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ON INTERPOLATION POLYNOMIALS OF THE HERMITE-FEJÉR TYPE II

S.J. GOODENOUGH AND T.M. MILLS

Given a real-valued function f on $[-1, 1]$, $n \in \mathbb{N}$, and the following partition of $[-1, 1]$:

$$-1 < x_n < x_{n-1} < \ldots < x_k := \cos\left((2k-1)\pi/2n\right) < \ldots < x_1 < 1 ,$$

there exists a unique polynomial $R_{4n-1}(f; x)$ of degree not exceeding $4n - 1$ such that

$$R_{4n-1}\left(f; x_k\right) = f\left(x_k\right) \quad \text{for} \quad k = 1, 2, 3, \ldots, n$$

and, for $j = 1, 2$ and 3,

$$R_{4n-1}^{(j)}\left(f; x_k\right) = 0 \quad \text{for} \quad k = 1, 2, 3, \ldots, n .$$

The polynomials $R_{4n-1}(f; x)$ for $n = 1, 2, 3, \ldots$ are called Hermite-Fejér type interpolation polynomials based on the zeros of $T_n(x) := \cos\left(n \cos^{-1}x\right)$.

In this article, the authors first outline the development of results pertaining to the (rate of) convergence of Hermite-Fejér type interpolation polynomials. They then extend this development by deriving a new pointwise error estimate. This estimate represents a marked improvement on all previous error estimates in that it reflects the interpolation conditions.

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Introduction

Let f be a real-valued function defined on $[-1, 1]$ and, for $k = 1, 2, 3, \ldots, n$, denote by

(1) $$x_k = \cos\big((2k-1)\pi/2n\big)$$

the zeros of the Chebyshev polynomial of the first kind

(2) $$T_n(x) = \cos n\theta \quad \text{where} \quad -1 \le x = \cos\theta \le 1 .$$

Then there exists a unique polynomial $R_{4n-1}(f; x)$ of degree at most $4n - 1$ satisfying the following conditions:

$$R_{4n-1}\big(f; x_k\big) = f\big(x_k\big) \quad \text{for} \quad k = 1, 2, 3, \ldots, n$$

and, for $j = 1, 2$ and 3,

$$R^{(j)}_{4n-1}\big(f; x_k\big) = 0 \quad \text{for} \quad k = 1, 2, 3, \ldots, n .$$

Henceforth, we shall refer to the polynomials $R_{4n-1}(f; x)$ as *Hermite-Fejér type interpolation polynomials*. These are so named due to their close connection with the Hermite-Fejér interpolation polynomials used by Fejér [*1*] in 1916 in presenting a new proof of the celebrated Weierstrass approximation theorem.

Indeed, the Hermite-Fejér type interpolation polynomials were first introduced by Krylov and Steuermann [*5*] (in 1922), who stated the following

THEOREM 1 [Krylov and Steuermann]. *If* $f \in C[-1, 1]$, *then*

$$\lim_{n\to\infty} \|R_{4n-1}(f)-f\|_\infty = 0 ,$$

where $\|\cdot\|_\infty$ *denotes the uniform norm on* $C[-1, 1]$.

Unfortunately, the proof given by Krylov and Steuermann was incorrect. A correct proof of this result was later furnished by Laden [*6*]. In 1959, Stancu [*10*] estimated the rate of convergence of the Hermite-Fejér type polynomials $R_{4n-1}(f; x)$, $n = 1, 2, 3, \ldots$ in terms of the modulus of continuity of f, denoted by $\omega(f; \delta)$, where

$$\omega(f; \delta) = \sup\{|f(x)-f(y)| : |x-y| \le \delta,\ x, y \in [-1, 1]\} .$$

THEOREM 2 [Stancu]. *There is a constant* $C_1 > 0$ *such that, if*

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$f \in C[-1, 1]$, *then*

$$\|R_{4n-1}(f)-f\|_\infty \le C_1 \omega(f;\ 1/\sqrt{n}) \quad for \quad n = 1, 2, 3, \ldots .$$

Stancu's result was subsequently improved by Florica [2], who proved the following

THEOREM 3 [Florica]. *There is a constant* $C_2 > 0$ *such that, if* $f \in C[-1, 1]$, *then*

$$\|R_{4n-1}(f)-f\|_\infty \le C_2 \omega\big(f;\ (\log n)/n\big) \quad for \quad n = 2, 3, 4, \ldots .$$

