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Cancellation of Cusp Forms Coefficients over Beatty Sequences on GL(m)

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Abstract. Let $A(n_1, n_2, \dots, n_{m-1})$ be the normalized Fourier coefficients of a Maass cusp form on GL(m). In this paper, we study the cancellation of $A(n_1, n_2, \dots, n_{m-1})$ over Beatty sequences.

1 Introduction

The size and oscillations of the Fourier coefficients of cusp forms on GL(m) have long been objects of special interest in the literature. In the case $m = 2$, these are investigated intensively when $a_f(n)$ are the normalized Fourier coefficients of a holomorphic cusp form or Maass cusp form on the upper half-plane. The Ramanujan–Petersson conjecture, proved by Deligne [4] for holomorphic cusp forms, states that $a_f(n) \ll n^{\varepsilon}$, for any $\varepsilon > 0$. In the holomorphic cusp form case, Good [7] showed that, for any $\varepsilon > 0$,

$$\sum_{n \leqslant X} a_f(n) \ll_{f,\varepsilon} X^{\frac{1}{3}+\varepsilon}.$$

Recently, Blomer [3] and Lao [11] also studied the cancellation behavior of $a_f(n)$ over special sequences. In the Maass cusp form case, Hafner [8] proved that

$$\sum_{n \leqslant X} a_f(n) e(\vartheta n) \ll_{f,\varepsilon} X^{\frac{1}{2}+\varepsilon}, \quad e(z) = e^{2\pi i z} \tag{1.1}$$

for any $\varepsilon > 0$ and any $\vartheta \in \mathbb{R}$.

In contrast, much less is known about the Fourier coefficients for cusp forms of higher rank. Let us recall some background on Maass cusp forms for GL(m), $m \geqslant 2$. We will follow the notations in Goldfeld [6] (see also Bump [2]). Let f be a Maass cusp form of type ν for SL($m, \mathbb{Z}$) with the Fourier–Whittaker expansion

$$f(z) = \sum_{\gamma \in U_{m-1} \backslash \mathrm{SL}_{m-1}} \sum_{n_1=1}^{\infty} \cdots \sum_{n_{m-2}=1}^{\infty} \sum_{n_{m-1} \neq 0} \frac{A(n_1, \dots, n_{m-1})}{\prod_{k=1}^{m-1} |n_k|^{\frac{k(m-k)}{2}}} W_J\left(M \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} z, \nu, \psi_{1,\dots,1} \right),$$

where $\mathrm{SL}_{m-1} = \mathrm{SL}(m-1, \mathbb{Z})$, $U_{m-1} = U_{m-1}(\mathbb{Z})$ is the group of

$$(m-1) \times (m-1)$$

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upper triangular matrices with integer entries and ones on the diagonal,

$$M = \operatorname{diag}(n_1 \cdots n_{m-2}|n_{m-1}|, \dots, n_1, 1)$$

and $W_J(z, \nu, \psi_{1,\dots,1})$ is the Jacquet–Whittaker function. It is well known that the Fourier coefficients are bounded (see [6, Lemma 9.1.3]), *i.e.*,

$$A(n_1, \dots, n_{m-1}) = O\Big(\prod_{k=1}^{m-1} |n_k|^{\frac{k(m-k)}{2}}\Big).$$

Thanks to the Rankin–Selberg theory, we know that the $A(n_1, \dots, n_{m-1})$ obey the Rananujan–Petersson conjecture on average. More precisely, the Rankin–Selberg L-function

$$L_{f\times f}(s) = \zeta(ms) \sum_{n_1=1}^{\infty} \cdots \sum_{n_{m-1}=1}^{\infty} \frac{|A(n_1, \dots, n_{m-1})|^2}{(n_1^{m-1} n_2^{m-2} \cdots n_{m-1})^s}$$

initially convergent for $\operatorname{Re} s$ large, has a meromorphic continuation to the whole plane with the only simple pole at $s = 1$. One can show that

$$\sum_{n_1^{m-1} n_2^{m-2} \cdots n_{m-1} \leqslant X} |A(n_1, \dots, n_{m-1})|^2 \ll_f X;$$

thus,

$$\sum_{n_{m-1} \leqslant X} |A(n_1, \dots, n_{m-1})|^2 \ll_{f, n_1, \dots, n_{m-2}} X. \tag{1.2}$$

In this paper, we are concerned with the cancellation of $A(n_1, \dots, n_{m-1})$ over Beatty sequences, *i.e.*,

$$S(\alpha, \beta, X) = \sum_{n \leqslant X} A(n_1, \dots, n_{m-2}, [\alpha n + \beta]),$$

where $\alpha, \beta \in \mathbb{R}$, $[x]$ denotes the greatest integer not exceeding x, and the so-called Beatty sequence is the sequence of integers defined by $\mathcal{B}_{\alpha,\beta} = \{[\alpha n + \beta]\}_{n=1}^{\infty}$. By the methods in [1, 11] we are able to establish the following.

