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STABILITY AND CATEGORICITY OF LATTICES

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Introduction. This paper is a contribution to applied stability theory. Our purpose is to investigate the complexity of lattices by determining the stability of their first order theories.

Stability measures the complexity of a theory T by counting the number of different "kinds" of elements in models of T. The notion of ω-stability was introduced by Morley [**26**] in 1965 and generalized by Shelah [**31**] in 1969. Shelah classified all first order theories according to their stability properties.

Stability and $\aleph_1$-categoricity are closely related (see [**26**] and [**1**]). In fact, the notions of stable, superstable and ω-stable can be regarded as successive approximations of $\aleph_1$-categorical. $\aleph_1$-categoricity is a very strong property while stability, superstability and ω-stability facilitate the classification of more "complex" theories.

The aim of applied stability and categoricity theory is to determine algebraic characterizations of those structures in an interesting class whose theories are $\aleph_0$-categorical, $\aleph_1$-categorical, ω-stable, superstable or stable. The class of Abelian groups is the only natural and interesting class we know of where a complete analysis of stability and categoricity has been given. (See [**8**].)

It follows from [**33**] that if $\mathfrak{A}$ is an infinite Boolean algebra, an infinite distributive lattice or in fact any partially ordered structure containing an infinite chain then $\mathfrak{A}$ is unstable. For such structures then, there is no stability or $\aleph_1$-categoricity. So it seems natural to consider the class of all lattices without infinite chains (in fact, without arbitrarily long finite chains). This class of lattices is a rich one and, as will be seen, has non-trivial stability and categoricity properties. Special cases of the results of this paper apply to the class of dimension $\leqq 2$ or "planar" lattices. Our main results yield:

(1) A characterization of stability and superstability in the class of all dimension $\leqq 2$ lattices;

(2) Characterizations of $\aleph_0$-categoricity, $\aleph_1$-categoricity, ω-stability,

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superstability and stability in the class of all height 4 and dimension $\leqq 2$ lattices.

These results were announced in [**34**] and [**35**].

0. Preliminaries. We will assume the reader is familiar with the basic elements of first-order logic (as in [**4**]).

Stability and Categoricity. Let $\mathfrak{A}$ be an L-structure and let $X \subset A$. A set of formulae $\Sigma(v)$, with at most v free, is a *type of* $\mathfrak{A}$ *over* X if

(i) if Σ_0 is a finite subset of Σ then

$$\mathfrak{A}_X \models (\exists v) \wedge (\sigma\colon \sigma \in \Sigma_0),$$

and

(ii) if $\phi(v)$ is a formula of L_x with at most v free then either $\Sigma \models \phi$ or $\Sigma \models \sim \phi$.

Let $S_{\mathfrak{A}}(X)$ denote the set of all types of $\mathfrak{A}$ over X.

A theory T is *κ-stable* if for every $\mathfrak{A} \models T$ and every subset X of A of power $\leqq \kappa$ we have $|S_{\mathfrak{A}}(X)| \leqq \kappa$. If T is not κ-stable then we say T is *κ-unstable*. T is *stable* if T is κ-stable for some infinite κ. Otherwise T is *unstable*. We say a structure $\mathfrak{A}$ is κ-stable when $Th(\mathfrak{A})$ is κ-stable. Note that every finite structure is κ-stable for every infinite cardinal κ.

THEOREM 0.1. [**26**]) *Let T be a complete theory in a countable language. T is ω-stable if and only if T is κ-stable for every infinite κ.*

THEOREM 0.2. ([**33**]) *For every complete countable theory T exactly one of the following occurs:*

(i) *T is κ-stable for every $\kappa \geqq 2^\omega$;*

(ii) *T is κ-stable if and only if $\kappa = \kappa^\omega$;*

(iii) *T is unstable.*

If T is κ-stable for all $\kappa \geqq 2^\omega$ we say T is *superstable*. We will sometimes refer to theories for which (ii) holds as "*merely*" *stable*.

THEOREM 0.3. ([**33**]) *T is unstable if and only if there is a formula $\Psi(\bar{v}, \bar{w})$ in $2n$ free variables and a model $\mathfrak{A} \models T$ with sequences $\bar{a}_i \in {}^nA$, $i \in \omega$, such that for all $i \neq j$*

$i < j$ if and only if $\mathfrak{A} \models \Psi[\bar{a}_i, \bar{a}_j]$.

A theory T is *κ-categorical* if every two models of T of power κ are isomorphic. We refer to a structure $\mathfrak{A}$ as κ-categorical when $Th(\mathfrak{A})$ is κ-categorical.

MORLEY'S THEOREM. (Theorem 7.1.14 of [**4**]) *Let T be a complete theory in a countable language. T is $\aleph_1$-categorical if and only if T is κ-categorical for every uncountable cardinal κ.*

Lattice Theory. It will usually be most convenient to view our lattices as relational structures.

Definition 0.4. Let $a, b \in A$ where $\mathfrak{A} = \langle A, \leqq \rangle$ is a lattice. We say a and b are *comparable* if $a \leqq b$ or $b \leqq a$. We write $a \| b$ when a and b are incomparable. $X \subset A$ is an *antichain* if $a \| b$ for all $a \neq b$ in X. X is a *chain* if a and b are comparable for all $a, b \in X$. $[a, b]$ denotes $\{c \in A: a \leqq c \leqq b\}$. Open and half open intervals are defined similarly. We say b *covers* a, and write $a \prec b$, if $a < b$ and $[a, b] = \{a, b\}$. The *dual* $\mathfrak{A}^d$ of $\mathfrak{A}$ is the lattice $\langle A, \leqq^1 \rangle$ where $a \leqq {}^1 b$ if and only if $b \leqq a$.

A *planar embedding* $e(\mathfrak{A})$ of $\mathfrak{A}$ is an injection $a \to \bar{a}$ from $\mathfrak{A}$ to $\mathbf{R}^2$ such that

(1) $\pi_2(e(a)) < \pi_2(e(b))$ whenever $a < b$ (π_2 is the second projection of $\mathbf{R}^2$ onto $\mathbf{R}$),

(2) The straight line segments $\bar{a}\bar{b}$ connecting $\bar{a}$ and $\bar{b}$ whenever $a \prec b$ in $\mathfrak{A}$ do not intersect, except possibly at their endpoints.

$\mathfrak{A}$ is *planar* if $\mathfrak{A}$ has a planar embedding. Intuitively, $\mathfrak{A}$ is planar if it can be drawn with no intersecting edges.

Dushnik and Miller [**7**] define the *dimension* of a poset $\mathfrak{A} = \langle A, \leqq \rangle$ as the least cardinal κ such that $\leqq$ is the intersection of κ linear orders on A. If $\mathfrak{A}$ is any partially ordered set of dimension $\leqq 2$, $\mathfrak{B} \subset \mathfrak{A}$ and $\mathfrak{B}$ is finite then $\mathfrak{B}$ is planar. (See for example [**19**], Proposition 5.2.)

Let $\mathfrak{A}$ and $\mathfrak{B}$ be lattices. We say $\mathfrak{A}$ *omits* $\mathfrak{B}$ if there is no subposet $\mathfrak{A}'$ of $\mathfrak{A}$ (i.e., submodel as a relational structure) which is isomorphic to $\mathfrak{B}$. Note that $\mathfrak{A}'$ need not be a sublattice of $\mathfrak{A}$. Kelly and Rival in [**19**] define a denumerable set of finite lattices, $\mathscr{L}$, and prove:

THEOREM 0.5. ([**19**], Theorem 1) *A finite lattice $\mathfrak{A}$ is planar if and only if $\mathfrak{A}$ omits every lattice in $\mathscr{L}$.*

(We will define some of the members of $\mathscr{L}$ as they are needed.) They also show how this provides a characterization of the dimension $\leqq 2$ lattices.

THEOREM 0.6. ([**19**], Theorem 6.1) *A lattice $\mathfrak{A}$ has dimension $\leqq 2$ if and only if $\mathfrak{A}$ omits every lattice in $\mathscr{L}$.*

PROPOSITION 0.7. *Suppose $\mathfrak{A}$ is a partially ordered set containing chains of every finite cardinality. Then $Th(\mathfrak{A})$ is unstable.*

Proof. Apply the Compactness Theorem and Theorem 0.3.

It follows that in stable posets there is a finite bound on the cardinalities of chains. We are led to the following definition.

Definition 0.8. Let $\mathfrak{A}$ be a partially ordered set. The *height* of $\mathfrak{A}$ is

$$\sup \{\kappa: \mathfrak{A} \text{ contains a chain of power } \kappa\}.$$

So in this terminology, Proposition 0.7 yields that all stable posets are of finite height. In particular, all stable lattices have finite height. The bulk of this paper is devoted to a study of lattices of height $\leqq 4$.

1. Lattices of height <4 and an unstable lattice of height 4. We first note the following well known fact about finite height lattices:

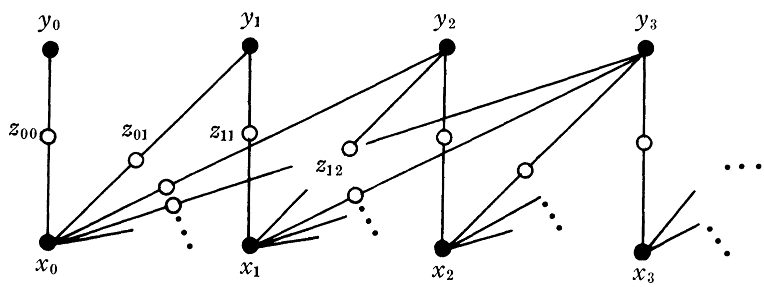
PROPOSITION 1.1. *Let $\mathfrak{A}$ be a lattice of finite height. Then $\mathfrak{A}$ has a minimum element*, 0, *and a maximum*, 1.

