Canad. Math. Bull. Vol. **20** (1), 1977

NUMERICAL-VALUED FOURIER TRANSFORMS

BY
RAIMOND A. STRUBLE

ABSTRACT. It is shown that the classical Fourier transform can be extended as an algebra isomorphism onto the algebra of all complex-valued functions, which are measurable and finite a.e., under pointwise addition and multiplication. The extended Fourier transform agrees with the distributional Fourier transform on the space of all distributions which have regular transforms. It is defined on an algebra of Mikusiński-type operators in which multiplication is convolution in the subspace of integrable distributions.

Only the one-dimensional case will be illustrated here. Let $\mathcal{K}$ denote the usual (pointwise) algebra of measurable (complex) functions which are finite-valued a.e. on the real line $\mathcal{R}$. The non-divisors of zero in $\mathcal{K}$ are the functions which are non-zero a.e. on $\mathcal{R}$. Let $\mathcal{K}_0$ denote the multiplicative subgroup of non-divisors of zero.

LEMMA. *If* $X \in \mathcal{K}$, *then* $X = W/Y$, *where* $W \in \mathcal{K}$, $Y \in \mathcal{K}_0$ *and* $|W(\tau)| \le 1/(1+\tau^2)$, $|Y(\tau)| \le 1/(1+\tau^2)$ *for* a.e. $\tau \in \mathcal{R}$.

Proof. *Let* $Y(\tau) = 1/[(1+\tau^2)(1+|X(\tau)|)]$ and $W(\tau) = X(\tau)Y(\tau)$.

The functions W and Y in this lemma have classical Fourier inverses, w and y, which are continuous functions on the real line R (to be distinguished from $\mathcal{R}$) and whose distributional Fourier transforms are W and Y. Let H denote the collection of all such continuous functions x on R which are Fourier inverses of measurable functions X with $(1+\tau^2)X(\tau)$ essentially bounded on $\mathcal{R}$. If $x, y \in H$, then $w = x * y$ is defined as that member of H which is the classical Fourier inverse of the product $W = XY$. Under pointwise addition and scalar multiplication and $*$, H becomes an algebra, and the Fourier transform is an algebra isomosphism from H into $\mathcal{K}$. For L_1 functions in H, this is the classical transform and $*$ is convolution. The subgroup H_0 of non-divisors of zero (with respect to $*$) in H are those $x \in H$ for which $X \in \mathcal{K}_0$.

Now let K be the (formal) algebra (with the usual identification) of fractions x/y with $x \in H$ and $y \in H_0$, and call the mapping which sends the fraction $x/y \in K$ to the function $X/Y \in \mathcal{K}$, the *extended* Fourier transform. The elements of K will be called *operators* and multiplication in K will be denoted by $*$.

Received by the editors October 8, 1975 and, in revised form, June 1, 1976.

https://doi.org/10.4153/CMB-1977-023-4 Published online by Cambridge University Press

Proposition 1. The extended Fourier transform is an algebra isomorphism from the algebra K of operators onto the algebra $\mathcal{K}$ of functions.

Proof. Obvious, from the lemma and the definitions.

Let $\mathcal{D}(R)$ denote the usual space of test functions for distributions on R and let $\mathcal{Z}$ denote the usual space of test functions for ultradistributions. The distributional Fourier transform is a vector space isomorphism from the space $\mathcal{D}'(R)$ of distributions onto the space $\mathcal{Z}'$ of ultradistributions.

Now $\mathcal{K}$ contains each regular ultradistribution (locally integrable function defining a continuous linear functional on $\mathcal{Z}$). Those distributions whose distributional Fourier transforms are regular ultradistributions will be identified with the corresponding operators whose extended Fourier transforms are the same regular ultradistributions. Let f be such a distribution. Then for each $\phi \in \mathcal{D}(R)$, the distributional Fourier transform of the convolution $f * \phi$ is the product $F\Phi$ of the distributional Fourier transform F of f and the classical Fourier transform Φ of ϕ. Normally, Φ is only considered a multiplier in $\mathcal{Z}'$, but in this case F is regular, and so this product is that in the ring $\mathcal{K}$. On the other hand, the extended Fourier transform of $f * \phi$ as an operator is also this same product $F\Phi$. Thus the identification of such distributions with operators preserves all of their convolution characteristics, and allows for their immediate recovery from the operators, through the Parseval relation, as linear functionals, i.e. $<f, \phi> = \int_{-\infty}^{\infty} F(\omega)\Phi(-\omega)\, d\omega$. In all but the last formula, ϕ can be replaced by any distribution with compact support. These considerations will now be formalized.

Proposition 2. *There exists a vector subspace N of K which is isomorphic with the space of all distributions with regular Fourier transforms. On N, the extended Fourier transform agrees with the distributional Fourier transform and multiplication* “$*$” *with the elements of $\mathcal{D}(R)$ (or more generally, of $\mathcal{E}'(R)$, the subspace of distributions with compact supports) is equivalent to convolution* “$*$”.

In [1] and [2] it is shown that if f and g are integrable distributions, then the convolution $f * g$ exists and is integrable and, furthermore, that their Fourier transforms F and G are regular and the product FG is the Fourier transform of $f * g$. This gives the following.

Proposition 3. *Let f and g be integrable distributions. Then $f, g, f * g \in N$ and $f * g = f * g$.*

The extended Fourier transform has been introduced into the operational calculus in [3].

AMS(MOS) *subject classifications* (1970). Primary 42A68 Secondary 44A40, 46F99.

References

1. B. Price, *On the Laplace transforms for distributions*, SIAM J. Math. Anal. **6** (1975), 49–80.

2. R. A. Struble, *Representations of Fourier transforms for distributions*, Bull. Inst. Acad. Sinica **2** (1974), 191–206.

3. R. A. Struble, *The numerical-valued Fourier transform in the two-sided operational calculus*, SIAM J. Math. Anal. (1977), 243–257.

Department of Mathematics
North Carolina State University
Raleigh, North Carolina 27607