Now suppose Ω is an increasing, subadditive, continuous function defined on $[0, \infty[$ with $\Omega(0) = 0$. Indeed, it may be helpful to think of Ω as a generalised modulus of continuity. Furthermore, for each fixed but arbitrary positive constant M , put

$$C_M(\Omega) := \{f \in C[-1, 1] : \omega(f;\ \delta) \le M\Omega(\delta) \text{ for all } \delta \ge 0\} .$$

One of the authors, Mills [7], focussed attention on $C_1(\Omega)$ to obtain an estimate (for the error incurred in replacing f by $R_{4n-1}(f)$) which is best possible in a certain sense.

THEOREM 4 [Mills]. *There exist positive constants* C_3 *and* C_4 *such that*

$$C_3/n \sum_{k=2}^{n} \Omega(1/k) \le \sup_{f\in C_1(\Omega)} \|R_{4n-1}(f)-f\|_\infty \le C_4/n \sum_{k=1}^{n} \Omega(1/k) \quad for \quad n = 2, 3, 4, \ldots .$$

In 1978, Prasad [8] established the following more general pointwise estimate.

THEOREM 5 [Prasad]. *There is a positive constant* C_5 *such that, if* $f \in C_M(\Omega)$, $n = 2, 3, 4, \ldots$ *and* $x \in [-1, 1]$,

$$|R_{4n-1}(f;\ x)-f(x)| \le C_5 M/n \sum_{k=1}^{n} \Omega\Big(\big(\sqrt{1-x^2}/k\big)+\big(1/k^2\big)\Big) .$$

It should be mentioned that this result was observed independently, but not proved, by Goodenough and Mills [3]. The main purpose of this

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paper is to establish

THEOREM 6. *There exist positive constants* c_1 *and* c_2 *such that, if* $f \in C[-1, 1]$, $n = 2, 3, 4, \ldots$ *and* $x \in [-1, 1]$,

$$(5)\quad \left|R_{4n-1}(f;\, x)-f(x)\right| \le \left(c_1 T_n(x)^4/n\right) \sum_{k=1}^{n} \omega\left(f;\, \left(\sqrt{1-x^2}/k\right)+\left(1/k^2\right)\right) + c_2\omega\left(f;\, \left(\left|T_n(x)\right|/n\right)\right) .$$

Note that the above theorem represents an improvement on Theorem 5 in that it reflects the interpolation conditions. More precisely, if $x = x_k = \cos\left((2k-1)\pi/2n\right)$, then both sides of inequality (5) vanish simultaneously.

Technical preliminaries

In this section, we shall unveil certain technical results which are needed for the proof of Theorem 6. Firstly, the formula for the Hermite-Fejér type interpolation polynomial $R_{4n-1}(f;\, x)$ is given by:

$$(6)\qquad R_{4n-1}(f;\, x) = \sum_{k=1}^{n} f\left(x_k\right)S_k(x)$$

where

$$(7)\qquad S_k(x) = F_k(x) + G_k(x) + H_k(x) ,$$

$$(8)\qquad F_k(x) = \left(1/n^4\right)\left[1-x_k^2\right]\left(1-x^2\right)\left(T_n(x)/\left(x-x_k\right)\right)^4 ,$$

$$(9)\qquad G_k(x) = \left(\left(4n^2-1\right)/6n^4\right)\left(x-x_k\right)^2\left(1-xx_k\right)\left(T_n(x)/\left(x-x_k\right)\right)^4 ,$$

$$(10)\qquad H_k(x) = \left(1/2n^4\right)\left[T_n(x)^2/\left(x-x_k\right)\right]^2$$

and

$$x_k = \cos\left((2k-1)\pi/2n\right) \quad \text{for} \quad k = 1, 2, 3, \ldots, n .$$

We have already reported that if $x = x_k$ for some $k = 1, 2, 3, \ldots, n$, then both sides of the inequality (5) vanish simultaneously. Accordingly, we may suppose that $x \ne x_k$ for

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$k = 1, 2, 3, \ldots, n$. In this case, we define x_j to be the node of interpolation which is closest to x : if there are two such nodes, then choose either one (but not both) to be x_j . Clearly $j = j(n)$, and provided $x \neq 1$, $j \to \infty$ as $n \to \infty$.

The following lemma is due to Kis ([4], p. 30).