It follows that only the one element lattice has height 1, and only the two element lattice, $\{0, 1\}$, has height 2. The theories of height 1 and height 2 lattices are therefore complete, have only finite models and are ω-stable.

Height 3 lattices are also very simple. Let $\mathfrak{A}$ be a height 3 lattice and let $a, b \in A\text{-}\{0, 1\}$. Suppose $a \leqq b$. Then $\{0, a, b, 1\}$ is a chain. Since $\mathfrak{A}$ contains no chains of cardinality >3, we have $a = b$. We have shown that $A = \{0, 1\}$ is an antichain in $\mathfrak{A}$. It follows that, up to isomorphism, there is exactly one height 3 lattice of each cardinality <2. So the complete theory of any infinite height 3 lattice is categorical in all infinite powers.

The class of height 4 lattices is much more interesting. In fact, there is a height 4 lattice whose complete theory is unstable.

Example 1.2. Let $\mathscr{U}$ consist of an antichain $x_0, x_1, \ldots, y_0, y_1, \ldots$ of elements which cover 0, and elements z_{ij}, $i \leqq j < \omega$, where z_{ij} is comparable only to the elements $0, x_i, y_j$, and 1; $z_{ij} > y_j$; $z_{ij} > x_1$, and $0 < z_{ij} < 1$. If we let '$\bullet$' represent an element covering 0, and '$\circ$' an element covered by 1, we can give an "aerial" representation of $\mathscr{U}\text{-}\{0, 1\}$ as follows:



THEOREM 1.3. *$Th(\mathscr{U})$ is unstable.*

Proof. Let $\Psi(v, w)$ be the formula:

$$(\exists x)[0 < x \wedge (\exists z)(x < z < 1 \wedge w < z) \wedge \sim(\exists z)(x < z < 1 \wedge v < z)].$$

Then for $i, j \in \omega$,

$$i < j \text{ if and only if } \mathscr{U} \models \Psi[y_i, y_j].$$

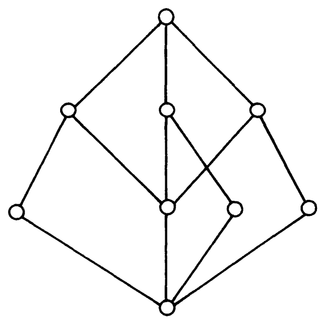
By Theorem 0.3, $Th(\mathscr{U})$ is unstable.

It can be shown that $\mathscr{U}$ has dimension 3 or 4, although we have been unable to determine which.

In the next three sections we study two classes of superstable height 4 lattices.

2. A structure theorem. There is a class of height 4 lattices with particularly nice structure containing the height 4, dimension $\leqq 2$ lattices.

Definition 2.1. Following [**19**], let B be the lattice:



and let B^d be the dual of B. Let $\mathscr{S}$ be the class of all lattices of height 4 which omit both B and B^d.

It follows from Theorem 0.6 ([**19**], Theorem 6.1) that the class $\mathscr{S}$ properly contains the class of all height 4, dimension $\leqq 2$ lattices.

Definition 2.2. Let $\mathfrak{A}$ be a finite height lattice, $n \in \omega$, $x, y, z_0, \ldots, z_n \in A$. $\langle z_0, \ldots, z_n \rangle$ is a *connecting sequence* if for $i < j \leqq n$, $z_i \neq z_j$; for $i < n$, $z_i < z_{i+1}$ or $z_i < z_{i+1}$; and for $i \leqq n$, $z_i \notin \{0, 1\}$. x is *connected to* y *in* n *steps* if there is a connecting sequence $\langle z_0, \ldots, z_n \rangle$ with $z_0 = x$ and $z_n = y$. x is *connected* to y if $x = y = 0$ or $x = y = 1$ or x is connected to y in n steps for some $n \in \omega$. We write $x\, C\, y$ for x is connected to y. We refer to the immediate successors of 0 as *level* 2, immediate successors of level 2 elements which are not equal to 1 as *level* 3.

LEMMA 2.3. *C is an equivalence relation on A.*

If $a \in A$, then $[a]$ denotes the C-class of a in A.

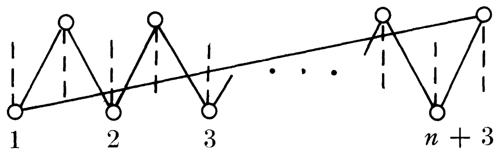
Definition 2.4. Let $\mathfrak{A}$ be a lattice in $\mathscr{S}$ and let $x, y \in A$-$\{0, 1\}$. We say x is a *dead end* off y if x is comparable to 0, 1 and y and to no other elements of $\mathfrak{A}$. We say y has κ dead ends if there are κ many dead ends off y.

THEOREM 2.5. *For any lattice* $\mathfrak{A}$ *in* $\mathscr{S}$, *each C-class of* $\mathfrak{A}$ *has exactly one of the following forms: (Dashed lines represent* $\geqq 0$ *dead ends.)*

(1) *the classes of* 0 *and* 1: $\{0\}$ *and* $\{1\}$;

(2) *lone elements L*: $\{x\}$ *where* $x \notin \{0, 1\}$;

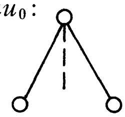
(3) *crowns* Cr_n:



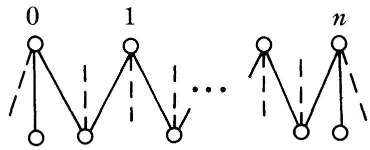
$n \geqq 0$;

(4) *finite length fences* —

(*up-up*) Fuu_0:

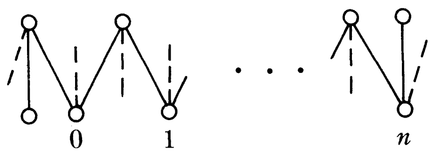


Fuu_n:



$n > 0$

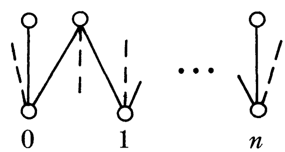
(*up-down*) Fud_n:



$n \geqq 0$

(*down-down*) Fdd_0:

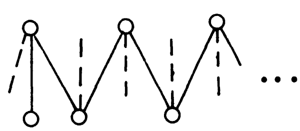
Fdd_n:



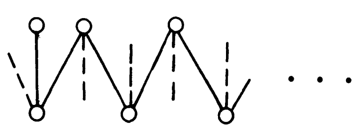
$n > 0$

(5) 1-*way infinite fences*

I^1u:



I^1d:



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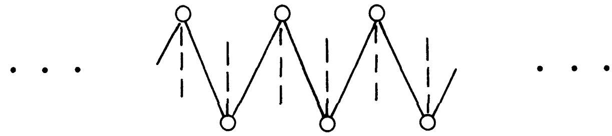
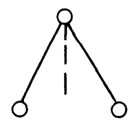
(6) *2-way infinite fences* I^2:



Proof. It is easy to check that none of these C-classes violate the axioms of $\mathscr{S}$. Given a class X from $\mathfrak{A}$, $\mathfrak{A}$ in $\mathscr{S}$, we must show that X has one of the forms in 1 to 6.

Suppose $X \neq \{0\}$, $X \neq \{1\}$ and there is no level 2 element $x \in X$ and elements $y_1 \neq y_2$ such that $x < y_1, y_2 < 1$. If w_1 and w_2 are level 3 elements in X then there is a connecting sequence $\langle z_0, \ldots, z_n \rangle$ with $z_0 = w_1$ and $z_n = w_2$. If $n > 0$, then z_1, w_1, w_2 violate the assumption above. So $n = 0$ and $w_1 = w_2$. So either X has no level 3 elements and has the form L, or X has exactly one level 3 element and has the form

Fdd_0  or Fuu_0 

Otherwise we can assume that $x_0 \in X$ and there are elements $y_1 \neq y_2$ such that $x_0 < y_1, y_2 < 1$.

For each $k \in \omega$ let

$$X_k = \{y \in A : y \text{ is connected to } x_0 \text{ in } \leqq k \text{ steps}\}.$$

We show by induction on k that the elements of each X_k are ordered in one of the forms in 2–4.

$k = 0$: $X_0 = \{x_0\}$. So X_0 has the form L.

$k = 1$: $X_1 = \{x_0\} \cup \{y \in A : x_0 < y < 1\}$. So X_1 has the form Fdd_0.

$k + 1$ where $k \geqq 1$: If $X_k = X_{k-1}$, then $X_{k+1} = X_k$ and the result follows from the induction hypothesis. So assume $X_k \neq X_{k-1}$. Notice that X_k is not of the form L or Fuu_0. There are 5 cases.

case (a). X_k has the form Cr_n.

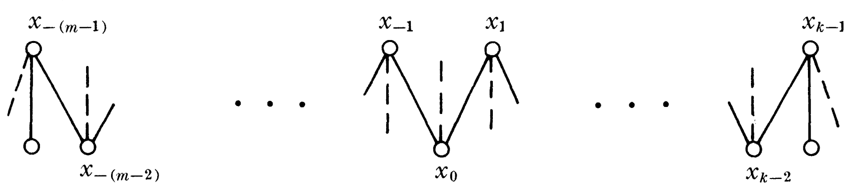
case (b). X_k has the form Fuu_n, $n > 0$.

case (c). X_k has form Fud_n.

case (d). X_k has form Fdd_0.

case (e). X_k has form Fdd_n, $n > 0$.

