotic properties of $\mu(t)u(t)^{-1}$. Kallianpur and Robbins [6] studied the case where X is Brownian motion in one or two dimensions. Darling and Kac [4] gave an elegant treatment for more general processes. These authors dealt entirely with convergence in distribution. It is clear from the case of Brownian motion that the convergence cannot be strengthened to almost everywhere convergence, or even convergence in measure.

Recently Brosamler [2] studied the pathwise asymptotic properties of additive functionals of Brownian motion. By using the idea of averaging with respect to certain log-scales he showed how to obtain almost everywhere convergence. Brosamler's results, specialized to the case of occupation times of a set A of finite strictly positive Lebesgue measure, imply that in dimension one $u(t)=c\sqrt{t}$ and

$$\lim_{t\to\infty}\,[\log t]^{-1}\int_1^t \mu(s)u(s)^{-1}\,d[\log s] = 1 \text{ a.e.,} \tag{1}$$

and in dimension two $u(t)=c\log t$ and

$$\lim_{t\to\infty}\,[\log\log t]^{-1}\int_e^t \mu(s)u(s)^{-1}\,d[\log\log s] = 1 \text{ a.e.} \tag{2}$$

The purpose of this paper is to generalize some of Brosamler's results to a wider class of Markov processes and to show how his log-averaging corresponds to the extreme case of a whole family of limit theorems for occupation times.

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To see how this is done, first note that if u is modified in a neighborhood of the origin so that $u(0)=1$, then (1) and (2) may be written in the form

$$\lim_{t\to\infty} [\log u(t)]^{-1} \int_0^t \mu(s)u(s)^{-1}\, d[\log u(s)] = 1 \text{ a.e.}$$

As Brosamler has observed

$$\begin{aligned}\int_0^t \mu(s)u(s)^{-1}\, d[\log u(s)] &= \int_0^t \mu(s)u(s)^{-2}u'(s)\, ds \\ &= \int_0^t \mu(s)\, d[-u(s)^{-1}] \\ &= \int_0^t u(s)^{-1}\, d\mu(s) - \mu(t)u(t)^{-1} \\ &= \int_0^t u(s)^{-1} I_A(X_s)\, ds - \mu(t)u(t)^{-1}.\end{aligned}$$

If $\mu(t)u(t)^{-1}$ converges in distribution, then we might hope that

$$\lim_{t\to\infty} [\log u(t)]^{-1}\mu(t)u(t)^{-1} = 0 \text{ a.e.} \tag{3}$$

Our problem would then be reduced to studying

$$[\log u(t)]^{-1} \int_0^t u(s)^{-1}\, d\mu(s). \tag{4}$$

More generally, we consider expressions of the form

$$\left[\int_0^t u^{\alpha-1}(s)\, du(s)\right]^{-1} \int_0^t u^{\alpha-1}(s)\, d\mu(s). \tag{5}$$

We obtain (4) when $\alpha=0$ and $\mu(t)u(t)^{-1}$ when $\alpha=1$. If $\alpha<0$, then $\int_0^t u^{\alpha-1}(s)\, d\mu(s)$ is bounded and there are no interesting limit theorems as $t\to\infty$.

Chung and Erdös [3, P. 16] have obtained the convergence of the discrete analogue of (4) for integer-valued random walks.

2. **The convergence theorems.** Recall that a function L is said to be slowly varying (see [5]) if $L>0$ in a neighborhood of infinity and

$$\lim_{t\to\infty} L(\delta t)L(t)^{-1} = 1 \tag{6}$$

for all $\delta>0$. It follows that (6) holds uniformly in δ when δ is restricted to a compact subset of $(0, \infty)$. A function of the form $L(t)t^{\beta}$ is said to be regularly varying of exponent β.

Darling and Kac proved convergence in distribution of $\mu(t)u(t)^{-1}$ under the assumptions that u is regularly varying of exponent β with $0\le\beta\le 1$ and

$$\int_0^t p(s, y, A)\, ds \sim u(t) \tag{7}$$

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uniformly for $y \in A$. Their argument, which involved Karamata's Tauberian theorem, does not carry over to expressions of the form (5) and we must strengthen (7) slightly.