LEMMA 1 [Kis]. *For* $-1 \le x = \cos\theta \le 1$,

$$|f(x_k)-f(x)| \le \begin{cases} 2\omega\big(f;\ (\sin\theta)/n\big) + 2\omega\big(f;\ 1/n^2\big) & \text{if } k = j , \\ \\ 5\omega\big(f;\ (i\sin\theta)/n\big) + 13\omega\big(f;\ i^2/n^2\big) & \text{if } i = |k-j| \ge 1 . \end{cases}$$

The next lemma will be needed to establish the term $C_2\omega\big(f;\ |T_n(x)|/n\big)$ which appears as part of the upper estimate for $|R_{4n-1}(f;\ x)-f(x)|$ in Theorem 6.

LEMMA 2. *If* $x = \cos\theta$, $x_k = \cos\theta_k$ *for* $k = 1, 2, \ldots, n$ *and* x_j *is the node closest to* x *, then* $|\theta-\theta_j| \le (\pi/2n)|\cos n\theta|$.

Proof. Suppose for example that $\theta_j < \theta \le \big(\theta_j+\theta_{j+1}\big)/2$. Then the establishment of the required inequality follows from the easy observation that the absolute value of the gradient of the line joining $\big(\theta_j,\ 0\big)$ and $\big(\big(\theta_j+\theta_{j+1}\big)/2,\ \cos n\big(\theta_j+\theta_{j+1}\big)/2\big)$ does not exceed the absolute value of the gradient of the line joining $\big(\theta_j,\ 0\big)$ and $(\theta,\ \cos n\theta)$. Other cases may be treated in a similar vein.

It is debatable whether the following lemma deserves its title. Nonetheless, the elementary inequalities contained therein will smooth the way for proving the all-important Lemma 4.

LEMMA 3. *If* $\alpha, \beta \in [0, \pi]$ *, then*

(i) $0 \le \sin\alpha \le 2\sin\frac{1}{2}(\alpha+\beta)$ *, and*

(ii) $\sin\frac{1}{2}(\alpha+\beta) \ge \sin\frac{1}{2}|\alpha-\beta|$.

LEMMA 4. *For* $k = 1, 2, \ldots, j-1, j+1, \ldots, n$, $F_k(x)$, $G_k(x)$ *and*

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$H_k(x)$ *are all expressible in the form* $O(1)T_n(x)^4/i^2$ *, where* $i = |k-j|$.

Proof. Now

$$F_k(x) = (1/n^4)\left(1-x_k^2\right)\left(1-x^2\right)\left[T_n(x)/\left(x-x_k\right)\right]^4$$
$$= \left\{T_n(x)^4/n^4\right\}\cdot\left\{\sin^2\theta_k/\left[4\sin^2\left(\left(\theta+\theta_k\right)/2\right)\right]\right\}\cdot\left\{\sin^2\theta/\left[4\sin^2\left(\left(\theta+\theta_k\right)/2\right)\right]\right\}$$
$$\cdot\left\{1/\left[\sin^4\left(\left(\theta-\theta_k\right)/2\right)\right]\right\}$$
$$\le \left\{T_n(x)^4/n^4\right\}\cdot\left\{1/\left[\sin^4\left(\left(\theta-\theta_k\right)/2\right)\right]\right\} \quad \text{by Lemma 3.}$$

But

(11) $$\left\{1/\left[\sin^4\left(\left(\theta-\theta_k\right)/2\right)\right]\right\} = O(1)/\left(\theta-\theta_k\right)^4 = O(1)n^4/i^4 \text{ , where } i = |k-j|$$