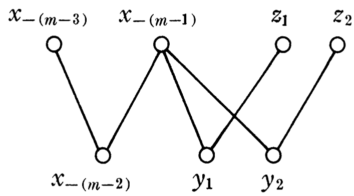
We shall give the proof of case (b). The other cases are proved similarly. So X_k looks like



$m \leqq k$, $n = (k + m)/2 - 1$. (Fuu_n must go k steps in at least one direction from x_0 by the assumption that $X_k \neq X_{k-1}$. It follows from the form of X_k that k, $m \geqq 2$.) Let $z \in X_{k+1} - X_k$. Either (i) $m = k$ and there is a $y < x_{-(m-1)}$ such that $z > y$; or (ii) there is a $y < x_{k-1}$ such that $z > y$; or (iii) both (i) and (ii). More than one such y in (i), (ii), or (iii) implies that $x_{-(m-1)}$ and z, or $x_{-(k-1)}$ and z have no infimum, so either (i′) $m = k$ and there is a unique $y < x_{-(m-1)}$ such that $z > y$; or (ii′) there is a unique $y < x_{k-1}$ such that $z > y$; or (iii′) both (i′) and (ii′).

If there is no z such that (iii′) then X_{k+1} has one of the forms Fuu_n (no new z at all), Fud_n, Fdd_{n+1}.

If there is a z such that (iii′) then it is unique, for suppose there were elements z_1, z_2, y_1, y_1', y_2, y_2' such that $y_1, y_2 < x_{-(m-1)}$; $y_1', y_2' < x_{k-1}$; $y_1, y_1' < z_1$; $y_2, y_2' < z_2$. If $y_1 \neq y_2$ then we have



and a copy of B^d in $\mathfrak{A}$. Similarly $y_1' \neq y_2'$ provides a copy of B^d in $\mathfrak{A}$. But if $y_1 = y_2$ and $y_1' = y_2'$, then z_1 and z_2 have no infimum. Now with a unique z satisfying (iii′), X_{k+1} has the form Cr_{n-1}.

The induction is complete. Note that $X = \bigcup (X_k : k \in \omega)$. If for some k, $X_{k+1} = X_k$ then $X = X_k$ and has one of the forms in (2) − (4). If for all k, $X_{k+1} \supsetneqq X_k$ then no X_k is a crown and X has form I^1u, I^1d or I^2.

Note that elements in $A - \{0, 1\}$ which are in different C-classes are not comparable. So Theorem 2.5 presents a very concrete structure for lattices in $\mathscr{S}$. This structure will be instrumental in characterizing the ω-stable models, the $\aleph_0$-categorical models, and the $\aleph_1$-categorical models in $\mathscr{S}$.

3. Stability and the class $\mathscr{S}$.

We now prove that all lattices in $\mathscr{S}$ are superstable. An alternate proof of this and a more general result are discussed in Section 7. The proof we give here is the most natural one, given the nice structure of the lattices in $\mathscr{S}$. It uncovers the "approximating formulae" for C-classes and exemplifies methods to be applied in more general settings.

LEMMA 3.1. *Let $\mathfrak{A}$ be in $\mathscr{S}$ and let $\mathfrak{B} > \mathfrak{A}$, $b \in B - A$. Either there is an element $a_0 \in A - \{0, 1\}$ such that b is a dead end off a_0, or b is not connected to any element of $\mathfrak{A}$.*

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Proof. This follows from the structure theorem and the first order theory of $\mathfrak{A}$. In fact, if b is a dead end off a_0, it is easy to see that a_0 has infinitely many dead ends.

LEMMA 3.2. *Let $\mathfrak{A}$ be a lattice in $\mathscr{S}$ and suppose $\mathfrak{B} \succ \mathfrak{A}$, $b \in B - A$ and b is a dead end off an element $\bar{a}$ in A. Let*

$$\Gamma(v) = Th(\mathfrak{A}_A) \cup \{v \neq a: a \in A\} \cup \{v \textit{ is a dead end off } \bar{a}\}.$$

Then $\Gamma(v)$ is a complete type in $\mathfrak{A}_A$.

Proof. Let $\mathfrak{C} \succ \mathfrak{A}$, $c \in C$ such that $\mathfrak{C}_A \models \Gamma[c]$. Since $b \in B - A$ and $\mathfrak{B} \succ \mathfrak{A}$ $\mathfrak{A} \models$ "$\bar{a}$ has $\geqq k$ dead ends" for each $k \in \omega$. It follows that in $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ there are infinitely many dead ends off $\bar{a}$. Using the method of games of [**9**] and [**12**] it is easy to show that $(\mathfrak{B}_A, b) \equiv (\mathfrak{C}_A, c)$.

LEMMA 3.3. *Let $\mathfrak{A}$ be a lattice in $\mathscr{S}$ and suppose $\mathfrak{B} \succ \mathfrak{A}$, $b \in B - A$ and b is not connected to any element of $\mathfrak{A}$. Let $\Sigma_b(v)$ be the type of b in $\mathfrak{B}$ and let*

$$\Gamma(v) = Th(\mathfrak{A}_A) \cup \Sigma_b(v)$$
$$\cup \{(v \textit{ is not connected to } a \textit{ in } n \textit{ steps}): a \in A, n \in \omega\}.$$

Then $\Gamma(v)$ is a complete type in $\mathfrak{A}_A$.

Proof. Let $\mathfrak{C} \succ \mathfrak{A}$, $c \in C$ such that $\mathfrak{C}_A \models \Gamma(c)$. We show that $(\mathfrak{B}_A, b) \equiv (\mathfrak{C}_A, c)$ using the method of games. Let $a_0, \ldots, a_{k-1} \in A$. We will give player II's winning strategy for the $((\mathfrak{B}, a_0, \ldots, a_{k-1}, b) \equiv (\mathfrak{C}, a_0, \ldots, a_{k-1}, c))$-game with m rounds. During the game player II will "protect" certain finite "neighbourhoods" of the elements $a_0, \ldots, a_{k-1}, b, c$ and the picks from $\mathfrak{B}$ and $\mathfrak{C}$ made in the game. Assume $B \cap C = A$.

Let $m_0 = m + k + 1$. Given $b_0, \ldots, b_{p-1} \in B$ and a non dead end $x \in B - \{0, 1\}$, we say a set P is an *approximation* of x in $(\mathfrak{B}, b_0, \ldots, b_{p-1})$ if P consists of:

(a) all dead ends off x in $\mathfrak{B}$, if there are $\leqq m_0$ of them; or

(b) m_0 dead ends off x including those among $\{b_0, \ldots, b_{p-1}\}$, if there are $> m_0$.

For each $n \leqq m$ let $\{x_0, \ldots, x_{k_n}\}$ be the set of all non dead ends connected to elements of $\{b_0, \ldots, b_{p-1}\}$ in $< 3^{m-n}$ steps. We say a set X is an *$(m$-$n)$-neighbourhood of $\{b_0, \ldots, b_{p-1}\}$ (in $\mathfrak{B}$)* if for each $i \leqq k_n$ there is an approximation P_i of x_i such that

$$X = \{x_0, \ldots, x_{k_n}\} \cup \bigcup(P_i: i \leqq k_n).$$

Remark 1. If X and Y are $(m$-$n)$-neighbourhoods of $\{b_0, \ldots, b_{p-1}\}$ then there is an isomorphism $f: X \rightarrow Y$ such that f is the identity map on $\{b_0, \ldots, b_{p-1}\}$, and $x \in X$ is an dead end in $\mathfrak{B}$ if and only if $f(x)$ is a dead end in $\mathfrak{B}$.

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We can similarly define $(m\text{-}n)$-neighbourhoods of finite subsets of $\mathfrak{A}$ and $\mathfrak{C}$.

We will use the formulae $\gamma_k(v)$ which say "v has $\geqq k$ dead ends".

Let B_0 be an $(m\text{-}0)$-neighbourhood of $\{a_0, \ldots, a_{k-1}, b\}$ in $\mathfrak{B}$ and let C_0 be an $(m\text{-}0)$-neighbourhood of $\{a_0, \ldots, a_{k-1}, c\}$ in $\mathfrak{C}$.

Claim. There is an isomorphism $f_0: B_0 \rightarrow C_0$ such that $f_0(a_i) = a_i$ for $i < k$, $f_0(b) = c$, and $x \in B_0$ is a dead end in $\mathfrak{L}$ if and only if $f_0(x)$ is a dead end in $\mathfrak{C}$.

Proof. The non dead ends of $\mathfrak{B}$ and $\mathfrak{C}$ connected to elements of $\mathfrak{A}$ are all in $\mathfrak{A}$ by Lemma 3.1. $Th(\mathfrak{A}_A)$ contains $\pm\gamma_k[a]$ for each $a \in A$ and $k \leqq m_0$, so we can easily define $f_0 \restriction_{B_0 - [b]}$.

Now let $x_0, \ldots, x_p$ be the non dead ends of $[b]$ which are connected to b in $<3^m$ steps. Let $\phi(v)$ be the formula:

$$\begin{aligned}
&(\exists v_0) \ldots (\exists v_p)[(\{v_0, \ldots, v_p\} \text{ is the set of all non dead ends} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{connected to } v \text{ in } <3^m \text{ steps}) \\
&\wedge \textstyle\bigwedge_{i,j \leqq p} (\pm v_i \leqq v_j{:}\ \pm x_i \leqq x_j) \\
&\wedge \textstyle\bigwedge_{i \leqq p} (\pm \gamma_k(v_i){:}\ \mathfrak{B} \models \pm\gamma_k[x_i],\ k \leqq m_0) \\
&\wedge \textstyle\bigwedge_{i \leqq p} (\pm v \leqq v_i{:}\ \pm b \leqq x_i) \wedge \textstyle\bigwedge_{i \leqq p} (\pm v_i \leqq v{:}\ \pm x_i \leqq b)].
\end{aligned}$$

$\phi(v)$ is in $\Sigma_b(v)$ and so $\mathfrak{C} \models \phi\,[c]$. Clearly we can extend f_0 to B_0 so that $f_0(b) = c$ and so x is a dead end in $\mathfrak{B}$ if and only if $f_0(x)$ is a dead end in $\mathfrak{C}$ for $x \in B_0$. This proves the claim.

