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Eisenstein Series Arising from Jordan Algebras

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Abstract. We describe poles and the corresponding residual automorphic representations of Eisenstein series attached to maximal parabolic subgroups whose unipotent radicals admit Jordan algebra structure.

1 Introduction

Let G be a simple, simply connected algebraic group defined over a global field k. Assume that G has a maximal parabolic subgroup $P = MN$ such that N is abelian and P is conjugated to the opposite parabolic $\overline{P} = M\overline{N}$. Then N admits structure of a Jordan algebra $(J, \circ)$. The main goal of this article is to study poles and residues of the degenerate Eisenstein series $E(s)$ attached to the parabolic P under an additional assumption that the algebra identity element $e \in J$ can be written as a sum $e = e_1 + \cdots + e_r$, for a system of perpendicular and absolutely indecomposable idempotent elements e_i. This assumption allows us to use the technique of Fourier–Jacobi series, due to Ikeda [11], and build an argument inductive on r. Examples of such Jordan algebras are $J_r(D)$, the algebras of $r \times r$ hermitian symmetric matrices with coefficients in a composition algebra D over k. In addition, for $r = 2$, there is a class of Jordan algebras $J_2(D)$ parameterized by quadratic spaces D over k. Let d denote the dimension of D. In order to understand the structure of residual automorphic representations, it is necessary to understand the structure of local degenerate principal series representations $I(s)$ attached to P at reducibility points. For real groups, in the setting of this paper, this was accomplished by Sahi in [20, 21]. On the other hand, for p-adic groups, Weissman [23] analyzed the structure of the degenerate principal series representations using a Fourier–Jacobi functor. In a nutshell, this method is a local analogue of Ikeda's method.

More precisely, the contents of this paper are as follows. In Section 2 we describe the groups and related Jordan algebras. Sections 3–5 are devoted to local results. Weissman looks only at the case of split, simply laced groups, so in Section 3 we generalize his results to non-split groups. In Section 5 we summarize the results of Sahi in the real case. In order to keep the exposition simple, we assume here that D is either split or totally anisotropic and $d \equiv 0 \pmod 4$. Section 6 is devoted to global results. The local Fourier–Jacobi functor works well with Ikeda's method, and we combine

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the two to obtain sharper results. Our final result is a complete description of poles and the corresponding residual representations in a natural family of cases that does not exhaust all possible cases that can be addressed by the methods of this paper. If we assume that $d \equiv 0 \pmod 4$ and some additional minor assumptions that are automatically satisfied if $r > 2$, then the Eisenstein series $E(s)$ has simple poles at the sequence of odd integers $1, 1 + d/2, \dots, 1 + (r-1)d/2$, and the residual representation is isomorphic to the co-socle of the global induced representation $I(s)$ at the same points.

Study of Eisenstein series attached to degenerate principal series has a long history, often intertwined with the classical theta correspondences and the Siegel–Weil formula ([15,16]). In particular, Ikeda's work deals with symplectic and unitary groups, which, in the language of this paper, are the cases $J = J_r(D)$ where $D = k$ or K, a quadratic extension of k. A different, more combinatorial approach is taken in [8–10]. Yamana [24] has taken Ikeda's work further, to quaternionic groups. This works goes beyond the confines of classical groups and is motivated by a Siegel–Weil formula in the setting of exceptional theta correspondences, a work in progress of the second author with Wee Teck Gan.

2 Groups

Following [14], we will describe the groups G and Jordan algebras appearing in this paper, starting with split groups. The general case is obtained by Galois descent.

2.1 Split Groups and Algebras

So assume that G is split *i.e.,* it is a Chevalley group. Let $\mathfrak{g}$ be the Lie algebra of G and let Φ be the root system arising from a maximal split Cartan subalgebra $\mathfrak{t} \subseteq \mathfrak{g}$. In particular, for every $\alpha \in \Phi$, we have the corresponding root space $\mathfrak{g}_\alpha \subseteq \mathfrak{g}$. Fix $\Delta = \{\alpha_1, \dots, \alpha_l\}$, a set of simple roots. Now every root can be written as a sum $\alpha = \sum_{i=0}^{l} m_i(\alpha)\alpha_i$ for some integers $m_i(\alpha)$. Every simple root α_j defines a maximal parabolic subalgebra $\mathfrak{p} = \mathfrak{p}_j = \mathfrak{m} \oplus \mathfrak{n}$ where the nilpotent radical $\mathfrak{n}$ is the direct sum of $\mathfrak{g}_\alpha$ such that $m_j(\alpha) > 0$. Let β be the highest root. The algebra $\mathfrak{n}$ is commutative if and only if $m_j(\beta) = 1$. In the following table we list of all possible pairs $(\mathfrak{g}, \mathfrak{m})$ with $\mathfrak{n}$ commutative and $\mathfrak{p}$ conjugate to the opposite parabolic by an element in G.

$\mathfrak{g}$	C_n	A_{2n-1}	D_{2n}	E_7	B_{n+1}	D_{n+1}
$\mathfrak{m}^{\text{der}}$	A_{n-1}	$A_{n-1} \times A_{n-1}$	A_{2n-1}	E_6	B_n	D_n
$\dim \mathfrak{n}$	$n(n+1)/2$	n^2	$n(2n-1)$	27	$2n+1$	$2n$
r	n	n	n	3	2	2
d	1	2	4	8	$2n-1$	$2n-2$

The integers r and d are invariants of the (split) Jordan algebra structure $(J, \circ)$ on $\mathfrak{n}$ that we will describe in a moment. Before that, we recall the definition of Jordan algebra structure (*cf.* [12, p. 6]). A Jordan algebra J is an algebra (not necessarily associative) over a field F of characteristic different from 2 in which the multiplication,

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denoted by $\circ$, satisfies the following identities for each $a, b \in J$:

$$a \circ b = b \circ a,$$
$$\big((a \circ a) \circ b\big) \circ a = (a \circ a) \circ (b \circ a).$$

Now we return to the table above. The integer r is the cardinality of any maximal set $S = \{\beta_1, \dots, \beta_r\}$ of strongly orthogonal roots α such that $\mathfrak{g}_\alpha \subseteq \mathfrak{n}$. Observe that $r = 1$ if and only if $G = \mathrm{SL}_2$. For every $\beta_i \in S$, take an $\mathfrak{sl}_2$-triple (f_i, h_i, e_i) where $e_i \in \mathfrak{g}_{\beta_i}$ and $f_i \in \mathfrak{g}_{-\beta_i}$. Let

$$f = \sum_{i=1}^r f_i, \quad h = \sum_{i=1}^r h_i, \quad \text{and} \quad e = \sum_{i=1}^r e_i.$$

Since the roots β_i are strongly orthogonal, (f, h, e) is also an $\mathfrak{sl}_2$-triple. The semi-simple element h preserves the decomposition $\mathfrak{g} = \overline{\mathfrak{n}} \oplus \mathfrak{m} \oplus \mathfrak{n}$. More precisely, $[h, x] = -2x$ for all $x \in \overline{\mathfrak{n}}$, $[h, x] = 0$ for all $x \in \mathfrak{m}$, and $[h, x] = 2x$ for all $x \in \mathfrak{n}$. The triple (f, h, e) lifts to a homomorphism $\varphi\colon \mathrm{SL}_2 \to G$. The element

$$w_0 = \varphi\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2.1}$$

normalizes M and conjugates $\mathfrak{n}$ into $\overline{\mathfrak{n}}$, and vice versa. The Jordan algebra multiplication $\circ$ on $J = \mathfrak{n}$ is defined by

$$x \circ y = \frac{1}{2}\big[x, [f, y]\big].$$

Note that e is the identity element. The elements e_i are mutually perpendicular ($e_i \circ e_j = 0$ if $i \neq j$) and idempotent ($e_i \circ e_i = e_i$) elements in J such that $e_1 + \cdots + e_r = e$. These idempotent elements give a Pierce decomposition of J,

$$J = \bigoplus_{1 \le i \le r} J_{ii} \oplus \bigoplus_{1 \le i < j \le r} J_{ij},$$

where

$$J_{ii} = \{x \in J \mid e_i \circ x = x\},$$
$$J_{ij} = \big\{x \in J \mid e_i \circ x = \tfrac{1}{2}x \text{ and } e_j \circ x = \tfrac{1}{2}x\big\}.$$