THEOREM 1. *Let $x \in S$ and $A \in \mathscr{B}$ be fixed. Assume that the function $t \to p(t, x, A)$ is regularly varying with exponent $\beta-1$ where $0 \leq \beta \leq 1$ and write $p(t, x, A) = L(t)t^{\beta-1}$. Assume further that*

$$\lim_{t\to\infty} p(t, y, A)p(t, x, A)^{-1} = 1 \tag{8}$$

uniformly for $y \in A$. Define

$$u(t) = 1+\int_0^t p(s, x, A)\, ds \tag{9}$$

and assume (in case $\beta=0$) that u is unbounded. If $\alpha \geq 0$ and $P\{X_0=x\}=1$, then

$$\left[\int_0^t u^{\alpha-1}(s)\, du(s)\right]^{-1} \int_0^t u^{\alpha-1}(s)\, d\mu(s)$$

converges in distribution as $t \to \infty$. In the extreme case where $\alpha=0$, the limiting distribution is the unit mass at 1.

Proof. The proof is a direct application of the method of moments. We will outline the main steps and leave several routine verifications to the reader. We write $p(t)=p(t, x, A)$ in what follows. The second moment is given by

$$\begin{aligned}
E\left\{\left(\int_0^t u^{\alpha-1}(s)\, d\mu(s)\right)^2\right\} &= E\left\{\left(\int_0^t u^{\alpha-1}(s) 1_A(X_s)\, ds\right)^2\right\} \\
&= E\left\{\int_0^t u^{\alpha-1}(r) 1_A(X_r)\, dr \int_0^t u^{\alpha-1}(s) 1_A(X_s)\, ds\right\} \\
&= 2\int_0^t u^{\alpha-1}(r)\, dr \int_A p(r, x, dy) \int_r^t u^{\alpha-1}(s) p(s-r, y, A)\, ds \\
&\sim 2\int_0^t u^{\alpha-1}(r) p(r)\, dr \int_r^t u(s)^{\alpha-1} p(s-r)\, ds \\
&= 2\int_0^t u^{\alpha-1}(s)\, ds \int_0^s u^{\alpha-1}(r) p(s-r) p(r)\, dr \\
&= 2\int_0^t u^{\alpha-1}(s) f_2(s)\, ds
\end{aligned}$$

where the asymptotic relation follows from (8). An elementary induction argument shows that

$$E\left\{\left(\int_0^t u^{\alpha-1}(s)\, d\mu(s)\right)^n\right\} \sim n! \int_0^t u^{\alpha-1}(s) f_n(s)\, ds \tag{10}$$

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where $f_1(s)=p(s)$ and

$$f_{n+1}(s) = \int_0^s u^{\alpha-1}(r)p(s-r)f_n(r)\,dr. \tag{11}$$

Now

$$f_n(s) \sim \begin{cases} c_n p(s)[\log u(s)]^{n-1} & \text{if} \quad \alpha = 0, \\ c_n p(s) u^{(n-1)\alpha}(s) & \text{if} \quad \alpha > 0 \end{cases} \tag{12}$$

where $c_1=1$ and

$$c_{n+1}c_n^{-1} = \begin{cases} n^{-1} & \text{if} \quad \alpha = 0, \\ \beta\Gamma(\beta)\Gamma(n\alpha\beta)\Gamma(\beta+n\alpha\beta)^{-1} & \text{if} \quad \alpha > 0, \beta > 0, \\ 1+(n\alpha)^{-1} & \text{if} \quad \alpha > 0, \beta = 0. \end{cases} \tag{13}$$