We can now begin the game. Player II's strategy will be to satisfy the following:

Induction hypothesis. Suppose after $n < m$ rounds $b_0, \ldots, b_{n-1}$, $c_0, \ldots, c_{n-1}$ have been chosen and there are $(m\text{-}n)$-neighbourhoods B_n of $\{a_0, \ldots, a_{k-1}, b, b_0, \ldots, b_{n-1}\}$ in $\mathfrak{B}$ and C_n of $\{a_0, \ldots, a_{k-1}, c, c_0, \ldots, c_{n-1}\}$ in $\mathfrak{C}$ and an isomorphism $f_n: B_n \rightarrow C_n$ such that

(i) $f_n(a_i) = a_i, f_n(b) = c, f_n(b_i) = c_i$,

(ii) for $x \in B_n$, x is a dead end in $\mathfrak{B}$ if and only if $f_n(x)$ is a dead end in $\mathfrak{C}$,

(iii) for every $x \in B_n$ connected to an element of $\{a_0, \ldots, a_{k-1}, b, b_0, \ldots, b_{n-1}\}$ in $<2.3^{m-(n+1)}$ steps either

(a) for every non dead end $\bar{a}$ of A: $x < \bar{a}$, $x = \bar{a}$, or $x > \bar{a}$ if and only if $f(x) < \bar{a}$, $f(x) = \bar{a}$, or $f(x) > \bar{a}$ respectively, or

(b) there are elements $\bar{a}_i \in A$, $i \in \omega$, such that $\bar{a}_i$ is not connected to $\bar{a}_j$ in $<2.3^{m-(n+1)}$ steps for $i \neq j$, and for each i there is an isomorphism g_i from an $(m - (n + 1))$-neighbourhood B' of x to an $(m - (n + 1))$-neighbourhood A_i of a_i with $g_i(x) = a_i$ and x dead end in $\mathfrak{B}$ if and only if $f(x)$ dead end in $\mathfrak{C}$.

(Note that the isomorphism f_n ensures that (iii) holds for $y \in C_n$ as well.)

The induction hypothesis is true with $n = 0$.

Round $n + 1$. Suppose player I picks b_n from $\mathfrak{B}$.

Case 1. b_n is connected to an element of $\{a_0, \ldots, a_{k-1}, b, b_0, \ldots, b_{n-1}\}$ in $<2.3^{m-(n+1)}$ steps. We can assume $b_n \in B_n$ and construct $B_{n+1} \subset B_n$, $f_{n+1} = f_n \upharpoonright_{B_{n+1}}$, $C_{n+1} = f_{n+1}(B_n)$, and let $c_n = f_{n+1}(b_n)$. The induction hypothesis for $n + 1$ follows easily.

Case 2. Not case 1, but b_n is connected to an element of $\{c_0, \ldots, c_{n-1}\}$ in $<2.3^{m-(n+1)}$ steps.

Suppose b_n is not a dead end. Then $b_n \in C_n$ and is therefore in A. But $f_n^{-1}(b_n) \neq b_n$ since $b_n \notin B_n$. So we have $f_n^{-1}(b_n)$ connected to an element of $\{a_0, \ldots, a_{k-1}, b, b_0, \ldots, b_{n-1}\}$ in $<2.3^{m-(n+1)}$ steps, $f_n(f_n^{-1}(b_n)) = b_n$, but $f_n^{-1}(b_n) \neq b_n$. So by (iii) of the induction hypothesis, we can find an element $c_n \in A \cap C$ which is not connected to an element of $\{a_0, \ldots, a_{k-1}, c, c_0, \ldots, c_{n-1}\}$ in $<2.3^{m-(n+1)}$ steps such that there is an isomorphism f' from an $(m - (n + 1))$-neighbourhood B' of $\{b_n\}$ in $\mathfrak{B}$ to an $(m - (n + 1))$-neighbourhood C' of $\{c_n\}$ in $\mathfrak{C}$, with $f'(b_n) = c_n$ and x a dead end in $\mathfrak{B}$ if and only if $f'(x)$ a dead end in $\mathfrak{C}$, for $x \in B'$. We can now construct an $(m - (n + 1))$-neighbourhood $B_n' \subset B_n$ of $\{a_0, \ldots, a_{k-1}, b, b_0, \ldots, b_{n-1}\}$ in $\mathfrak{B}$ and let

$$B_{n+1} = B_n' \cup B', \; C_{n+1} = f_n(B_n') \cup C' \text{ and } f_{n+1} = f' \cup f_n \upharpoonright_{B_n'}.$$

The induction hypothesis, in particular condition (iii), now holds for $n + 1$.

Similarly if b_n is a dead end.

Case 3. Not case 1 or 2, but b_n is either

(i) equal to a non dead end $\bar{a} \in A$, or

(ii) a dead end off an element $\bar{a} \in A$.

We let $c_n = \bar{a}$ if (i), and let c_n be any dead end off $\bar{a}$ in $\mathfrak{C}$ if (ii). Since $\bar{a} \in A$, $(m - (n + 1))$-neighbourhoods B' of $\{b_n\}$ and C' of $\{c_n\}$ are isomorphic and are disjoint from any $(m - (n + 1))$-neighbourhoods of $\{a_0, \ldots, a_{k-1}, b, b_0, \ldots, b_{n-1}\}$ and $\{a_0, \ldots, a_{k-1}, c, c_0, \ldots, c_{n-1}\}$. So we can easily construct B_{n+1}, $C_{n+1} f_{n+1}$ to satisfy the induction hypothesis for $n + 1$.

Case 4. Not case 1, 2 or 3. Then by Lemma 3.1, b_n is not connected to any element of A.

Let $x_0, \ldots, x_p$ be the non dead ends of $[b_n]$ which are connected to b_n in $<3^{m-(n+1)}$ steps. Let $\phi(v)$ be the formula:

$$(\exists v_0) \ldots (\exists v_p)[(\{v_0, \ldots, v_p\} \text{ is the set of all non dead ends connected to } v \text{ in } <3^{m-(n+1)} \text{ steps})$$
$$\wedge \bigwedge_{i,j \leqq p} (\pm v_i \leqq v_j \colon \pm x_i \leqq x_j)$$
$$\wedge \bigwedge_{i \leqq p} (\pm \gamma_k(v_i) \colon \models \pm \gamma_k[x_i], k \leqq m_0)$$
$$\wedge \bigwedge_{i \leqq p} (\pm v \leqq v_i \colon \pm b_n \leqq x_i) \wedge \bigwedge_{i \leqq p} (\pm v_i \leqq v \colon \pm x_i \leqq b_n)].$$

Since $b_n \in B - A$ and $\mathfrak{B} > \mathfrak{A}$, the formula

$$(\exists w_0) \ldots (\exists w_k)[\bigwedge_{i \leqq k} \phi(w_i) \wedge \bigwedge_{i<j\leqq k} (w_i \text{ is not connected to } w_j \text{ in } \leqq l \text{ steps})]$$

is true in $\mathfrak{A}$ for each $k, l \in \omega$. So let c_n be an element of A such that c_n is not connected to any element of $\{a_0, \ldots, a_{k-1}, c, c_0, \ldots, c_{n-1}\}$ in $< 2.3^{m-(n+1)}$ steps and $\mathfrak{A} \models \phi[c_n]$. $\mathfrak{C} > \mathfrak{A}$, so $\mathfrak{C} \models \phi[c_n]$ and we can construct $(m - (n + 1))$-neighbourhoods B' of $\{b_n\}$ and C' of $\{c_n\}$ and an isomorphism $f' : B' \rightarrow C'$. We are assured from the existence of $[b_n]$ in $B - A$ that $B_{n+1}, C_{n+1}, f_{n+1}$ constructed from B', C', B_n, C_n, f_n will satisfy the induction hypothesis, specifically condition (iii), for $n + 1$.

Suppose player I picks c_n from $\mathfrak{C}$. Then player II chooses a $b_n \in B$ similarly.

After m rounds, the isomorphism f_m wins the game for player II.

THEOREM 3.4. *Let $\mathfrak{A}$ be a lattice in $\mathscr{S}$. Then $Th(\mathfrak{A})$ is superstable.*

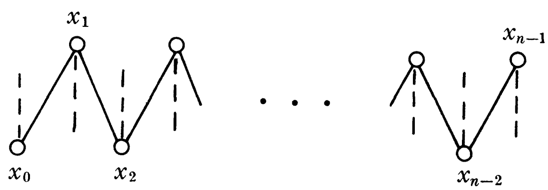
Proof. It will suffice to show for every $\mathfrak{A}$ in $\mathscr{S}$ that if $|A| \leqq \kappa$ then

$$|S_{\mathfrak{A}}(A)| \leqq \kappa + 2^{\omega}.$$

By Lemma 3.2 there are at most κ many types which include a formula "v is connected to a in n steps" for some $a \in A$, $n \in \omega$. By Lemma 3.3 there are at most 2^{ω} types which exclude all these formulae (at most one for each type $\Sigma_b(v)$ in the countable language with no constants).

There are lattices in $\mathscr{S}$ which do not have ω-stable theories. The following example is, in fact, dimension 2.