The space J_{ii} is one-dimensional and spanned by e_i. The dimension of J_{ij}, for $i < j$, is d. It is independent of $i < j$. Let $D = J_{12}$. Then D is a quadratic space with a split quadratic form

$$q(x) = \tfrac{1}{2}\kappa\big([f_1, x], [f_2, x]\big),$$

where $\kappa(\cdot, \cdot)$ is the Killing form normalized by $\kappa(f_1, e_1) = 1$. If $r > 2$, then one can identify all J_{ij} with D and, using $J_{12} \circ J_{23} \subset J_{23}$, endow D with a multiplication such that $q(xy) = q(x)q(y)$, for all $x, y \in D$, *i.e.*, D is a composition algebra.

Each triple (f_i, h_i, e_i) lifts to a homomorphism of algebraic groups $\varphi_i\colon \mathrm{SL}_2 \to G$. By restricting φ_i to the torus of diagonal matrices in SL_2, we obtain a homomorphism (a co-character) $\omega_i^\vee\colon \mathbb{G}_m \to M$,

$$\omega_i^\vee(t) = \varphi_i\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}. \tag{2.2}$$

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Let $T_r \subseteq M$ be the torus generated by all $\omega_i^\vee(t)$. Any element in $T_r(k)$ is uniquely written as a product of $\omega_i^\vee(t_i)$ for some $t_i \in k^\times$. One checks that the restricted root system with respect to T_r is of the type C_r. Since G is simply connected, the group of characters $\mathrm{Hom}(M, \mathbb{G}_m) \cong \mathbb{Z}$ has a canonical generator ω, which, when restricted to the torus T, is the fundamental weight corresponding to the simple root α_j, defining M. Moreover, the kernel of ω is M^{der}, the derived group of M. A simple computation with the root data shows that

$$\omega\big(\,\omega_i^\vee(t)\big) = t. \tag{2.3}$$

2.2 Fourier–Jacobi Tower

Let $Q = LV$ be the standard parabolic subgroup of G such that the roots of L^{der} are perpendicular to the highest root β. In other words, the Lie algebra of L^{der} is the centralizer in $\mathfrak{g}$ of the $\mathfrak{sl}_2$-triple corresponding to β. The unipotent radical V is a Heisenberg group with the center $Z = \exp(\mathfrak{g}_\beta)$. The group L^{der} is semi-simple and, by inspection, has a unique simple factor G_1 that is not contained in M. Let $M_1 = G_1 \cap M$ and $N_1 = G_1 \cap N$. Then $P_1 = M_1 N_1$ is maximal parabolic subgroup of G_1. The unipotent radical N_1 has a Jordan algebra J_1 structure that is easily related to J. Indeed, if we pick the set $S = \{\beta_1, \dots, \beta_r\}$ of strongly orthogonal roots such that $\beta_1 = \beta$, then J_1 is the sum of the pieces in the Pierce decomposition where all the indexes are greater than 1. This process can be continued, and will give a sequence of simple groups $G, G_1, \dots, G_{r-1} \cong \mathrm{SL}_2$ and maximal parabolic groups with unipotent radicals $N, N_1, \dots, N_{r-1} \cong k$. Moreover, this process gives us a canonical choice of S such that β_i is the the highest root of G_{i-1}. This sequence is the Fourier–Jacobi tower referred to in the title, and the last group SL_2 is the terminal group. This process is summarized by the following table:

$\mathfrak{g}$	C_n	A_{2n-1}	D_{2n}	E_7	B_{n+1}	D_{n+1}
$\mathfrak{m}^{\mathrm{der}}$	A_{n-1}	$A_{n-1} \times A_{n-1}$	A_{2n-1}	E_6	B_n	D_n
$\mathfrak{l}^{\mathrm{der}}$	C_{n-1}	A_{2n-3}	$A_1 \times D_{2n-2}$	D_6	$A_1 \times B_{n-1}$	$A_1 \times D_{n-1}$
$\mathfrak{g}_1$	C_{n-1}	A_{2n-3}	D_{2n-2}	D_6	A_1	A_1
$\mathfrak{m}_1^{\mathrm{der}}$	A_{n-2}	$A_{n-2} \times A_{n-2}$	A_{2n-3}	D_5	0	0

We need the following remark. Let $\varphi\colon \mathrm{SL}_2 \to G$ arising from this S. Then w_0, defined by the equation (2.1), permutes the simple roots of M.

2.3 Non-split Groups

In [14] it was proved that the centralizer of $\varphi(\mathrm{SL}_2)$ in $\mathrm{Aut}(G)$ is precisely $\mathrm{Aut}(J)$. Therefore, by functoriality of Galois cohomology, a class $c \in H^1(k, \mathrm{Aut}(J))$ defines a class in $H^1(k, \mathrm{Aut}(G))$. Hence, the class c defines a Jordan algebra J_c, a form of J, and a form G_c of G whose Lie algebra contains the triple (f, h, e). Hence, G_c contains a form P_c of P whose unipotent radical is isomorphic to J_c. Moreover, if the form J_c arises from a form of D, *i.e.*, J_c contains the absolutely indecomposable idempotents e_i, then the Lie algebra of G_c contains the triples (f_i, h_i, e_i), and G_c contains the split torus T_r, defined above. This torus is maximal if D is anisotropic.

Henceforth, we shall omit the subscript c, and G will denote a group arising from a Jordan algebra $J \cong J_r(D)$, where D is a composition algebra if $r \geq 3$, and simply a quadratic space if $r = 2$. In particular, G contains the maximal parabolic subgroup $P = MN$ such that $N \cong J$, and the Heisenberg parabolic subgroup $Q = LV$, such that the center of V is $Z \cong J_{11}$, and G is the first term of a Fourier–Jacobi tower where the next group is G_1 with the the maximal parabolic subgroup $P_1 = M_1N_1$ such that $N_1 \cong J_1$, where $J_1 \cong J_{r-1}(D)$, etc.

3 Representations of p-adic Groups

In this section, k is a p-adic field. The goal of this section is to extend the results of Weissman in [23] to non-split groups.

3.1 Fourier–Jacobi Functor

We fix a non-trivial additive character ψ of $Z \cong k$, the center of the unipotent radical V of the parabolic $Q = LV$. Let ω_ψ be the corresponding irreducible representation of V with the central character ψ. Note that $L^{\mathrm{der}} = [L, L]$ (or its 2-fold cover) acts on ω_ψ, via the Weil representation. Let π be a smooth representation of G and $\pi_{Z,\psi}$ the maximal quotient of π on which Z acts as ψ. Then $\pi_{Z,\psi}$ is a multiple of ω_ψ, and

$$FJ(\pi) = \mathrm{Hom}_V(\omega_\psi, \pi_{Z,\psi})$$

is naturally a L^{der}-module. The Fourier–Jacobi functor $\pi \mapsto FJ(\pi)$ is exact [23].