Theorem 1.1 *Let $\alpha > 1$ be of type 1. Assume that*

$$\sum_{n_{m-1} \leqslant X} A(n_1, \dots, n_{m-1}) e(n_{m-1}\vartheta) \ll_{f,\varepsilon} X^{\theta+\varepsilon}, \quad \theta < 1, \tag{1.3}$$

for any $\varepsilon > 0$ and any $\vartheta \in \mathbb{R}$. Then we have

$$S(\alpha, \beta, X) = \sum_{n \leqslant X} A(n_1, \dots, n_{m-2}, [\alpha n + \beta]) \ll_{f,\alpha,\varepsilon,n_1,\dots,n_{m-2}} X^{\theta+\varepsilon}$$

uniformly in $\beta \in \mathbb{R}$.

Theorem 1.1 tells us that the cancellation behavior of cusp form coefficients over Beatty sequences is nearly the same as that in linear exponential sums. By Theorem 1.1 and (1.1), we easily obtain the following corollary.

Corallary 1.2 *Let $\alpha > 1$ be of type* 1*. Let $a_f(n)$ be the Fourier coefficients of a Maass cusp form f on* GL(2)*. Then we have*

$$\sum_{n \leqslant X} a_f([\alpha n + \beta]) \ll_{f,\alpha,\varepsilon} X^{1/2+\varepsilon}$$

uniformly in $\beta \in \mathbb{R}$.

Recently, Miller [12] established the nontrivial bound

$$\sum_{n_2 \leqslant X} A(n_1, n_2) e(n_2 \vartheta) \ll_{f,\varepsilon,n_1} X^{3/4+\varepsilon},$$

for any $\varepsilon > 0$ and any $\vartheta \in \mathbb{R}$. We then have the following corollary.

Corallary 1.3 *Let $\alpha > 1$ be of type* 1*. Let $A(n_1, n_2)$ be the Fourier coefficients of a Maass cusp form f on* GL(3)*. Then we have*

$$\sum_{n \leqslant X} A\left(n_1, [\alpha n + \beta]\right) \ll_{f,\alpha,\varepsilon,n_1} X^{3/4+\varepsilon}$$

uniformly in $\beta \in \mathbb{R}$.

Remark. By Theorem 1.1, we see that the upper bound of $S(\alpha, \beta, X)$ depends on the type of α. Recall that ([9, p. 121] for more details) the irrational number α is said to be of *type* η if η is the supremum of all ι for which $\liminf_{q\to\infty} q^{\iota} \|q\alpha\| = 0$, where q runs through the positive integers. The celebrated theorems of Khintchine (see, for example, [10, p. 23]) and Roth [13] tell us that almost all $\alpha \in \mathbb{R}$ and all algebraic irrational numbers are of type 1.

2 Preliminaries

In order to prove the theorem, we need the definition of the discrepancy. Suppose that we are given a sequence $u_n, n = 1, 2, \ldots, N$ of points of $\mathbb{R}/\mathbb{Z}$. Then the *discrepancy* $D(N)$ of the sequence is

$$D(N) = \sup_{\mathcal{I} \in [0,1)} \left| \frac{\mathcal{V}(\mathcal{I}, N)}{N} - |\mathcal{I}| \right|, \tag{2.1}$$

where the supremum is taken over all subintervals $\mathcal{I} = (c, d)$ of the interval $[0, 1)$, $\mathcal{V}(\mathcal{I}, N)$ is the number of positive integers $n \leqslant N$ such that $u_n \in \mathcal{I}$ and $|\mathcal{I}| = d - c$ is the length of the interval $\mathcal{I}$.

Now let $D_{\alpha,\beta}(N)$ denote the discrepancy of the sequence $\{\alpha n + \beta\}, n = 1, \ldots, N$, where $\{x\} = x - [x]$. The following result is Theorem 3.2 in [9].

Lemma 2.1 *Let α be of type 1. Then for all $\beta \in \mathbb{R}$, we have $D_{\alpha,\beta} \ll_\alpha N^{-1}$.*

We also need the following elementary result (see [1, Lemma 3.4]).

Lemma 2.2 *Let $\alpha > 1$. An integer n' has the form $n' = [\alpha n + \beta]$ if and only if*

$$0 < \{\alpha^{-1}(n' - \beta + 1)\} \leqslant \alpha^{-1}.$$

Let $0 < \gamma < 1$ and

$$\psi(x) = \begin{cases} 1 & \text{if } 0 < x \leqslant \gamma, \\ 0 & \text{if } \gamma < x \leqslant 1. \end{cases} \tag{2.2}$$

Then we have (see [14, Chapter 2, Lemma 2]) the following lemma.