The proof of these facts breaks up naturally into several cases, depending on whether α and β are positive or zero. If $\beta=0$, then u is slowly varying with $L(t)=o(u(t))$, and if $\beta>0$, then u is regularly varying of exponent β with $u(t)\sim\beta^{-1}L(t)t^{\beta}$ (see [5, p. 281]). Using these facts and (11), we now obtain by induction the asymptotic behavior of the functions $\{f_n\}$. If $\alpha=0$ and $0<\delta<1$, then

$$\begin{aligned} f_{n+1}(s) &\sim c_n \int_0^s u^{-1}(r)p(s-r)p(r)[\log u(r)]^{n-1}\,dr \\ &\sim c_n \int_0^{\delta s} u^{-1}(r)p(s-r)p(r)[\log u(r)]^{n-1}\,dr \\ &\sim c_n \int_0^{\delta s} u^{-1}(r)L(s-r)(s-r)^{\beta-1}p(r)[\log u(r)]^{n-1}\,dr. \end{aligned}$$

If δ is chosen sufficiently small, then the above expression may be replaced by

$$\begin{aligned} c_n L(s)s^{\beta-1} \int_0^{\delta s} u^{-1}(r)p(r)[\log u(r)]^{n-1}\,dr &= c_n n^{-1} p(s)[\log u(\delta s)]^n \\ &\sim c_n n^{-1} p(s)[\log u(s)]^n. \end{aligned}$$

If $\alpha>0$, $0=\beta$ and $0<\delta<\frac{1}{2}$, then

$$\begin{aligned} f_{n+1}(s) &\sim c_n \int_0^s u^{n\alpha-1}(r)p(s-r)p(r)\,dr \\ &\sim c_n\left(\int_0^{\delta s} + \int_{(1-\delta)s}^{s}\right) \\ &\sim c_n \int_0^{\delta s} u^{n\alpha-1}(r)L(s-r)(s-r)^{-1}p(r)\,dr + c_n \int_{(1-\delta)s}^{s} u^{n\alpha-1}(r)p(s-r)L(r)r^{-1}\,dr. \end{aligned}$$

Again choosing δ sufficiently small, we obtain

$$\begin{aligned} f_{n+1}(s) &\sim c_n L(s)s^{-1} \int_0^{\delta s} u^{n\alpha-1}(r)p(r)\,dr + c_n L(s)s^{-1}u^{n\alpha-1}(s) \int_{(1-\delta)s}^{s} p(s-r)\,dr \\ &\sim c_n p(s)u^{n\alpha}(s)(1+(n\alpha)^{-1}). \end{aligned}$$

In the final case where $\alpha>0$ and $\beta>0$ we have

$$\begin{aligned} f_{n+1}(s) &\sim c_n\int_0^s u^{n\alpha-1}(r)p(s-r)p(r)\,dr \\ &\sim c_n\beta^{1-n\alpha}\int_0^s L(r)^{n\alpha}r^{n\alpha\beta-1}L(s-r)(s-r)^{\beta-1}\,dr \\ &\sim c_n\beta^{1-n\alpha}L(s)^{n\alpha+1}\int_0^s (s-r)^{\beta-1}r^{n\alpha\beta-1}\,dr \\ &= c_n\beta^{1-n\alpha}L(s)^{n\alpha+1}s^{\beta-1+n\alpha\beta}\int_0^1 (1-v)^{\beta-1}v^{n\alpha\beta-1}\,dv \\ &= c_n\beta\Gamma(\beta)\Gamma(n\alpha\beta)\Gamma(\beta+n\alpha\beta)^{-1}p(s)u^{n\alpha}(s). \end{aligned}$$

It follows from (10), (12) and (13) that

$$E\left\{\left(\int_0^t u^{\alpha-1}(s)\,d\mu(s)\right)^n\right\} \sim d_n\left(\int_0^t u^{\alpha-1}(s)\,du(s)\right)^n$$

where $d_n=1$ if $\alpha=0$ and $d_n=\alpha^{n-1}(n-1)!c_n$ if $\alpha>0$. In either case it follows that $\lim_{n\to\infty}\sup(d_n)^{1/n}n^{-1}<\infty$ and our theorem follows from a standard result on the unique determination of a distribution by its moments [1, p. 182].