Let ω be the character of M giving the isomorphism of $M/M^{\mathrm{der}} \cong \mathbb{G}_m$ (*cf.* (2.3)). Let χ be a quadratic character and $|\cdot|^s$ the absolute value character, taken to the power $s \in \mathbb{C}$, of $k^\times = \mathbb{G}_m(k)$. We can pull back these two characters to M via ω. Let $I(\chi, s) = \mathrm{Ind}_P^G(\chi \otimes |\cdot|^s)$ be the degenerate principal series representation of G. The modular character ρ_P can be expressed in terms of ω using the relation $\omega(\omega_1^\vee(t)) = t$. The conjugation action of $\omega_1^\vee$ on $N \cong J_r(D)$ is given by multiplication by t^2 on $J_{11} \cong k$, and by t on each $J_{1,i} \cong D$, for $1 < i \leq r$. Thus, $\rho_P(m) = |\omega(m)|^{2+(r-1)d}$, where $d = \dim D$. The trivial representation is a quotient of $I(1, 1 + (r-1)\frac{d}{2})$.

We now compute the action of the Fourier–Jacobi functor on $I(\chi, s)$. In order to state the result, we need some additional data arising from the Weil representation ω_ψ^D of $\mathrm{SL}_2(k)$ on $C_0^\infty(D)$ (*cf.* [22]). For every $t \in k^\times$, let $h(t)$ be the element in the universal central extension of $\mathrm{SL}_2(k)$, introduced by Steinberg, projecting to $\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1}\end{smallmatrix}\right)$. In particular, $h(t)h(s) = h(ts)(t, s)$, where (t, s) is the Steinberg symbol. Then, for every $f \in C_0^\infty(D)$,

$$\omega_\psi^D(h(t))(f)(x) = \chi_D(t)|t|^{d/2} f(tx),$$

where $\chi_D(t)$ is a fourth root of 1. If $\dim D$ is odd, then $\chi_D(t)\chi_D(s) = \chi_D(ts)(t, s)_2$ where $(t, s)_2$ is the Hilbert symbol. If $\dim D = 2n$, then χ_D is independent of ψ. It is a quadratic character of $k^\times$ that corresponds to the quadratic algebra $K = k(\sqrt{\Delta})$, by the local class field theory, where Δ is the discriminant of the quadratic form q on D. More precisely, if $q = a_1x_1^2 + \cdots + a_{2n}x_{2n}^2$, then $\Delta = (-1)^n a_1 \cdot \cdots \cdot a_{2n}$. If (D, q) is a direct sum of n-hyperbolic planes, or if the anisotropic kernel of (D, q) is a quaternion algebra, then χ_D is trivial. However, if the anisotropic kernel of (D,q) is a quadratic field K, then χ_D corresponds to K by the local class field theory.

Theorem 3.1 *Let $I(\chi,s)$ be the degenerate principal series corresponding to the pair (G,P) where the unipotent radical of P is isomorphic to the Jordan algebra $J_r(D)$. Let (G_1,P_1) be the next pair in the Fourier–Jacobi tower. Then*

$$FJ(I(\chi,s)) = I_1(\chi\chi_D,s).$$

Proof This is proved by Weissman in [23] if G is split and simply laced. The proof given there extends easily to the more general class of groups considered here. (Note that G is split and simply laced precisely when D, considered as a quadratic space, is isomorphic to a direct sum of hyperbolic planes.) ∎

3.2 Decomposition of Degenerate Principal Series

Let π be an irreducible representation of G. If π is not the trivial representation, then there exists a non-trivial character ψ of Z such that $\pi_{Z,\psi} \neq 0$. The group L acts on Z by conjugation and, therefore, on non-trivial characters of Z. If this action is transitive, then, without loss of generality, we can fix a non-trivial character ψ of Z, and then π is non-trivial if and only if $FJ(\pi) \neq 0$. In this case Theorem 3.1 can be used to compute points of reducibilities of the degenerate principal series $I(\chi,s)$ using the induction on r.

Since the Pontrjagin dual of Z is isomorphic to Z, over local fields, the group L acts transitively on non-trivial characters if and only if acts transitively on non-trivial elements in Z. This is true over the algebraic closure of k, and it holds over k if $H^1(k,C)$ is trivial, where C is the centralizer of e_1 in L. (We use $Z \cong J_{11} = k \cdot e_1$.) If G is not of the absolute type C_{2r} or A_{2r-1}, then $C = L^{\mathrm{der}}$. Since k is p-adic, the Galois cohomology of simply-connected groups is trivial, and the transitivity holds. It also holds for $G = \mathrm{SL}_{2r}$. Hence it fails only if $G = \mathrm{Sp}_{2r}$ or SU_{2r} *i.e.*, $J = J_r(k)$ or $J_r(K)$, respectively, where K is a quadratic field extension of k. In these two cases, $C/L^{\mathrm{der}} \cong \mu_2$ and K^1, respectively, where K^1 is the group of elements of norm one in $K^\times$. Thus the orbits of non-trivial characters are parameterized by the classes of squares in $k^\times$ and $k^\times/N_{K/k}(K^\times)$, respectively.

Thus, in Theorems 3.2 and 3.3, we can use Theorem 3.1 to get the length of the degenerate principal series.

Theorem 3.2 *Let $I(\chi,s)$ be the principal series of G arising from the maximal parabolic subgroup P whose radical N is isomorphic to the Jordan algebra $J_2(D)$ where* $\dim D > 2$. *Assume that χ is a quadratic character and s real.*

(i) $\dim D = 2n-2$, *and the discriminant of the quadratic form q is trivial,* i.e., $\chi_D = 1$. *If $\chi \neq 1$, then $I(\chi,s)$ is irreducible unless $s = 0$, and then it is a direct sum of two non-isomorphic irreducible representations. If $\chi = 1$, then $I(\chi,s)$ is irreducible unless $s = \pm 1, \pm n$ and then it has a non-split composition series of two non-isomorphic irreducible representations.*

(ii) $\dim D = 2n-2$, *and the discriminant is non-trivial,* i.e., $\chi_D = \chi_K$ *where K is a quadratic field extension of k. If $\chi \neq \chi_K$, then $I(\chi,s)$ is irreducible unless $s = 0$,*

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and then it is a direct sum of two non-isomorphic irreducible representations, or $\chi = 1$ *and* $s = \pm n$ *where the trivial representation occurs. If* $\chi = \chi_K$*, then* $I(\chi, s)$ *is irreducible unless* $s = \pm 1$*, and then it has a non-split composition series of two non-isomorphic irreducible representations.*

(iii) $\dim D = 2n - 1$. *Then* $I(\chi, s)$ *is irreducible unless* $s = \pm 1/2$*, and then it has a non-split composition series of two non-isomorphic irreducible representations, or* $\chi = 1$ *and* $s = \pm(n + 1/2)$ *where the trivial representation occurs.*

Proof We shall use the Fourier–Jacobi functor, note that $G_1 = \mathrm{SL}_2(k)$ (or its two-fold cover) in all three cases, since $r = 2$. Consider the first case. The location of the trivial representation is at $\chi = 1$ and $s = \pm n$, as previously noted. The Fourier–Jacobi functor is exact, takes $I(\chi, s)$ to $I_1(\chi, s)$, and kills only the trivial representation. Hence, if $(\chi, s) \neq (1, \pm n)$, then a non-trivial composition series in $I(\chi, s)$ will give one for $I_1(\chi, s)$. The decomposition of the principal series of $\mathrm{SL}_2(k)$ is well known. It follows that $I(\chi, s)$ reduces (possibly) only for $\chi = 1$ and $s = \pm 1$, or $\chi \neq 1$ and $s = 0$. Irreducibility of $I(0, \chi)$ implies existence of the complementary series, which must end before the points where the trivial representation is contained. This forces reducibility for $\chi \neq 1, s = 0$, and $1, s = \pm 1$. Parts (ii) and (iii) are proved similarly; for (iii) one uses that the principal series of $\widetilde{\mathrm{SL}}_2(k)$ reduces at $s = \pm 1/2$. ■