Lemma 2.3 *For any $\triangle \in \mathbb{R}$ such that $0 < \triangle < \frac{1}{8}$ and $\triangle \leqslant \frac{1}{2}\min\{\gamma, 1-\gamma\}$, there exists a periodic function $\psi_\triangle(x)$ of period 1 satisfying the following properties:*

(i) $0 \leqslant \psi_\triangle(x) \leqslant 1$ *for all* $x \in \mathbb{R}$;
(ii) $\psi_\triangle(x) = \psi(x)$ *if* $\triangle \leqslant x \leqslant \gamma - \triangle$ *or* $\gamma + \triangle \leqslant x \leqslant 1 - \triangle$;
(iii) $\psi_\triangle(x)$ *can be represented as a Fourier series*

$$\psi_\triangle(x) = \gamma + \sum_{j=1}^{\infty} (g_j e(jx) + h_j e(-jx)),$$

where the coefficients g_j and h_j satisfy the uniform bound

$$\max\{|g_j|, |h_j|\} \ll \min\{j^{-1}, j^{-2}\triangle^{-1}\}, \qquad (j \geqslant 1).$$

3 Proof of Theorem 1.1

For brevity, we denote $A(n_{m-1}) := A(n_1, \dots, n_{m-1})$. Let $\alpha > 1$ be of type 1 and $\gamma = \alpha^{-1}$. Put $\delta = \alpha^{-1}(1 - \beta)$, $N_0 = [\alpha + \beta - 1]$ and $N = [\alpha X + \beta]$. Then by Lemma 2.2 and the definition of $\psi(x)$ in (2.2) , we have

$$S(\alpha, \beta, X) = \sum_{n \leqslant X} A([\alpha n + \beta]) = \sum_{\substack{N_0 \leqslant n \leqslant N \\ 0 < \{\gamma n + \delta\} \leqslant \gamma}} A(n) = \sum_{N_0 \leqslant n \leqslant N} A(n)\psi(\gamma n + \delta).$$

Applying Lemma 2.3, we obtain

$$S(\alpha, \beta, X) = \sum_{N_0 \leqslant n \leqslant N} A(n)\psi_\triangle(\gamma n + \delta) + O\Big(\sum_{\substack{N_0 \leqslant n \leqslant N \\ \{\gamma n + \delta\} \in \mathcal{I}}} |A(n)| \Big), \tag{3.1}$$

where $\mathcal{I} = [0, \triangle) \cup (\gamma - \triangle, \gamma + \triangle) \cup (1 - \triangle, 1)$ with $|\mathcal{I}| \ll \triangle$.

Let $\mathcal{V}(\mathcal{I}, N_0, N) = \sharp\{N_0 \leqslant n \leqslant N \mid \{\gamma n + \delta\} \in \mathcal{I}\}$. Then by (2.1) and Lemma 2.1, we have $\mathcal{V}(\mathcal{I}, N_0, N) \ll_\alpha 1 + |\mathcal{I}|X \ll_\alpha 1 + \triangle X$. Thus, by Cauchy's inequality and (1.2), we see that the O-term in (3.1) is bounded by

$$\ll \Big(\sum_{N_0 \leqslant n \leqslant N} |A(n)|^2 \Big)^{1/2} \Big(\sum_{\substack{N_0 \leqslant n \leqslant N \\ \{\gamma n + \delta\} \in \mathcal{I}}} 1 \Big)^{1/2} \ll_{f,\alpha,n_1,\dots,n_{m-2}} X^{1/2}(1 + \triangle X)^{1/2}.$$

Therefore, by Lemma 2.3 (iii) and (1.3), we deduce that

$$\begin{aligned} S(\alpha, \beta, X) &= \sum_{N_0 \leqslant n \leqslant N} A(n)\psi_\triangle(\gamma n + \delta) + O_{f,\alpha,n_1,\dots,n_{m-2}}(X^{1/2}(1 + \triangle X)^{1/2}) \\ &= \gamma \sum_{N_0 \leqslant n \leqslant N} A(n) + \sum_{j=1}^{\infty} g_j e(\delta j) \sum_{N_0 \leqslant n \leqslant N} A(n) e(\gamma j n) \\ &\quad + \sum_{j=1}^{\infty} h_j e(-\delta j) \sum_{N_0 \leqslant n \leqslant N} A(n) e(-\gamma j n) + O_{f,\alpha,n_1,\dots,n_{m-2}}(X^{1/2}(1 + \triangle X)^{1/2}) \\ &\ll_{f,\alpha,\varepsilon,n_1,\dots,n_{m-2}} X^{\theta+\varepsilon} + X^{\theta+\varepsilon}\Big\{ \sum_{j \leqslant \triangle^{-1}} j^{-1} + \sum_{j > \triangle^{-1}} \triangle^{-1} j^{-2} \Big\} + \triangle^{1/2} X \\ &\ll_{f,\alpha,\varepsilon,n_1,\dots,n_{m-2}} X^{\theta+\varepsilon} + X^{\theta+\varepsilon} \log \triangle^{-1} + \triangle^{1/2} X. \end{aligned}$$

On taking $\triangle = X^{2(\theta-1)}$, we conclude that $S(\alpha, \beta, X) \ll_{f,\alpha,\varepsilon,n_1,\dots,n_{m-2}} X^{\theta+\varepsilon}$. This finishes the proof of Theorem 1.1.

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