so it follows that

$$F_k(x) = O(1)T_n(x)^4/i^4 .$$

In particular,

$$F_k(x) = O(1)T_n(x)^4/i^2 .$$

Secondly,

$$G_k(x) = \left(\left(4n^2-1\right)/6n^4\right)\cdot\left(x-x_k\right)^2\left(1-xx_k\right)\left(T_n(x)/\left(x-x_k\right)\right)^4$$
$$< \left\{2T_n(x)^4/3n^2\right\}\cdot\left\{\left[1-x_k^2+x_k^2-xx_k\right]/\left(x-x_k\right)^2\right\}$$
$$= \left\{2T_n(x)^4/3n^2\right\}\cdot\left\{\sin^2\theta_k/\left[4\sin^2\left(\left(\theta+\theta_k\right)/2\right)\right]\right\}\cdot\left\{1/\left[\sin^2\left(\left(\theta-\theta_k\right)/2\right)\right]\right\}$$
$$+ \left\{T_n(x)^4/3n^2\right\}\cdot\left\{\left(\cos\theta_k\right)/\left(\sin\left(\left(\theta+\theta_k\right)/2\right)\sin\left(\left|\theta-\theta_k\right|/2\right)\right)\right\}$$
$$\le \left\{2T_n(x)^4/3n^2\right\}\cdot\left\{1/\left[\sin^2\left(\left(\theta-\theta_k\right)/2\right)\right]\right\}$$
$$+ \left\{T_n(x)^4/3n^2\right\}\cdot\left\{1/\left[\sin^2\left(\left(\theta-\theta_k\right)/2\right)\right]\right\} \quad \text{by Lemma 3}$$
$$\le \left\{2T_n(x)^4c_1/3i^2\right\} + \left\{T_n(x)^4c_1/3i^2\right\} \quad \text{for some absolute constant } c_1 .$$

Thus

$$G_k(x) = O(1)T_n(x)^4/i^2 .$$

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Finally,

$$\begin{aligned} H_k(x) &= (1/2n^4)\left[T_n(x)^2/(x-x_k)\right]^2 \\ &= \left(T_n(x)^4/2n^4\right)\cdot\left(1/\left(4\sin^2((\theta+\theta_k)/2)\sin^2((\theta-\theta_k)/2)\right)\right) \\ &\le \left(T_n(x)^4/8n^4\right)\cdot\left(1/\left(\sin^4((\theta-\theta_k)/2)\right)\right) \quad \text{by Lemma 3 .} \end{aligned}$$

Thus

$$H_k(x) = O(1)T_n(x)^4/i^4 \quad \text{by (11).}$$

In particular,

$$H_k(x) = O(1)T_n(x)^4/i^2$$

and the lemma is proved.

We now have the necessary machinery for the

Proof of Theorem 6

From the definition of $R_{4n-1}(f;\, x)$, we have

$$R_{4n-1}(1;\, x) = \sum_{k=1}^{n} S_k(x) \equiv 1 \text{ .}$$

It follows that

(12) $|R_{4n-1}(f;\, x)-f(x)|$

$$\begin{aligned} &= \left|\sum_{k=1}^{n} |f(x_k)-f(x)|S_k(x)\right| \\ &\le \sum_{k=1}^{n} |f(x_k)-f(x)|S_k(x) \\ &= \sum_{k=1}^{j-1} |f(x_k)-f(x)|S_k(x) + |f(x_j)-f(x)|S_j(x) + \sum_{k=j+1}^{n} |f(x_k)-f(x)|S_k(x) \end{aligned}$$

where, as before, x_j is the node closest to x ,

$$= W_1 + W_2 + W_3 \text{ , say.}$$

We now estimate W_1, W_2 and W_3 in turn.

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Firstly,

$$(13)\quad W_1 = \sum_{k=1}^{j-1} |f(x_k)-f(x)|S_k(x)$$

$$= O(1)T_n(x)^4 \sum_{k=1}^{j-1} (1/i^2)\left[5\omega(f;\ (i \sin \theta)/n)+13\omega(f;\ i^2/n^2)\right]$$

by Lemmas 1 and 4, where $i = j - k$

$$= O(1)\left\{T_n(x)^4/n\right\} \sum_{k=1}^{n} \omega\left(f;\ \left(\sqrt{1-x^2}/k\right)+\left(1/k^2\right)\right)$$

by the methods of Saxena [9].

Similarly we may show that

$$(14)\quad W_3 = O(1)\left\{T_n(x)^4/n\right\} \sum_{k=1}^{n} \omega\left(f;\ \left(\sqrt{1-x^2}/k\right)+\left(1/k^2\right)\right) .$$

It remains only to estimate W_2 . Now

$$(15)\quad \begin{aligned} W_2 &= |f(x_j)-f(x)|S_j(x) \\ &\le |f(x_j)-f(x)| \\ &\le \omega(f;\ |x_j-x|) \\ &\le \omega(f;\ |\theta_j-\theta|) \\ &\le 2\omega(f;\ |T_n(x)|/n) \quad \text{by Lemma 2.} \end{aligned}$$

We conclude that (12), (13), (14) and (15) prove Theorem 6.

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