Theorem 3.3 *Let* $I(\chi, s)$ *be the principal series of* G *arising from the maximal parabolic subgroup* P *whose radical* N *is isomorphic to the Jordan algebra* $J_r(D)$ *such that* $\chi_D = 1$*. Let* $d = \dim D$*. Assume that* χ *is quadratic and* s *real. If* $\chi \neq 1$*, then* $I(\chi, s)$ *is irreducible unless* $s = 0$*, and then it is a direct sum of two non-isomorphic representations. If* $\chi = 1$*, then* $I(\chi, s)$ *is irreducible unless* $s = \pm 1, \pm(1 + d/2), \dots, \pm(1 + (r-1)d/2)$*, and then it has a non-split composition series of two non-isomorphic irreducible representations.*

Proof In view of the previous theorem, we can assume that $r \geq 3$. Hence D is a composition algebra. Moreover, the condition $\chi_D = 1$ implies that D is either split (and even dimensional) or a quaternion algebra. In the former case G is split and simply laced, so this case was covered by Weissman. Hence, it remains to do the case when D is a quaternion algebra. This is proved by induction on r. Assuming the result for $r - 1$, the Fourier–Jacobi functor implies that reducibility points are possibly only those listed, and the length of the composition series is not longer than two. In the next section we will show that, for $\chi = 1$, the spherical representation is a proper subquotient at the indicated points; see Corollary 4.6. ■

4 Intertwining Operators

We collect some facts we need about principal series representations. In this section k is a local field.

4.1 c-function for Split Groups

Here we assume that G is split and simply laced; *i.e.,* D is a sum of hyperbolic planes, and let B be the Borel subgroup, corresponding to our choice of simple roots. Let χ

be an unramified character of T, for w in the Weyl group of G we have standard local intertwining operators $A(\chi,w)$,

$$A(\chi,w)\colon \operatorname{Ind}_B^G(\chi) \longrightarrow \operatorname{Ind}_B^G(w(\chi)).$$

In this definition, the choice of the Haar measure on k is such that, in the p-adic case, the measure of the ring of integers is 1. Let $f_\chi \in I(\chi)$ be the unique spherical vector normalized so that $f_\chi(1) = 1$. Then $A(\chi,w)(f_\chi) = c(\chi,w) f_{w(\chi)}$ where the factor for $c(\chi,w)$ is given by the Gindikin–Karpelevič formula (*cf.* [4] for archimedean fields and [17] for p-adic fields)

$$c(\chi,w) = \prod_{\substack{\alpha\in\Phi^+,\\ w(\alpha)<0}} \frac{L(0,\chi\circ\alpha^\vee)}{L(1,\chi\circ\alpha^\vee)},$$

where $\alpha^\vee$ is the co-root corresponding to a root α and L-functions are Iwasawa–Tate's L-functions. If $\chi\circ\alpha^\vee = |\cdot|^s$, then $L(0,\chi\circ\alpha^\vee) = \zeta(s)$ and $L(1,\chi\circ\alpha^\vee) = \zeta(s+1)$, where

$$\zeta(s) = (1-q^{-s})^{-1}$$

if k is a p-adic field with the residual field of order q, and

$$\zeta(s) = \pi^{-\frac{s}{2}}\Gamma(s/2)$$

if $k \cong \mathbb{R}$. We use this formula to determine the action of the standard intertwining operator $A(s)\colon I(s) \to I(-s)$,

$$A(s)(f)(g) = \int_N f(w_0 ng)\, dn$$

on the spherical vector, where w_0 is the element defined by equation (2.1). It permutes simple roots of M and maps the roots that span N to the roots that span $\overline{N}$. Concretely, it is the product of the longest Weyl group elements of G and M. Let $f_s \in I(s)$ be the spherical vector normalized by $f_s(1) = 1$. Let χ_s be an unramified character of T such that $I(s)$ is a subrepresentation of $\operatorname{Ind}_B^G(\chi_s)$. Then $A(s)(f_s) = c(\chi_s,w_0)f_{-s}$, which reduces the computation to a combinatorial exercise. We summarize the result in the following lemma.

Lemma 4.1 *Let $I(s)$ be the degenerate principal series for split, simply laced G arising from a parabolic $P = MN$ such that $N \cong J_r(D)$. Let $d = \dim D$. Let $f_s \in I(s)$ be the normalized spherical vector, and $c(s)$ the complex function defined by $A(s)(f_s) = c(s)f_{-s}$. Then*

$$c(s) = \prod_{i=0}^{r-1} \frac{\zeta(s-id/2)}{\zeta(s+id/2+1)},$$

where $\zeta(s)$ is the local zeta function as above.

Now one can easily understand the poles and zeroes of $c(s)$, using the poles of $\zeta(s)$. In the p-adic case, for s real, $\zeta(s)$ never vanishes and has a simple pole at $s = 0$. It follows that $c(s)$ has simple poles at $s = 0, d/2, \dots, (r-1)d/2$ and simple zeros at $s = -1, -1-d/2, \dots, -1-(r-1)d/2$. Thus, at these points, $I(s)$ has a composition series of length two and the spherical representation is a unique irreducible submodule. In the real case, $\zeta(s)$ has a simple pole at negative even integers. Thus, if d is divisible by 4, $c(s)$ has zeros at negative odd integers and poles at even positive integers. We summarize with the following corollary.

Corollary 4.2 *If G is split, the local field k is p-adic or real, and $d \equiv 0 \pmod 4$ then $c(s)$ is not vanishing at odd positive integers, in particular, at*

$$s = 1, 1 + d/2, \ldots, 1 + (r-1)d/2.$$

For split p-adic groups, the Satake parameters of the irreducible spherical quotients, at positive reducibility points, have a nice description. As previously, let $S = \{\beta_1, \ldots, \beta_r\}$ be a maximal set of strongly orthogonal roots spanning $\overline{N}$. Let $\varphi_i\colon \mathrm{SL}_2 \to G$ be the homomorphism corresponding to β_i, for every i. For $j = 2, \ldots, r$, let $\psi_j\colon \mathrm{SL}_2 \to G$, be the homomorphism given by the product of $\varphi_1, \ldots, \varphi_{j-1}$, and ψ_1 is the trivial homomorphism. (The actual choice of S is not important, since for different choices of S, resulting ψ_j are G-conjugated.) The corresponding unipotent class in G is $(j-1)A_1$ in the Bala–Carter notation. Let $\widehat{G}(\mathbb{C})$ be the Langlands dual group. Let $\widehat{\psi}_j\colon \mathrm{SL}_2 \to \widehat{G}(\mathbb{C})$ be the homomorphism that corresponds to ψ_j via the Spaltenstein order reversing map from unipotent orbits of G to unipotent orbits of $\widehat{G}(\mathbb{C})$ [2]. Then the Satake parameter of the spherical quotient of $I(s)$ at the reducibility point $s_0 = 1 + (r-j)d/2$ is

$$\widehat{\psi}_j \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}.$$

If $d \equiv 0 \pmod 4$, then each $\widehat{\psi}_j$ corresponds to a distinguished unipotent orbit in $\widehat{G}(\mathbb{C})$ *i.e.,* one that it is not contained in a proper Levi. It follows that the Aubert dual of the spherical representation is a square integrable representation. For the definition of the Aubert duality, we refer the reader to [1]; for the unitarity of the Aubert duals of square-integrable representations of classical groups, we refer the reader to [6, 7, 18]. We record the following proposition.

Proposition 4.3 *If G is split, the local field k is p-adic or real, and $d \equiv 0 \pmod 4$ then the spherical quotients at $s = 1, 1+d/2, \ldots, 1+(r-1)d/2$ are Aubert duals of square integrable representations.*

4.2 c-function for Non-split Groups

In this section k is a p-adic field with the residual field of order q and D is a quaternion algebra over k. Let O be the maximal order in D and π a prime element in O. For every $x \in D$ let $|x|$ be the reduced norm composed with the usual absolute value on k. In particular, $|\pi| = q^{-1}$. Let $I(s)$ be the principal series for $\mathrm{SL}_2(D)$ defined as the set of all smooth functions on $\mathrm{SL}_2(D)$ such that

$$f\left(\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) g\right) = |a/c|^{\frac{s}{2}+1} f(g)$$

for all choices of data. Consider the intertwining map $A(s)\colon I(s) \to I(-s)$ defined by

$$A(s)(f)(g) = \int_D f\left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) g\right)\, dx,$$

where dx is the invariant measure on D normalized so that the volume of O is 1.