Further identification of the limiting distribution is possible in certain cases. If $\alpha>0$ and $\beta=0$, then the limiting distribution is a Gamma distribution with density

$$[\alpha^{1/\alpha}\Gamma(1/\alpha)]^{-1}e^{-x/\alpha}x^{(1-\alpha)/\alpha}.$$

If $\alpha>0$ and $\beta=1$, then the limiting distribution is the unit mass at 1. The case $\alpha=1$ involves the Mittag–Leffler distributions as in [4].

THEOREM 2. *Let $x\in S$ and $A\in\mathscr{B}$ be fixed. Assume that $p(t,x,A)>0$ for sufficiently large t and*

$$(14)\qquad \limsup_{t\to\infty}\left[\sup_{y\in A} p(t,y,A)p(t,x,A)^{-1}\right]\le 1.$$

Let

$$\varepsilon(t)=\sup_{y\in A}\,[p(t,y,A)-p(t,x,A)]^+.$$

Define u by (9) *and assume*

$$(15)\qquad \int_0^t \varepsilon(s)u(s)^{-1}\,ds = 0([\log u(t)]^\lambda)$$

for some $\lambda<1$. If $P\{X_0=x\}=1$, then

$$(16)\qquad \lim_{t\to\infty}\,[\log u(t)]^{-1}\int_0^t u^{-1}(s)\,d\mu(s)=1 \text{ a.e.}$$

Proof. Write

$$\begin{aligned} E\left\{\left(\int_0^t u(s)^{-1}\,d\mu(s)\right)^2\right\} &= 2\int_0^t u(r)^{-1}\,dr\int_A p(r,x,dy)\int_r^t u(s)^{-1}p(s-r,y,A)\,ds \\ &= I_1+I_2+I_3 \end{aligned}$$

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where

$$I_1 = 2\int_0^t u(r)^{-1}p(r)\,dr\int_r^t u(s)^{-1}p(s)\,ds,$$

$$I_2 = 2\int_0^t u(r)^{-1}p(r)\,dr\int_r^t u(s)^{-1}[p(s-r)-p(s)]\,ds,$$

$$I_3 = 2\int_0^t u(r)^{-1}\,dr\int_A p(r,x,dy)\int_r^t u(s)^{-1}[p(s-r,y,A)-p(s-r)]\,ds.$$

Now $I_1=[\log u(t)]^2$, and according to the second mean value theorem for integrals, I_2 can be dominated by

$$\begin{aligned} 2\int_0^t d[\log u(r)]u(r)^{-1}\sup_{r\le v\le t}\int_r^v [p(s-r)-p(s)]\,ds \\ &\le 2\int_0^t d[\log u(r)]u(r)^{-1}\sup_{r\le v\le t}[u(v-r)-u(v)+u(r)] \\ &\le 2\int_0^t d[\log u(r)] \\ &= 2\log u(t) \end{aligned}$$

where the second inequality follows from the fact that $u(v-r)-u(v)\le 0$. If (15) holds, then I_3 is dominated by

$$\begin{aligned} 2\int_0^t u(r)^{-1}p(r)\,dr\int_r^t u(s)^{-1}\varepsilon(s-r)\,ds &\le 2\int_0^t u(r)^{-1}p(r)\,dr\int_0^t u(s)^{-1}\varepsilon(s)\,ds \\ &= O([\log u(t)]^{1+\lambda}). \end{aligned}$$

Hence

$$\text{(17)}\qquad \operatorname{var}\left\{[\log u(t)]^{-1}\int_0^t u^{-1}(r)\,d\mu(s)\right\} = O([\log u(t)]^{\lambda-1}).$$

For $n\ge 1$ let

$$t_n = \inf\{t:\log u(t)\ge n^{2/(1-\lambda)}\}.$$

It follows from (17) that (16) holds along the sequence $\{t_n\}$. Using the fact that

$$\log u(t_n) = n^{2/(1-\lambda)} \sim \log u(t_{n+1}),$$

and

$$\int_0^{t_n} u(s)^{-1}\,d\mu(s) \le \int_0^t u(s)^{-1}\,d\mu(s) \le \int_0^{t_{n+1}} u(s)^{-1}\,d\mu(s)$$

for $t_n\le t\le t_{n+1}$, we obtain the almost everywhere convergence in (16) and the proof is complete.