Example 3.5. For each even $n \in \omega$ and each $s \in {}^n2$, let $(X_s, \leqq)$ be the C-class:



where x_i has $s(i)$ dead ends for each $i < n$. Let

$$\mathfrak{A} = \{0, 1\} \cup \bigcup (X_s : n \text{ even}, s \in {}^n2).$$

For each $s \in {}^{\omega}2$ consider the set of formula

$$\Sigma_s(v_0) = \{(\exists v_1) \ldots (\exists v_n)[\bigwedge_{1 \leqq i \leqq n} (v_i \text{ is the unique non dead end connected to } v_0 \text{ in } i \text{ steps}) \wedge \bigwedge_{i \leqq n} (v_i \text{ has } s(i) \text{ dead ends})] : n \in \omega\}.$$

Clearly each $\Sigma_s(v_0)$ is consistent with $Th(\mathfrak{A})$ and $s \neq s'$ implies $\Sigma_s(v_0) \cup \Sigma_{s'}(v_0)$ is inconsistent. So $|S_{\mathfrak{A}}(\phi)| = 2^\omega$ and $Th(\mathfrak{A})$ is not ω-stable.

In the proof of Lemma 3.3 we used certain formulae to describe finite "neighbourhoods" of elements. We will see that these formulae actually determine C-classes up to (∞, ω)-equivalence and hence elementary equivalence. We will use this fact to determine which models in $\mathcal{S}$ have ω-stable theories, and later, in Section 6, to characterize the models of $\mathcal{S}$ with $\aleph_0$-categorical theories and the models with $\aleph_1$-categorical theories.

Definition 3.6. Recall the formulae $\gamma_k(v) \equiv$ "v has $\geqq k$ dead ends". Given a lattice $\mathfrak{A}$ in $\mathcal{S}$ and an element a in $A - \{0, 1\}$, we define formulae $\psi_{n,m}{}^{\mathfrak{A},a}$ and $\phi_{n,m}{}^{\mathfrak{A},a}$ for each $n, m \in \omega$. Let $a_0, \ldots, a_{p_n}$ be the non dead ends connected to a in $\leqq n$ steps.

$$\begin{aligned}\psi_{n,m}{}^{\mathfrak{A},a}(v, v_0, \ldots, v_{p_n}) \equiv [(&\{v_0, \ldots, v_{p_n}\} \\ &\text{is the set of non dead ends connected to } v \text{ in} \leqq n \text{ steps}) \\ \wedge \textstyle\bigwedge_{i,j \leqq p_n} &(\pm v_i \leqq v_j \colon \pm a_i \leqq a_j) \\ \wedge \textstyle\bigwedge_{i \leqq p_n} \bigwedge_{k \leqq m} &(\pm \gamma_k(v_i) \colon \mathfrak{A} \vDash \pm \gamma_k[a_i]) \\ \wedge \textstyle\bigwedge_{i \leqq p_n} &(\pm v \leqq v_i \colon \pm a \leqq a_i) \wedge \textstyle\bigwedge_{i \leqq p_n} (\pm v_i \leqq v \colon a_i \leqq a)].\end{aligned}$$

Note that $\Psi_{n,m}{}^{\mathfrak{A},a}$ is uniquely defined modulo the names of the variables v_i and the order of the conjuncts. We let

$$\phi_{n,m}{}^{\mathfrak{A},a}(v) \equiv (\exists v_0) \ldots (\exists v_{p_n}) \psi_{n,m}{}^{\mathfrak{A},a}(v, v_0, \ldots, v_{p_n}).$$

LEMMA 3.7. *Let $\mathfrak{A}$ be a lattice in $\mathcal{S}$, $a \in A - \{0, 1\}$. Suppose $\mathfrak{B}$ is in $\mathcal{S}$ and $b \in B$ such that*

$$\mathfrak{B} \vDash \phi_{n,m}{}^{\mathfrak{A},a}[b] \quad \textit{for all } n, m \in \omega.$$

Then

$$\langle [b]; \leqq, b \rangle \equiv_{\infty,\omega} \langle [a], \leqq, a \rangle.$$

Proof. We break the proof into three cases.

Case 1. The set of non dead ends connected to a is finite.

Case 2. $[a]$ is a 1-way infinite fence.

Case 3. $[a]$ is a 2-way infinite fence.

We give the proof in the most difficult case.

Proof of case 3. First note that for each $n' \leqq n$, $k' \leqq k$

$$(*) \quad \vDash \psi_{n,k}{}^{\mathfrak{A},a}(v, v_0, \ldots, v_p) \rightarrow \psi_{n',k'}{}^{\mathfrak{A},a}(v, v_0, \ldots, v_p).$$

Let $\ldots, a_{-1}, a_0, a_1, \ldots$ enumerate the non dead ends of $[a]$ so that $a_z < a_{z+1}$ or $a_z > a_{z+1}$ for all $z \in Z$ and $a = a_0$ or a is a dead end off a_0. By renaming variables we can assume that v_z is associated with a_z in each $\psi_{n,m}{}^{\mathfrak{A},a}$ with v_z free. $\mathfrak{B} \vDash \phi_{n,0}{}^{\mathfrak{A},a}[b]$ for each $n \in \omega$, so $[b]$ is a 2-way

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infinite fence. Choose b_0 so that $b = b_0$ if b is non dead end, or so that b is a dead end off b_0 if b us a dead end. Clearly for each $n > 0$, b_0 is the unique element x of B such that

$$\mathfrak{B} \models (\exists v_{(p_n-1)/2}) \ldots (\exists v_{-1})(\exists v_1) \ldots (\exists v_{(p_n-1)/2})\psi_{n,0}{}^{\mathfrak{A},a}[b, x].$$

For each n, there are 2 elements $x \in B$ such that

$$\mathfrak{B} \models (\exists v_{(p_n-1)/2}) \ldots (\exists v_{-1})(\exists v_2) \ldots (\exists v_{(p_n-1)/2})\psi_{n,0}{}^{\mathfrak{A},a}[b, b_0, x].$$

It follows from (*) that there is at least one element b_1 such that

$$\mathfrak{B} \models (\exists v_{(p_n-1)/2}) \ldots (\exists v_{-1})(\exists v_2) \ldots (\exists v_{(p_n-1)/2})\psi_{n,k}{}^{\mathfrak{A},a}[b, b_0, b_1]$$

for all n and k.

The pattern is now set. For each n and each z such that $-(p_n - 1)/2 \leqq z \leqq (p_n - 1)/2$, there is a unique element b_z such that $\psi_{n,0}{}^{\mathfrak{A},a}[b, b_0, b_1, b_z]$ is satisfiable in $\mathfrak{B}$, because there is exactly one non dead end connected to b_0 and b_1 in the number of steps specified by $\psi_{n,0}{}^{\mathfrak{A},a}$. Since

$$\mathfrak{B} \models (\exists v_{(p_n-1)/2}) \ldots (\exists v_{-1})(\exists v_2) \ldots (\exists v_{(p_n-1)/2})\psi_{n,k}{}^{\mathfrak{A},a}[b, b_0, b_1]$$

for all $k, n \in \omega$, we can enumerate the non dead ends of $[b]$ as $\ldots, b_{-1}, b_0, b_1, \ldots$ so that

$$\mathfrak{B} \models \psi_{n,k}{}^{,a}[b, b_{(p_n-1)/2}, \ldots, b_{(p_n-1)/2}]$$

for all $n, k \in \omega$. It follows that $b_z \leqq b_{z'}$, if and only if $a_z \leqq a_{z'}$; $\gamma_k[b_z]$ if and only if $\gamma_k[a_z]$; $b = b_0$ and $a = a_0$, or b is a dead end off b_0 and a is a dead end off a_0. Using the method of games it is now easy to show that

$$\langle [b], \leqq, b \rangle \equiv_{\infty,\omega} \langle [a], \leqq, a \rangle.$$

We will now use the "approximating formulae", $\phi_{n,m}{}^{\mathfrak{A},a}\ (v)$, to characterize the ω-stable models in $\mathscr{S}$ in terms of "approximating C-classes".

Definition 3.8. Let $\mathfrak{A}$ be in $\mathscr{S}$ and let $a \in A - \{0, 1\}$ be a non dead end. Let $\#_{\mathfrak{A}}a$ denote the number of dead ends off a in $\mathfrak{A}$. We say a is *end of fence* (*in* $\mathfrak{A}$) if there is at most one other non dead end in $A - \{0, 1\}$ comparable to a.

LEMMA 3.9. *Let $X_0, X_1, \ldots$ be finite subclasses of C-classes of $\mathfrak{A}$, $\mathfrak{A}$ in $\mathscr{S}$. (i.e., $x, y \in X_i \rightarrow$ there is a connecting sequence $\langle z_0, \ldots, z_n \rangle$ in X_i with $x = z_0$ and $y = z_n$.) Let*

$$F = \{f_{ij}: (X_i, \leqq) \rightarrow (X_j, \leqq): i \leqq j\}$$

be a family of embeddings such that for $i \leqq j \leqq k$, $f_{ii} = \mathrm{id}_{X_i}$ and $f_{jk} \circ f_{ij} = f_{ik}$. Form the direct limit $(X, \leqq)$ of $(\langle (X_i, \leqq): i \in \omega \rangle, F)$ with embeddings $f_i: X_i \rightarrow X$. Then $\bar{X} = (X \cup \{0, 1\}, \leqq)$ is in $\mathscr{S}$ and X is a C-class in $\bar{X}$.

Proof. The proof is straightforward.

Definition 3.10. Let $(X, \leqq)$ with embeddings $f_i: X_i \to X$, $i \in \omega$ be the direct limit of $(\langle (X_i, \leqq): i \in \omega \rangle, F)$, where each X_i is a finite subclass of $\mathfrak{A}$, $\mathfrak{A}$ in $\mathscr{S}$. Suppose for each $i \in \omega$, $x \in X_i$

(1) $f_i(x)$ end of fence in $\bar{X}$ implies x end of fence in $\mathfrak{A}$,

(2) $f_i(x)$ dead end in $\bar{X}$ if and only if x dead end in $\mathfrak{A}$,

(3) for each $k < i$, $\#_X f_i(x) = k$ implies $\#_{\mathfrak{A}} x = k$.