Lemma 4.4 *Let $f_s \in I(s)$ be the unique $\mathrm{SL}_2(O)$-invariant function such that $f_s(1) = 1$, and let $c(s)$ be the function defined by $A(s)(f_s) = c(s)f_{-s}$. Then*

$$c(s) = \frac{\zeta(s)}{\zeta(s+2)},$$

where $\zeta(s) = (1 - q^{-s})^{-1}$.

Proof This is surely well known, but we include a short proof for convenience. The value $c(s)$ is equal to $A(s)f_s(1)$. Write D as a union

$$O \cup \pi^{-1}(O \smallsetminus (\pi)) \cup \pi^{-2}(O \smallsetminus (\pi)) \cup \cdots.$$

The function $x \mapsto f\left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\right)$ is equal to 1 on O and to $q^{-n(s+2)}$ on $\pi^{-n}(O \smallsetminus (\pi))$. This leads us to the sum

$$c(s) = 1 + (q^2 - 1)q^{-(s+2)} + (q^4 - q^2)q^{-2(s+2)} + (q^6 - q^4)q^{-3(s+2)} + \cdots,$$

which can be easily summed to give the claimed result. ∎

Now we can compute the c-function for the degenerate principal series $I(s)$ for the group G corresponding to the Jordan algebra $J_r(D)$, by factoring the standard intertwining map $A(s) \colon I(s) \to I(-s)$ as a product of intertwining maps corresponding to simple root reflections in the restricted root system (of type C_r). We summarize the computation in the following lemma.

Lemma 4.5 *Assume that G corresponds to $J_r(D)$ where D is a quaternion algebra. Let $c(s)$ be the function such that $A(s)f_s = c(s)f_{-s}$ where $A(s) \colon I(s) \to I(-s)$ is the standard intertwining operator. Then, up to a non-zero constant,*

$$c(s) = \prod_{i=0}^{r-1} \frac{\zeta(s - 2i)}{\zeta(s \pm (2i+1))},$$

where the signs in the denominator alternate in i, so that the sign is $+$ for $i = r - 1$. In words, $c(s)$ has a simple pole at even integers $0, 2, \ldots, 2(r-1)$ and a simple zero at odd integers $-1 - 2(r-1), 1 + 2(r-2), -1 - 2(r-3), \ldots$

Corollary 4.6 *If G corresponds to $J_r(D)$ where D is a quaternion algebra, then the spherical vector generates a proper submodule of $I(s)$ for*

$$s = -1 - 2(r-1), 1 + 2(r-2), -1 - 2(r-3), \ldots.$$

Proof The standard intertwining operator is always non-zero, and at these points the c-function vanishes. ∎

Remark In particular, the quotient at $s = 1 + 2(r-2)$ (a minimal representation) is not spherical. This agrees with results of Gan and Savin [3] where it was observed that minimal representations of tame non-quasi split groups are not spherical.

5 Real Groups

In this section we assume that k is a real or complex field and $J = J_r(D)$ and $d = \dim D$ is even. If $k = \mathbb{R}$, then D is assumed to be either split or an anisotropic quadratic space of dimension divisible by 4. (Either of these conditions will assure that $\chi_D = 1$.) If D and hence G are split, then the principal series $I(s)$ reduces at the points $1, 1 + d/2, \dots 1 + (r-1)d/2$, and the spherical representation is the unique irreducible quotient [21].

Now we move to G corresponding to $J_r(D)$ where D is an anisotropic quadratic space of dimension divisible by 4. We need the following facts from [20]; the notation is taken from Section 2. There is a maximal split torus T_r in G that gives rise to a restricted root system of type C_r. A maximal compact subgroup $K \subset G$ is the centralizer of an involution given by conjugation action of

$$\varphi\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

where $\varphi\colon \mathrm{SL}_2 \to G$ arises from a set $S = \{\beta_1, \dots, \beta_r\}$ of strongly orthogonal roots. Note that the matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ is conjugated to the matrix $\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$ in $\mathrm{SL}_2(\mathbb{C})$ by the classical Cayley transform matrix. Since the centralizer of

$$\varphi\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

in $\mathfrak{g}_{\mathbb{C}}$ is $\mathfrak{m}_{\mathbb{C}}$, it follows that the Cayley transform conjugates $K_{\mathbb{C}}$ to $M_{\mathbb{C}}$. Let γ_i be the weights for $K_{\mathbb{C}}$ obtained by transporting the weights β_i by the Cayley transform. Now for every r-tuple of integers $a_1 \geq \cdots \geq a_r$, we have a K-type that corresponds to the irreducible representation of $K_{\mathbb{C}}$ with the highest weight

$$a_1\gamma_1 + \cdots + a_r\gamma_r.$$

These types appear in the degenerate principal series $I(s)$ with multiplicity one. Sahi [20] has described the composition series of $I(s)$ as well as the Jantzen filtration which, in this case, simply measures the order of vanishing of the intertwining map $A(s)\colon I(s) \to I(-s)$ on each K-type. We shall look only at the reducibility points $1, 1 + d/2, \dots, 1 + (r-1)d/2$. If we fix a reducibility point $1 + (i-1)d/2$, then the composition series of $I(s)$ has a shape of a truncated pyramid consisting of irreducible subquotients $V_{p,q}$ such that $p, q \geq 0$ and $r - i \leq p + q \leq r$. The subquotients of the Jantzen filtrations are the floors of the pyramid, in particular, they are isomorphic to direct sums of $V_{p,q}$ where $p + q = t$, a constant. The socle is the bottom floor *i.e.,* $t = r$, and the co-socle is the top floor, *i.e.,* $t = r - i$. The types of the irreducible quotients $V_{p,q}$ of $I(1 + (i-1)d/2)$, in particular $p + q = r - i$, are particularly nice. They form a cone

$$(p-q)\tfrac{d}{4}(\gamma_1 + \cdots + \gamma_r) + (a_1\gamma_1 + \cdots + a_r\gamma_r),$$

where $a_{p+1} = \cdots = a_{n-q} = 0$.

6 Global Results

In this section k is a global field unless otherwise specified.

6.1 Global Fourier–Jacobi Method

We follow here the work of Ikeda [11]. Let $Z \cong \mathbb{G}_a$ be the root subgroup corresponding to the highest root β. Recall that $Q = LV$ is the standard parabolic subgroup such that the Levi factor L corresponds to the simple roots perpendicular to β. The unipotent radical V is a Heisenberg group with the center Z. Recall that $P = MN$ is a maximal parabolic subgroup in a standard position such that the unipotent radical N is abelian. In particular, P contains V, and

$$V = (V \cap M) \cdot (V \cap N).$$

One checks that $V \cap N$ is a maximal abelian subgroup of V. Write $Y = V \cap M$. Let X be the unique abelian subgroup of V, normalized by the torus T, trivially intersecting Z, such that $V \cap N = XZ$. Thus, we can write $V = XYZ = YXZ$, where XZ and YZ are maximal abelian subgroups of V. Let ψ denote a global or local, non-trivial additive character of Z. The group commutator gives a Z-valued pairing between X and Y. We recall the p-adic analogue of the Stone-von Neumann theorem. There is a unique smooth irreducible representation ω_ψ of V with central character ψ. We call it the Heisenberg representation (*cf.* for example in [19]; the treatment in [23, Section 2] is particularly suited to our purposes). The pairing and ψ define a Fourier transform from $S(X)$ and $S(Y)$ the spaces of Schwartz functions. Each of the two spaces realizes the Heisenberg representation of V, locally and globally (*cf.* [23, p. 283]). We will use $S(X)$ unless specified otherwise. Let $G_1 = [L, L]$ or an appropriate factor. Let $J = G_1 V = V G_1$ be the Jacobi group. Then the Weil representation ω_ψ of J is the unique extension of the Heisenberg representation of V to J. Let $\mathbb{A}$ be the ring of adelés over k. Let Λ be a functional on $S(X(\mathbb{A}))$ defined by