COROLLARY 1. *If* (14) *but not* (15) *holds in Theorem* 2, *then the convergence in* (16) *holds in* L^2.

Proof. If only (14) holds, then $\varepsilon(t)=o(p(t))$ and $I_3=o([\log u(t)]^2)$. Hence the left side of (17) is $o(1)$ and the proof is complete.

3. **Examples and discussion.** For many processes X the hypotheses of Theorem 2 are easy to verify. In fact, (8) implies (14) and if for some $\eta>0$

$$p(t, y, A) - p(t, x, A) = O([\log u(t)]^{-\eta} p(t)) \tag{18}$$

uniformly for $y \in A$, then (15) holds. For example, let $A \subset R$ be a compact set of strictly positive Lebesgue measure and let X be a symmetric stable process with index $\alpha \in [1, 1]$. The density of X is known (see [5]) and

$$p(t, y, A) = ct^{-1/\alpha} + O(t^{-3/\alpha})$$

uniformly for $y \in A$. Hence (8) holds and (18) holds for all $\eta>0$. For two dimensional Brownian motion we have

$$p(t, y, A) = ct^{-1} + O(t^{-2})$$

and (18) holds for all $\eta>0$.

We now show that Brosamler's log-averaging follows under the hypotheses of Theorem 2. If suffices to verify (3). If (14) holds, then

$$\begin{aligned} E\{\mu(t)^2\} &= O\left(\int_0^t p(r)\, dr \int_r^t p(s-r)\, ds\right) \\ &= O(u(t)^2). \end{aligned}$$

Thus the L^2-convergence of (3) follows. Chebyshev's inequality implies that for $\varepsilon>0$

$$P\{[\log u(t)]^{-1} \mu(t) u(t)^{-1} \geq \varepsilon\} = O([\varepsilon \log u(t)]^{-2}). \tag{19}$$

For $n \geq 1$ let

$$t_n = \inf\{t : \log u(t) \geq n\}.$$

It follows from (19) that (3) holds along the sequence $\{t_n\}$. If $t_n \leq t \leq t_{n+1}$, then

$$\begin{aligned} [u(t) \log u(t)]^{-1} \mu(t) &\leq [u(t_n) \log u(t_n)]^{-1} \mu(t_{n+1}) \\ &\leq 2e[u(t_{n+1}) \log u(t_{n+1})]^{-1} \mu(t_{n+1}) \end{aligned}$$

and our result follows.

In conclusion we remark that it is certainly possible to try to prove theorems like Theorems 1 and 2 for processes other than occupation times. A trivial example is given by a standard one dimensional Brownian motion process X. Let

$$u(t) = 1 + (E\{X(t^2)\})^{1/2} = 1 + t^{1/2}.$$

It follows from the time-change property of the Ito integral that

$$\int_0^t u^{\alpha-1}(s)\, dX(s) = X^*\left(\int_0^t u^{2(\alpha-1)}(s)\, ds\right) \tag{20}$$

where X^* is a standard Brownian motion process. If $\alpha>0$, then (20) implies that

$$\left[\int_0^t u(s)^{\alpha-1}\, du(s)\right]^{-1} \int_0^t u(s)^{\alpha-1}\, dX(s)$$

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converges to a normal distribution of mean zero and variance α. If $\alpha=0$, then the strong law of large numbers implies that

$$\lim_{t\to\infty}\left[\int_0^t u(r)^{-1}\,du(r)\right]^{-1}\int_0^t u(r)^{-1}\,dX(r) = 0 \text{ a.e.}$$

For a further refinement in this case, note that if $\ell(t)=t^{1/\log\log t}$ for sufficiently large t, then the law of the iterated logarithm and some straight forward asymptotic estimates imply that the quantity

$$\left[\int_0^t u^{-1}(s)\ell(s)\,du(s)\right]^{-1}\int_0^t u^{-1}(s)\ell(s)\,dX(s)$$

converges to zero in distribution but has the cluster set $[-2, 2]$ on a.e. path.

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