Then we say $(X, \leqq)$ is a *limit C-class of* $\mathfrak{A}$.

Notice that limit C-classes are countable, and if X is a finite limit C-class of $\mathfrak{A}$ then $\mathfrak{A}$ contains a C-class isomorphic to X.

LEMMA 3.11. *Let $(X, \leqq)$ be a countably infinite C-class of a lattice $\mathfrak{B}$ in $\mathscr{S}$ and let $\mathfrak{A}$ be in $\mathscr{S}$. $(X, \leqq)$ is a limit C-class of $\mathfrak{A}$ if and only if for some $x_0 \in X$, $\{\phi_{n,m}^{\mathfrak{B},x_0}(v): n, m \in \omega\}$ is consistent with $Th(\mathfrak{A})$.*

Proof. The proof is routine given the information expressed in the formulae $\phi_{n,m}^{\mathfrak{B},x_0}(v)$.

THEOREM 3.12. *Let $\mathfrak{A}$ be a lattice in $\mathscr{S}$. $Th(\mathfrak{A})$ is ω-stable if and only if $\mathfrak{A}$ has only countably many non-isomorphic limit C-classes.*

Proof. Assume $\mathfrak{A}$ has $>\omega$ non-isomorphic limit C-classes. For each such class X, by Lemma 3.11, we can choose $x \in X$ such that

$$\Sigma_X(v) = \{\phi_{n,m}{}^{\bar{X},x}(v): n, m \in \omega\}$$

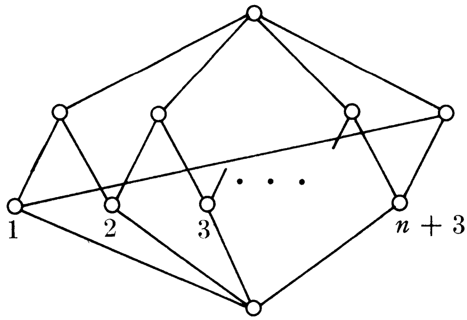
is consistent with $Th(\mathfrak{A})$. For countable classes X and Y, $X \equiv_{\infty,\omega} Y$ if and only if $X \simeq Y$ so it follows from Lemma 3.7 that if X and Y are non-isomorphic limit C-classes of $\mathfrak{A}$, then $\Sigma_X(v) \cup \Sigma_Y(v)$ is inconsistent. Hence we have $>\omega$ types in $S_{\mathfrak{A}}(\phi)$ and $Th(\mathfrak{A})$ is not ω-stable. The other direction is similar.

Example 3.13. In the class $\mathscr{S}$, ω-stability is a strictly weaker property than $\aleph_1$-categoricity. Consider a lattice with 2 level 2 elements each with $\geqq\omega$ dead ends.

We leave to Section 6 the task of classifying the lattices in $\mathscr{S}$ with $\aleph_1$-categorical theories.

4. Superstability and the class $\mathscr{T}$. We now turn to our second class of lattices containing the height 4, dimension $\leqq 2$ lattices.

Following [**19**], for each $n \in \omega$ we let A_n be the following lattice, called a *crown* (see also Theorem 2.5(3)): $\mathscr{T}$ is the class of all height 4 lattices which omit A_n for every $n \in \omega$. We will show that each lattice in $\mathscr{T}$ has a superstable theory. So a height 4 lattice which is not superstable must contain both a crown and a copy of either B or B^d.



The following obvious fact about lattices in $\mathscr{T}$ is crucial to this section.

LEMMA 4.1. *If a_1 and a_2 are elements of a lattice $\mathfrak{A}$ in $\mathscr{T}$ and a_1 is connected to a_2, then there is a unique connecting sequence between a_1 and a_2; i.e., there is a unique n and a unique connecting sequence $\langle z_0, \ldots, z_n \rangle$ such that $z_0 = a_1$ and $z_n = a_2$.*

Definition 4.2. Let $\mathfrak{A}$ be a lattice in $\mathscr{T}$ and let X be a finite set of elements from $A - \{0, 1\}$. For each $k \in \omega$, let

$$X_k = \{x \in A: x \text{ is connected to an element of } X \text{ in } k \text{ steps, but not to any element of } X \text{ in } {<k} \text{ steps}\}.$$

Fix $n, m \in \omega$. We will construct sets $X^{n,m,k}$, $k \leqq n$, by induction on k.

$$X^{n,m,0} = \bigcup(X_i: i \leqq n).$$

Now for $k < n$ and $x \in X_{n-k}$ let

$$P_x = \{z \in X^{n,m,k} - \bigcup(X_i: i < n - k): \text{there exist } i_0 \leqq j_0 \leqq n \text{ and a connecting sequence } \langle w_0, \ldots, w_{j_0} \rangle \text{ such that } w_0 \in X, w_{i_0} = x \text{ and } w_{j_0} = z\}.$$

Define a tree ordering $\leqq_x$ on P_x by $z \leqq_x z'$ if and only if there exist $i_0 \leqq j_0 \leqq n$ and a connecting sequence $\langle w_0, \ldots, w_{j_0} \rangle$ such that $w_0 = x$, $w_{i_0} = z$ and $w_{j_0} = z'$. This is a well defined partial ordering by Lemma 4.1. Let $\mathscr{P}_x = \langle P_x, \leqq_x \rangle$. We will discard "surplus" copies of $\mathscr{P}_x$. For each $y \in X_{n-(k+1)}$ there are cardinals $\lambda_0, \ldots, \lambda_p$ so that the elements of X_{n-k} comparable to y can be enumerated as $x_\alpha{}^i$: $i \leqq p$, $\alpha < \lambda_i$, where for $i, j < p$, $\mathscr{P}_{x_\alpha{}^i} \simeq \mathscr{P}_{x_\beta{}^i}$ if and only if $i = j$; and for $i \leqq p$, $x_\alpha{}^i$ is in a connecting sequence of length $\leqq 2n + 2$ between elements of X if and only if $i = p$. Note that λ_p is finite. Let

$$X_y = \bigcup(P_{x_\alpha{}^i}: i = p \text{ or } \alpha < \min\{\lambda_i, m\}).$$

Finally let

$$X^{n,m,k+1} = \bigcup(X_i: i \leqq n - (k + 1)) \cup \bigcup(X_y: y \in X_{n-(k+1)}).$$

Any set constructed in the manner of $X^{n,m,n}$ above will be called an $(n+1, m)$-neighbourhood of X (in $\mathfrak{A}$). These neighbourhoods are what is needed to carry out an analysis paralleling Lemmas 3.1, 3.2 and 3.3. We refer the reader to [**36**] for the remaining details.

THEOREM 4.3. *If* $\mathfrak{A}$ *is a lattice in* $\mathscr{T}$, *then* $Th(\mathfrak{A})$ *is superstable.*

5. A "merely" stable lattice of height 4. In Section 1 we saw that there are height 4 lattices with unstable theories. In this section we construct a height 4 lattice with a stable but not superstable theory. From Sections 3 and 4, we know that such a lattice must contain a crown and a copy of either B or B^d.

The example we give owes a debt to Example 7.1.32 of [**4**].

Definition 5.1. $\langle z_0, \ldots, z_n \rangle$ is a *direct connecting sequence* if $\langle z_0, \ldots, z_n \rangle$ is a connecting sequence and for $0 < i < n$, z_i is comparable to exactly 2 elements other than 0 and 1. We say z_0 is *directly connected to* z_n *in* n *steps*.

Example 5.2. We describe our example $\mathscr{M}$ as follows:

(i) $\mathscr{M}$ has a set $\{y_i: i \in \omega\}$ of *range elements* which are level 2 elements with exactly 1 dead end.

(ii) $\mathscr{M}$ has a set $\langle f_s: s \in {}^{\omega}\omega\}$ of *function elements* which are level 2 elements with exactly 2 dead ends.

(iii) For each $s \in {}^{\omega}\omega$ and each $i \in \omega$ there is a unique direct connecting sequence $\langle f_s, z_1, \ldots, z_{2i+1}, y_{s(i)} \rangle$ from f_s to $y_{s(i)}$.

In order to investigate the stability of $Th(\mathscr{M})$, we will need to know something about the other models of $Th(\mathscr{M})$.

Let $\mathfrak{A} \equiv \mathscr{M}$. A number of facts about $\mathfrak{A}$ follow from the first order properties of $\mathscr{M}$.

Property 1. $\mathfrak{A}$ has an infinite set of range elements; level 2 elements with exactly 1 dead end.

Property 2. $\mathfrak{A}$ has an infinite set of function elements; level 2 elements with exactly 2 dead ends.

Property 3. The function elements of $\mathfrak{A}$ "behave like" the function elements of $\mathscr{M}$. For each $n \in \omega$,

$$\mathfrak{A} \models (\forall v)[(v \text{ is a function element}) \to (\exists\, !y)(y \text{ is a range element and } f \text{ is directly connected to } y \text{ in } 2n+2 \text{ steps})].$$

Moreover, for each $n \in \omega$,

$$\mathfrak{A} \models (\forall x)(\forall y)[\text{there is at most 1 connecting sequence } \langle x, z_1, \ldots, z_n, y \rangle \text{ such that } \textstyle\bigwedge_{1 \leqq i \leqq n} (z_i \text{ has no dead ends})].$$

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We let $f(n)$ denote the unique range element connected to the function element f in $2n + 2$ steps.