$$\Lambda(\phi) = \sum_{x \in X(k)} \phi(x) = \sum_{x \in X(k)} (\omega_\psi(x)(\phi))(0)$$

for every $\phi \in S(X(\mathbb{A}))$. Now every ϕ defines an automorphic function $\Theta^\phi = \Lambda(\omega_\psi(g)\phi)$ on J. Let $f_s \in I(\chi, s)$ be a global holomorphic section. Then the Eisenstein series

$$E(s)(g) = \sum_{\gamma \in P(k)\backslash G(k)} f_s(\gamma g)$$

converges for $\Re(s)$ large enough. If f is a smooth function on $Z(k)\backslash G(\mathbb{A})$, define

$$f_\psi(g) = \int_{Z(k)\backslash Z(\mathbb{A})} f(zg)\overline{\psi}(z)\, dz.$$

Write $G(k) = \bigcup_{w \in S} P(k)wQ(k)$, as a union of double cosets, and $E(s) = \sum_{w \in S} E(s)^w$ by breaking up the sum over individual cosets. It follows, from [23, Lemma 4.2.2], that $E(s)^w_\psi = 0$ except for $w = w_\beta$, representing the open double coset. It follows that, for

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$g_1 \in G_1$,

$$\int_{V(k)\backslash V(\mathbb{A})} E_\psi(vg_1)\overline{\Theta^\phi}(vg_1)\,dv = \int_{V(k)\backslash V(\mathbb{A})} \sum_{\gamma\in P(k)\backslash P(k)w_\beta Q(k)} f_s(\gamma v g_1)\overline{\Theta^\phi}(vg_1)\,dv.$$

Recall that $P_1 = G_1 \cap P$. It is easy to compute $w_\beta^{-1}Pw_\beta \cap Q$ and verify that

$$P\backslash Pw_\beta Q = w_\beta \cdot (YZ \times P_1\backslash G_1).$$

Now the above integral can be written as

$$\int_{V(k)\backslash V(\mathbb{A})} \sum_{\gamma_2\in Y(k)Z(k)} \sum_{\gamma_1\in P_1(k)\backslash G_1(k)} f_s(w_\beta\gamma_2\gamma_1 v g_1)\overline{\Theta^\phi}(\gamma_2\gamma_1 v g_1)\,dv,$$

where we used $\Theta^\phi(vg_1) = \Theta^\phi(\gamma_2\gamma_1 v g_1)$. After the change of integration $v := \gamma_1^{-1}v\gamma_1$, we can contract the integral and the first sum, giving

$$\int_{X(k)\backslash V(\mathbb{A})} \sum_{\gamma_1\in P_1(k)\backslash G_1(k)} f_s(w_\beta v\gamma_1 g_1)\overline{\Theta^\phi}(v\gamma_1 g_1)\,dv.$$

Finally, using the definition Θ^ϕ, we arrive to

$$\int_{V(\mathbb{A})} \sum_{\gamma_1\in P_1(k)\backslash G_1(k)} f_s(w_\beta v\gamma_1 g_1)\overline{\omega_\psi(v\gamma_1 g_1)(\phi)(0)}\,dv.$$

For $g_1 \in G_1(\mathbb{A})$, let

$$F_s(g_1) = \int_{V(\mathbb{A})} f_s(w_\beta v g_1)\overline{\omega_\psi(vg_1)(\phi)(0)}\,dv.$$

Then $F_s \in I_1(\chi\chi_D, s)$, and we have shown that

$$\int_{V(k)\backslash V(\mathbb{A})} E_\psi(vg_1)\overline{\Theta^\phi}(vg_1)\,dv = \sum_{\gamma_1\in P_1(k)\backslash G_1(k)} f_{1,s}(\gamma_1 g_1) = E_F(s)(g_1)$$

the Eisenstein series on G_1 attached to F_s. Of course, so far, this works for $\Re(s)$ large enough. We now show that F_s extends to a holomorphic section for $\Re(s) > 0$. Consider first the local situation. For $\Re(s)$ large enough, the local integral

$$F_s(g_1) = \int_V f_s(w_\beta v g_1)\overline{\omega_\psi(vg_1)(\phi)(0)}\,dv$$

defines an intertwining a map from $I(\chi, s) \otimes \omega_{\overline{\psi}}$ onto $I_1(\chi\chi_D, s)$, intertwining the action of $J = G_1 V$.

Lemma 6.1 *Recall the notation from Section 2.1. The local intertwining map $(f_s, \phi) \to F_s$ from $I(\chi, s) \otimes \omega_{\overline{\psi}}$ to $I_1(\chi\chi_D, s)$ extends holomorphically to the region $\Re(s) > -(r-1)\frac{d}{2}$. The map is non-zero for every s in the region.*

Proof We construct the continuation by writing the integral as an iterated integral, first integrating over X. We can assume that $g_1 = 1$, and write $v = xyz$. Note that $w_\beta^{-1}xw_\beta \in P$, so $f_s(w_\beta xyz) = f_s(w_\beta yz)$. Since $\omega_\psi(v)(\phi)(0)$ is equal to

$$\phi(x)\psi([y, -x])\psi(z),$$

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integrating over X amounts to taking the Fourier transform of ϕ and evaluating at y. Thus the local integral is equal to

$$\int_{YZ} f_s(w_\beta yz)\overline{\psi}(z)\overline{\widehat{\phi}(y)}\,dydz.$$

Next, by an easy SL_2-computation, the integral of $f_s(w_\beta yz)$ over Z is absolutely converging if $\Re(s) > -(r-1)\frac{d}{2}$, and the output depends polynomially on $|y|$. Since $\widehat{\phi}(y)$ is rapidly decreasing, it follows that the integral over YZ is absolutely converging in the same range of s. By analytic continuation, the map $(f_s,\phi) \to F_s$ given by the iterated integral, intertwines the actions of $J = G_1V$, since it does so for large $\Re(s)$. The map is easily seen to be non-zero, as $\widehat{\phi}$ can be arbitrary. ∎

We continue in the setting of the previous lemma, and compute the local integral explicitly in the p-adic case, assuming that all data are unramified. In that case, $\widehat{\phi}$ is the characteristic function of $Y(O)$, where O is the ring of integers in the local field. Hence we can assume that $y \in Y(O)$; then $f_s(w_\beta yz) = f_s(w_\beta z)$, and the integral reduces to

$$\int_Z f_s(w_\beta z)\overline{\psi}(z)dz = 1 - \chi(\varpi)\frac{1}{q^{s+1+(r-1)\frac{d}{2}}} = L\Big(s+1+(r-1)\frac{d}{2},\chi\Big)^{-1},$$

where the quantity on the right hand side is obtained by a very easy SL_2-computation (*cf. e.g.*, [5, Proposition 1.6.5]). Here q is the order of the residue field of O and ϖ is the uniformizer. The product of these local factors is convergent and non-zero if $\Re(s) > -(r-1)\frac{d}{2}$. Hence, we have an analytic continuation of the global integral as well.

Lemma 6.2 *Assume that $\Re(s) > -(r-1)\frac{d}{2}$, and we are in a local, p-adic, situation. Assume that the assumptions of Theorems 3.2 or 3.3 are met. Let $f_s \in I(\chi,s)$ and $\phi \in S(X)$. The map $(f_s,\phi) \to F_s$ from $I(\chi,s)\otimes\omega_{\overline{\psi}}$ to $I_1(\chi\chi_D,s)$ is surjective. Moreover, if f_s is contained in the unique irreducible submodule of $I(\chi,s)$, then F_s is contained in the unique irreducible submodule of $I_1(\chi\chi_D,s)$.*

Proof We remark that Ikeda proves surjectivity by an explicit calculation. Here we present another argument in the p-adic case that gives the additional information about submodules. The map is clearly non-zero, and it intertwines the actions of G_1V where V acts trivially on $I_1(\chi\chi_D,s)$. In particular, Z acts trivially on $I_1(\chi\chi_D,s)$ and the map descends to a non-zero map from $I(\chi,s)_{Z,\psi}\otimes\omega_{\overline{\psi}}$ to $I_1(\chi\chi_D,s)$. By [23, Proposition 3.2] and Theorem 3.1, $I(\chi,s)_{Z,\psi} \cong I_1(\chi\chi_D,s)\otimes\omega_\psi$, as G_1V-module. Since V acts trivially on $I_1(\chi\chi_D,s)$, the map descends to a map from $I_1(\chi\chi_D,s)\otimes(\omega_\psi\otimes\omega_{\overline{\psi}})_V$ to $I_1(\chi\chi_D,s)$ intertwining the actions of G_1. Here $(\omega_\psi\otimes\omega_{\overline{\psi}})_V$ denotes the maximal quotient on which V acts trivially. It is one-dimensional by Schur's lemma. Hence, we get a non-trivial map from $I_1(\chi\chi_D,s)$ to $I_1(\chi\chi_D,s)$. By our assumption on G, $I_1(s)$, if reducible, is of length 2, multiplicity free and indecomposable, this map must be a multiple of the identity map. From this description of the map $(f_s,\phi)\to F_s$, it is straightforward to check the lemma. ∎

Proposition 6.3 *Assume that G arises from a Jordan algebra $J_r(D)$. Let $d = \dim D$. Let χ be a Grossencharacter satisfying $\chi^2 = 1$. Let $E(s)$ be the Eisenstein series arising from a holomorphic section f_s of the principal series $I(\chi, s)$.*

(i) *Assume that $\chi_D = 1$. Then $E(s)$ is holomorphic at $s_0 > 0$ except, possibly, when $\chi = 1$ and $s_0 = 1, 1 + d/2, \dots, 1 + (r-1)d/2$.*

(ii) *Assume that $r = 2$.*

  (a) *Assume that $d = 2n - 1$. Then $E(s)$ is holomorphic at $s_0 > 0$ except, possibly, when $s_0 = 1/2$, or $\chi = 1$ and $s_0 = n + \frac{1}{2}$, where the trivial representation is the residue.*

  (b) *Assume that $d = 2n - 2$. Let χ_D be the Grossencharacter attached globally to D. Then $E(s)$ is holomorphic at $s_0 > 0$ except, possibly, when $\chi = \chi_D$ and $s_0 = 1$, or $\chi = 1$ and $s_0 = n$, where the trivial representation is the residue.*

The possible poles are at most simple. Moreover, if the local component of f_{s_0} at any p-adic place is contained in the unique submodule of $I(\chi, s_0)$, then $E(\chi, s)(f)$ is holomorphic at s_0.

Proof For notational convenience we deal with the first case. Fix $\Re(s_0) > 0$, and expand

$$E(s)(g) = \frac{f(g)}{(s - s_0)^l} + \text{higher powers of } (s - s_0)$$

where $f(g)$ is the residual form. Then for $s_0 = 1 + (r-1)d/2$ and $\chi = 1$, we have $l = 1$ and the residual representation is the trivial representation; this is a well known case. So assume that $s_0 \neq 1 + (r-1)d/2$ or $\chi \neq 1$. Let $\mathcal{A}$ be the residual automorphic representation. (We work with spaces of K-finite functions.) We claim that there exists $f \in \mathcal{A}$ such that the global Fourier coefficient $f_\psi(1)$ is non-zero. Assume that $f_\psi(g) = 0$ for all non-trivial characters ψ and $g \in G(\mathbb{A})$. Then f is left $Z(\mathbb{A})$-invariant. Let v be a local place and let $\mathbb{A}_v$ be the ring of adelés with the local factor k_v removed. By the weak approximation theorem, f is determined by its restriction to $G(\mathbb{A}_v)$. Since $G(\mathbb{A}_v)$ and $Z(k_v)$ commute, it follows that f is left $Z(k_v)$-invariant. If this is true for every f, then the v-adic component of $\mathcal{A}$ is the trivial representation, a contradiction to the assumption on s. Hence, there exists $f \in \mathcal{A}$ and $g \in G(\mathbb{A})$ such that $f_\psi(g) \neq 0$. We write $g = g_\infty g_f$, where g_∞ denotes the archimedean part, and g_f the part belonging to the finite adeles. We easily get rid of the g_f–part by a right translation, so we can assume that $f_\psi(g) \neq 0$ for $g \in G(\mathbb{A}_\infty)$ (*i.e.*, $G(\mathbb{R})$, if we are working over $\mathbb{Q}$). Since f is K–finite it is analytic, and then f_ψ is analytic as well. So we expand this non-trivial f_ψ near identity, and there exists an element of the universal enveloping algebra, say $\mathfrak{D}$, such that $\mathfrak{D}(f_\psi)(1) \neq 0$, but $\mathfrak{D}(f_\psi)(1) = (\mathfrak{D}f)_\psi(1)$, so we have found $h \in \mathcal{A}$ such that $h_\psi(1) \neq 0$.

So let $f \in \mathcal{A}$ such that $f_\psi(1) \neq 0$. There exists $\phi \in S(X(\mathbb{A}))$ such that

$$g_1 \longmapsto \int_{V(k)\backslash V(\mathbb{A})} f_\psi(vg_1)\overline{\Theta^\phi}(vg_1)\, dv$$

is a non-trivial function on G_1. It follows that $E_F(s)$ has a pole of order l. By the induction assumption, $l = 0$ or 1 and $l = 1$ only if s_0 is one of the listed values. Furthermore, by the induction assumption and Lemma 6.2, $E(s)$ has no pole if the local component of f_{s_0} is in the irreducible submodule of $I(s_0)$. ∎

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6.2 Main Result

To prove existence of the poles *i.e.,* the if and only if result, one could argue as Ikeda and prove that the local integral is surjective at the archimedean places. Instead we compute the constant term of the Eisenstein series along the unipotent radical of the minimal parabolic. The full constant term involves a complicated sum over the Weyl group; however, we shall look only at the summand where the intertwining operator $A(s)$ appears. This will give us not only existence of the pole, but also a control of the structure of the residual representation. In order to keep arguments as simple as possible, we shall henceforth work with $J_r(D)$ such that $d \equiv 0 \pmod 4$ and D has trivial discriminant *i.e.,* $\chi_D = 1$. Then Proposition 6.3 simply says that $E(s)$ has possible simple poles at odd integers $1, 1 + d/2, \dots, 1 + (r-1)d/2$.

Theorem 6.4 *Assume G corresponds to $J_r(D)$ such that $d \equiv 0 \pmod 4$. In addition, assume that*

(i) *the discriminant of the quadratic space D is trivial,* i.e., $\chi_D = 1$*;*
(ii) *the quadratic space D is either split or totally anisotropic;*
(iii) *for every real place v, D_v is either split or totally anisotropic.*

Then the Eisenstein series $E(s)$ has simple poles at $s_0 = 1, 1 + d/2, \dots, 1 + (r-1)d/2$. At each s_0 the residual representation is square integrable and isomorphic to the co-socle of the global degenerate principal series $I(s_0)$.

Observe that conditions (i)–(iii) are automatically satisfied if $r \geq 3$.