Property 4. $\mathfrak{A}$ must have enough function elements to satisfy the following formula (true in $\mathscr{M}$):

$$(\forall x_0) \dots (\forall x_{n-1})[\wedge_{i<n} (x_i \text{ is a range element} \rightarrow (\exists f_0) \dots (\exists f_{k-1})(\bigwedge_{i<k} (f_i \text{ is a function element}) \wedge \bigwedge_{j<k, i<n} (f_j(i) = x_i) \wedge \bigwedge_{i<j<k} (f_i \neq f_j))]$$

for each $n, k \in \omega$.

Property 5. The elements of $\mathfrak{A}$ which are not range or function elements, dead ends off range or function elements, or elements on the unique direct connecting sequences between function elements f and their range elements $f(n)$, $n \in \omega$, are either

(i) members of infinite sequences $\langle z_i : i \in \omega \rangle$ where z_0 is comparable to a range or function element and for all i, $z_i \notin \{0, 1\}$, z_i is comparable to z_{i+1} and z_i has no dead ends, or

(ii) members of 2-way infinite fences $\langle z_i : i \in Z \rangle$, $z_i \notin \{0, 1\}$, where for all $i \in Z$, z_i has no dead ends.

The following lemma states that all height 4 lattices satisfying properties 1 to 5 above are elementarily equivalent. Since $\mathscr{M}$ satisfies 1 to 5, this shows that properties 1 to 5 are necessary and sufficient conditions for a height 4 lattice to be elementarily equivalent to $\mathscr{M}$.

LEMMA 5.3. *Let $\mathfrak{A}$ and $\mathfrak{B}$ satisfy* 1 *to* 5 *above. Then* $\mathfrak{A} \equiv \mathfrak{B}$.

Proof. In the $(\mathfrak{A} \equiv \mathfrak{B})$-game with m rounds, player II will play so that after n rounds, $0 < n \leqq m$, the following induction hypothesis is satisfied.

Induction hypothesis. For each $i < n$, $a_i \in A$ and $b_i \in B$ have been chosen in round $i + 1$ so that there exist $A_n \subset A$, $B_n \subset B$, $X_n \subset \{0, \dots, n - 1\}$, $\phi_n : A_n \rightarrow B_n$ and for each $i \in X_n$ function elements $f_i \in A$, $g_i \in B$ such that

(1) $i \in X_n$ if and only if there is a function element f in A and $k < 3^m$ such that a_i is a dead end off f, $a_i = f$, a_i is in the direct connecting sequence from f to $f(k)$, $a_i = f(k)$, a_i is a dead end off $f(k)$, or a_i is directly connected to f or $f(k)$ in $<2 \cdot 3^{m-(i+1)}$ steps; if and only if as above with B, b_i, g for A, a_i, f;

(2) For each $i \in X_n$, f_i and g_i fill the roles of f and g in (1);

(3) For all $i, j \in X_n$ and $k, k' < 3^m$, $f_i(k) = f_j(k')$ if and only if $g_i(k) = g_j(k')$, and $f_i = f_j$ if and only if $g_i = g_j$;

(4) $x \in A_n$ if and only if

(i) for some $i < n$, $i \in X_n$ and there is an $l < 3^m$ such that x is a dead end off f_i, $x = f_i$, x is in the direct connecting sequence from f_i to

$f_i(l)$, $x = f_i(l)$, or x is a dead end off $f_i(l)$; or x is in the direct connecting sequence from a_i to f_i or from a_i to $f_i(l)$; or

(ii) for some $i < n$, a_i is not a function or range element and x is directly connected to a_i in $<3^{m-n}$ steps.

$x \in B_n$ if and only if as above with g_i, b_i for f_i, a_i.

(5) ϕ_n: $(A_n, \leqq) \simeq (B_n, \leqq)$; for each $i \in X_n$, $\phi_n(f_i) = g_i$; and for each $i < n$, $\phi_n(a_i) = b_i$.

The details of player II's strategy are similar in spirit to those of Lemma 3.3 so we omit them.

COROLLARY 5.4. *$Th(\mathscr{M})$ is not superstable.*

Proof. For each infinite cardinal κ, let

$$T = \{t \in {}^{\omega}\kappa\colon t \text{ is eventually } 0\}.$$

$|T| = \kappa^{\underline{\omega}} = \kappa$. Let $\mathfrak{A}$ consist of κ range elements $\{y_\alpha\colon \alpha \in \kappa\}$ and a set of function elements $\{f_i\colon t \in T\}$ connected so that $f_t(n) = y_{t(n)}$ for each $n \in \omega$. It follows easily from Lemma 5.3 that $\mathfrak{A} \equiv \mathscr{M}$. Now for each $s \in {}^{\omega}\kappa$, let

$$\begin{aligned}\Sigma_s(v) = &\{v \text{ is a function element}\}\\ &\cup \{v(n) = f_t(n)\colon t(n) = s(n),\ t \in T\}\\ &\cup \{v(n) \neq f_t(n)\colon t(n) \neq s(n),\ t \in T\}.\end{aligned}$$

Each $\Sigma_s(v)$ is consistent with $(\mathfrak{A}, f_t)_{t\in T}$, and if $s \neq s'$ then $s(n) \neq s'(n)$ for some n, so for $t \in T$ such that $s\restriction_{n+1} \subseteq t$,

$$(v(n) = f_t(n)) \in \Sigma_s(v)$$

while

$$(v(n) \neq f_t(n)) \in \Sigma_{s'}(v).$$

So there are at least κ^ω complete types in $S_{\mathfrak{A}}(\{f_t\colon t \in T\})$.

The proof of Lemma 5.3 is more important than the lemma itself in what remains. As we count types over models of $Th(\mathscr{M})$ we can repeatedly use player II's winning strategy to show that certain types are complete. In each case we shall state the appropriate lemma but omit the straightforward proof.

Let $\mathfrak{C} \equiv \mathscr{M}$ and let $\kappa = |C|$. Let $\Sigma(v)$ be a complete type in $S_{\mathfrak{C}}(C)$ and we will examine the possibilities for $\Sigma(v)$. Note that for any function element $f \in C$ and any $k \in \omega$, $\Sigma(v)$ cannot place v in the direct connecting sequence from f to $f(k)$, or as a dead end off f or $f(k)$, because in each case $Th(\mathfrak{C}_C)$ says there are only finitely many such elements.

Case 1. $(v = c_0) \in \Sigma(v)$ for some $c_0 \in C$. Then $\Sigma(v)$ is the type of c_0. There are κ of these types.

Case 2. $\Sigma(v)$ says "$v \notin C$ and v is a range element".

LEMMA 5.5. *Let* $\mathfrak{C} \equiv \mathscr{M}$. *Let*

$$\Gamma(v) = Th(\mathfrak{C}_C) \cup \{v \neq c: c \in C\} \cup \{(\textit{there is exactly one dead end off } v)\}.$$

Then $\Gamma(v)$ *is a complete type in* $S_{\mathfrak{C}}(C)$.

It follows that we get only 1 complete type in case 2.

Case 3. $\Sigma(v)$ says "$v \notin C$ and v is a function element".

LEMMA 5.6. *Let* $\mathfrak{C} \equiv \mathscr{M}$ *and let* C_0 *be the range elements of* $\mathfrak{C}$. *Suppose* $Y \subset \omega$, $s_0 \in {}^YC_0$, *and* R *is an equivalence relation on* $(\omega - Y)$. *Let*

$$\begin{aligned}\Gamma(v) \vDash\ & Th(\mathfrak{C}_C) \cup \{v \neq c: c \in C\}\\ &\cup \{(\textit{there are exactly 2 dead ends off } v)\}\\ &\cup \{v(k) = s_0(k): k \in Y\} \cup \{v(k) \neq c: k \notin Y, c \in C\}\\ &\cup \{\pm (v(k) = v(n)): k, n \in \omega - Y, \pm(\langle k, n\rangle \in R)\}.\end{aligned}$$

Then $\Gamma(v)$ *is a complete type in* $S_{\mathfrak{C}}(C)$.

It follows that we get $\leqq 2^\omega \cdot \kappa^\omega \cdot 2^\omega = \kappa^\omega$ complete types from case 3.

Case 4. $\Sigma(v)$ says "$v \notin C$ and v is a dead end off a range element". It follows that the range element is also not in $\mathfrak{C}$.

LEMMA 5.7. *Let* $\mathfrak{C} \equiv \mathscr{M}$. *Let*

$$\Gamma(v) = Th(\mathfrak{C}_C) \cup \{v \neq c: c \in C\} \cup \{(\exists y)(v \textit{ is the unique dead end off } y)\}.$$

Then $\Gamma(v)$ *is a complete type in* $S_{\mathfrak{C}}(C)$.

It follows that case 4 yields 1 complete type.

Case 5. $\Sigma(v)$ says "$v \notin C$ and v is a dead end off a function element". It follows that the function element is also not in $\mathfrak{C}$.

LEMMA 5.8. *Let* $\mathfrak{C} \equiv \mathscr{M}$ *and let* C_0 *be the set of range elements of* $\mathfrak{C}$. *Suppose* $Y \subset \omega$, $s_0 \in {}^YC_0$, *and* R *is an equivalence relation on* $(\omega - Y)$. *Let*

$$\begin{aligned}\Gamma(v) =\ & Th(\mathfrak{C}_C) \cup \{v \neq c: c \in C\}\\ &\cup \{(v \textit{ is a dead end off an element with 2 dead ends})\}\\ &\cup \{(\exists f)(0 < f < v \wedge f(k) = s_0(k)): k \in Y\}\\ &\cup \{(\exists f)(0 < f < v \wedge f(k) \neq c): k \notin Y, c \in C\}\\ &\cup \{(\exists f)(0 < f < v \wedge \pm (f(k) = f(n))):\\ &\qquad\qquad k, n \in \omega - Y, \pm(\langle k, n\rangle \in R)\}.\end{aligned}$$

Then $\Gamma(v)$ *is a complete type in* $S_{\mathfrak{C}}(C)$.