Proof Let $E(s)$ be the Eisenstein series attached to a holomorphic section $f(s) = \otimes_v f_v(s)$ in $I(s) = \otimes_v I_v(s)$. Let s_0 be one of the points and let v be a p-adic place where G is split. (G is split at almost all primes, as we shall argue in a moment.) If $f_v(s_0)$ belongs to the irreducible submodule of $I_v(s_0)$ then $E(s)$ is holomorphic at s_0. In particular, only the irreducible spherical quotient at the place v can contribute to the residual representation. By Proposition 4.3, the spherical quotient of $I_v(s_0)$ is the Aubert dual of a square integrable representation. It follows that the residual representation is square integrable. Hence, it decomposes as a direct sum of irreducible representations, so it must be a quotient of the co-socle of $I(s_0)$. In order to show that the residual representation is the full co-socle, we need to show that the pole is achieved as the section $f(s)$ passes through types belonging to irreducible representations in the co-socle.

Let Φ denote the root system of G relative to a maximal split torus. If D is split, then G is a split (Chevalley) group; if D is anisotropic, then for the maximal split torus we can take is T_r as in Section 2. Let W be the corresponding Weyl group. Let $P_0 = M_0 N_0$ be a minimal parabolic subgroup containing the split torus, corresponding to a choice of positive roots Φ^+ in Φ, and we can assume that the parabolic group $P = MN$ is in the standard position *i.e.,* $M_0 \subseteq M$ and $N \subseteq N_0$. Let $\Phi_M^+ \subseteq \Phi^+$ be the positive roots for M. Let $W(M) = \{w \in W : w(\Phi_M^+) > 0\}$. The element w_0, the product of the longest Weyl group elements for G and M, belongs to $W(M)$. We shall use that w_0 permutes Φ_M^+ and that $w_0(\omega) = \omega^{-1}$. The degenerate principal series $I(s) = \mathrm{Ind}_P^G(|\omega|^s)$ is naturally embedded in the principal series $\mathrm{Ind}_{P_0}^G(\chi_s)$, where χ_s is a character of M_0.

The just mentioned properties of w_0 imply that $w_0(\chi_s) = \chi_{-s}$. If $E(s)$ is the Eisenstein series built from a holomorphic section $f(s)$, its constant term along N_0 is naturally a function on M_0. As such, it is a sum

$$\sum_{w \in W(M)} d_w(s) w(\chi_s),$$

where $d_w(s)$ are meromorphic functions that depend on $f(s)$. We look at the summand corresponding to w_0. Assume first that G is split. Let $f_v(s)$ be the normalized spherical vector in the local principal series representations $I_v(s)$. If $s > 0$, then $I_v(s)$ is generated by $f_v(s)$. Let $E(s)$ be the Eisenstein series corresponding to $f(s) = \otimes_v f_v(s)$. The contribution of w_0 to the constant term is the restriction to M_0 of $A(s)(f(s)) = c(s)f(-s)$, where, by Lemma 4.1

$$c(s) = \prod_{i=0}^{r-1} \frac{\zeta(s - id/2)}{\zeta(s + id/2 + 1)}.$$

Here $\zeta(s)$ is the global Dedekind ζ-function corresponding to the the number field k. It is well known that $\zeta(s)$ has a simple pole at $s = 1$, hence $d_{w_0}(s) = c(s)$ has simple poles at the points of interest. We now look at the case of anisotropic D.

Lemma 6.5 *For almost all places v, the quadratic space D_v is split.*

Proof Assume that v is a p-adic place. Since the discriminant of D_v is trivial, D_v is either split or has a 4-dimensional anisotropic kernel isomorphic to a quaternion algebra. The isomorphism class is determined by the isomorphism class of the Clifford algebra attached to D_v. But this algebra is a localization of the global Clifford algebra attached to D. The Clifford algebra is a central simple algebra and localizes to a matrix algebra for almost all places. This proves the lemma. ■

So let S be the finite set of places such that D_v is split for $v \notin S$. We consider the Eisenstein series $E(s)$ corresponding to the constant section $f(s) = \otimes_v f_v(s)$ where, for all $v \notin S$, $f_v(s)$ is the spherical vector, while for $v \in S$, $f_v(s)$ is arbitrary. Then the contribution of w_0 to the constant term is again given by $A(s)f(s)$. Since we know how to compute the action of the intertwining operator on the spherical vector at the places $v \notin S$, we have

$$A(s)f(s) = c(s)\Big(\bigotimes_{v \in S} c_v(s)^{-1} A_v(s) f_v(s)\Big) \otimes \Big(\bigotimes_{v \notin S} f_v(-s)\Big).$$

Since $c_v(s)^{-1}$ are non-zero by Corollary 4.2 and the local intertwining operators are always non-zero, we see $d_{w_0}(s)$ has poles at the points of interest. In fact, since the holomorphic properties of $A_v(s)$ reflect the Jantzen filtration, we see that the pole is achieved for f_v in any K-type belonging to the co-socle of $I_v(s)$.

It remains to show that the pole, at the point s_0, of the w_0-summand in the constant term is not cancelled out by a pole of the w-summand for some other $w \in W(M)$. The cancellation can happen only if

$$w(\chi_{s_0}) = w_0(\chi_{s_0}).$$

Thus, we need to show that there is no such $w \in W(M)$. This is an easy check left to the reader in the case of split groups, but we provide details if D is totally anisotropic.

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In this case Φ is of the type C_r and Φ_M of the type A_{r-1}. Any real character of M_0 is determined by the restriction to the maximal split torus T_r. Recall that any element in T_r is uniquely written as a product of $\omega_i(t_i)$, where $\omega_i^\vee$ are the co-characters defined by equation (2.2). Thus any real character χ of T_r is determined by an r-tuple $(s_1,\dots,s_r)$ of real numbers defined by $\chi(\omega_i) = |\cdot|^{s_i}$. In these coordinates the modular character is

$$\rho = \big(1 + (r-1)d, \dots, 1 + d, 1\big).$$

Note that the difference between the consecutive entires is d, which reflects the fact that short root spaces are d-dimensional. In order to compute χ_s we observe that the (group) root spaces corresponding to $\pm\alpha$ where α is a short simple root (*i.e.*, a simple root of M) generate a group isomorphic to $\mathrm{Spin}(H \oplus D)$ where H is a 2-dimensional hyperbolic plane. The degenerate principal series for this group, with respect to the maximal parabolic subgroup whose unipotent radical is the root space of α, contains the trivial representation as a submodule for $s = -d$. It follows that $s_i - s_{i+1} = -d$ for the coordinates of χ_s. These equations pin down a line, and the linear parameter s is fixed by demanding that $w_0(\chi_s) = \chi_{-s}$ and $\chi_{s_0} = -\rho$ for $s_0 = -1 - (r-1)d/2$. (At this point, both series of representations contain the trivial representation as a submodule.) Putting everything together yields

$$\chi_s = (s, s, \dots, s) + \tfrac{d}{2}(1-r, 3-r, \dots, r-1).$$

We claim that χ_{s_0} is regular at the reducibility points. To that end, recall that a character $\chi = (s_1,\dots,s_r)$ is singular if it is contained in a wall $s_i = 0$, $s_i - s_j =$ or $s_i + s_j = 0$. Since the coordinates of χ_{s_0} form a strictly increasing sequence of odd integers, it is clear that χ_{s_0} cannot satisfy the first two equations. Since d is divisible by 4, the coordinates of χ_{s_0} are congruent modulo 4. The equation $s_i + s_j = 0$ implies that s_i and s_j are opposite integers. But two opposite odd integers are never congruent modulo 4, hence $s_i + s_j = 0$ cannot hold. ∎

If $k = \mathbb{Q}$ and $J = J_3(\mathbb{O})$ where $\mathbb{O}$ is the Cayley-Graves octonion algebra, then the residual representation at $s = 5$ and $s = 1$ contains the singular modular form on the exceptional tube domains of weights 4 and 8, respectively, discovered by Kim [13].

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