So there are $\leqq \kappa^\omega$ types from case 5.

Case 6. $\Sigma(v)$ says "$v \notin C$ and v is on a direct connecting sequence from a function element to a range element".

It follows that the function element is not in $\mathfrak{C}$, but we have 2 subcases.

Case 6a. For some range element $y_0 \in C$, $\Sigma(v)$ says "$v \notin C$ and v is in a direct connecting sequence from a function element to y_0".

LEMMA 5.9. *Let $\mathfrak{C} \equiv \mathscr{M}$ and let C_0 be the set of range elements of $\mathfrak{C}$. Suppose $Y \subset \omega$, $s_0 \in {}^Y C_0$, $k_0 \in Y$, $l_0 < 2k_0 + 2$, and R is an equivalence relation on $(\omega - Y)$. Let*

$$\begin{aligned}
\Gamma(v) = {} & Th(\mathfrak{C}_C) \cup \{v \neq c: c \in C\} \\
& \cup \{(\exists f)[(f \text{ is a function element}) \\
& \wedge (v \text{ is directly connected to } f \text{ in } l_0 \text{ steps}) \\
& \wedge (v \text{ is directly connected to } s_0(k_0) \text{ in } 2k_0 + 2 - l_0 \text{ steps})]\} \\
& \cup \{(\exists f)[(f \text{ is a function element}) \\
& \wedge (v \text{ is directly connected to } f \text{ in } l_0 \text{ steps}) \\
& \wedge (f(k) = s_0(k))]: k \in Y\} \\
& \cup \{(\exists f)[(f \text{ is a function element}) \\
& \wedge (v \text{ is directly connected to } f \text{ in } l_0 \text{ steps}) \\
& \wedge (f(k) \neq c) \\
& \wedge \pm (f(k) = f(n))]: c \in C_0, k, n \in \omega - Y, \pm (\langle k, n\rangle \in R)\}.
\end{aligned}$$

Then $\Gamma(v)$ is a complete type in $S_{\mathfrak{C}}(C)$.

So case 6a yields $\leqq 2^\omega \cdot \kappa^\omega \cdot \omega \cdot 2^\omega = \kappa^\omega$ complete types.

Case 6b. $\Sigma(v)$ says "$v \notin C$ and v is in a direct connecting sequence from a function element to a range element, but v is not directly connected to any element of C".

Lemma 5.9 adapts easily to this case. In $\Gamma(v)$ replace (v is directly connected to $s_0(k_0)$ . . .) by (v is directly connected to a range element . . .) and add the set of formulae $\{(v \text{ is not directly connected to } c): c \in C_0\}$.

So case 6b yields $\leqq \kappa^\omega$ complete types.

Case 7. $\Sigma(v)$ says $v \notin C$ and v is directly connected to a range element, but not to any function element".

There are 2 subcases.

Case 7a. There is a range element $y_0 \in C$ such that $\Sigma(v)$ says "$v \notin C$, v is directly connected to y_0, but v is not directly connected to any function element".

LEMMA 5.10. *Let $\mathfrak{C} \equiv \mathscr{M}$ and let C_0 be the set of range elements of $\mathfrak{C}$. Let*

$y_0 \in C_0$, $k_0 \in \omega$ and let

$$\begin{aligned}\Gamma(v) = {} & Th(\mathfrak{C}_C) \cup \{v \neq c: c \in C\} \\ & \cup \{(v \text{ is directly connected to } y_0 \text{ in } k_0 \text{ steps})\} \\ & \cup \{(v \text{ is not directly connected to a function element in } l \text{ steps}): \\ & \qquad l \in \omega\}.\end{aligned}$$

Then $\Gamma(v)$ is a complete type in $S_{\mathfrak{C}}(C)$.

So case 7a yields $\leqq \kappa \cdot \omega = \kappa$ complete types.

Case 7b. $\Sigma(v)$ says "$v \notin C$, v is not directly connected to a function element, v is connected to a range element but not to one in $\mathfrak{C}$".

This time we can show that for each $k_0 \in \omega$,

$$\begin{aligned}\Gamma(v) = {} & Th(\mathfrak{C}_C) \cup \{v \neq c: c \in C\} \\ & \cup \{(v \text{ is directly connected to a range element in } k_0 \text{ steps})\} \\ & \cup \{(v \text{ is not directly connected to a function element in} \\ & \qquad l \text{ steps}): l \in \omega\} \\ & \cup \{(v \text{ is not directly connected to } c \text{ in } l \text{ steps}): c \in C, l \in \omega\}\end{aligned}$$

is a complete type in $S_{\mathfrak{C}}(C)$.

So case 7b yields ω complete types.

Case 8. $\Sigma(v)$ says "$v \notin C$ and v is directly connected to a function element, but not to any range element".

This breaks into 2 subcases as in case 7. In each subcase the types are determined by the number of steps v is from the function element (as in case 7), and the description of the function (as in case 6). There will be $\leqq \kappa \cdot \omega = \kappa$ complete types where the function element is in $\mathfrak{C}$, and $\leqq \kappa^{\omega} \cdot \omega = \kappa^{\omega}$ where the function element is not in $\mathfrak{C}$.

Case 9. $\Sigma(v)$ says "v is not connected to any function or range element".

This type is completed by adding one of the formulae (v is level 2) or (v is level 3).

So case 9 yields only 2 complete types.

This exhausts the possibilities for $\Sigma(v)$ in $S_{\mathfrak{C}}(C)$. Altogether we have counted

$$\leqq \kappa + 1 + \kappa^{\omega} + 1 + \kappa^{\omega} + (\kappa^{\omega} + \kappa^{\omega}) + (\kappa + \omega) + (\kappa + \kappa^{\omega}) + 2 = \kappa^{\omega}$$

complete types in $S_{\mathfrak{C}}(C)$. So we have proven

THEOREM 5.11. *$Th(\mathscr{M})$ is stable but not superstable.*

Although we have been unable to determine the exact dimension of $\mathscr{M}$, it can be shown (as with Example 1.3) that $\dim(\mathscr{M}) \in \{3, 4\}$.

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6. Categoricity and the class $\mathscr{S}$. We now return to the lattices in $\mathscr{S}$. We will apply our "approximating formulae" for the C-classes of models in $\mathscr{S}$ to give concrete algebraic characterizations of the $\aleph_0$-categorical and the $\aleph_1$-categorical models in $\mathscr{S}$.

THEOREM 6.1. *Let $\mathfrak{A}$ be a lattice in $\mathscr{S}$. Then $Th(\mathfrak{A})$ is $\aleph_0$-categorical if and only if there are bounds $N, M \in \omega$ such that*

(i) *$\mathfrak{A}$ has no connecting sequences of length $> N$, and*

(ii) *for all non dead ends $a \in A$, if a has $\geqq M$ dead ends then a has $\geqq \omega$ dead ends.*

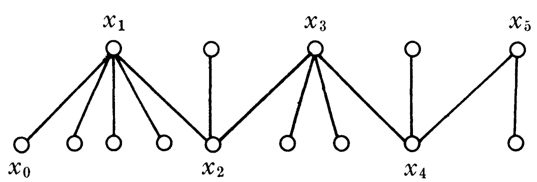
Proof. Necessity follows from the Ryll-Nardzewski Theorem (see [**4**]). To prove sufficiency, make use of the approximating formulae $\phi_{N,M}{}^{\mathfrak{B},b}$ from Definition 3.6.

The lattices in $\mathscr{S}$ with $\aleph_1$-categorical theories come in 3 forms.

Definition 6.2. We shall say $\mathfrak{A}$ is *type* 1 if there is a finite bound on the cardinalities of the C-classes of $\mathfrak{A}$ and there is a C-class X of $\mathfrak{A}$ such that for all C-classes Y, if $Y \neq X$ then there are only finitely many C-classes in $\mathfrak{A}$ isomorphic to Y.

$\mathfrak{A}$ is *type* 2 if there is a non dead end $a \in A$ such that a has infinitely many dead ends, and only finitely many elements of $\mathfrak{A}$ are not dead ends off a.

For each $s \in {}^n(\omega \times \{2, 3\})$ such that $(s(i))_1 \neq (s(i+1))_1$ for $i < n-1$, let s represent a connecting sequence $\langle x_i: i \in n\rangle$ together with $(s(i))_0$ dead ends (other than x_{i-1} or x_{i+1} when $i = 1 = 0$ or $i+1 = n-1$), off x_i, where x_i is level $(s(i))_1$. For example, $\langle\langle 0, 2\rangle, \langle 3, 3\rangle, \langle 1, 2\rangle, \langle 2, 3\rangle, \langle 1, 2\rangle, \langle 1, 3\rangle\rangle$ represents



By s^{-1} we mean the sequence $\langle s(n-1), \ldots, s(0)\rangle$. If $t \in {}^m(\omega \times \{2, 3\})$ and $(t(i))_1 \neq (t(i+1))_1$ for $i < m-1$, then the poset represented by the concatenation $s \frown t$ is well defined provided $(s(n-1))_1 \neq (t(0))_1$. In this way we can represent C-classes in lattices in $\mathscr{S}$ which have only finitely many dead ends off each element as (possibly infinite) concatenations of such sequences. We say $\mathfrak{A}$ is *type* 3 if $\mathfrak{A}$ is infinite and there is an even $n \in \omega - \{0\}$ and an $s \in {}^n(\omega \times \{2, 3\})$, $(s(i))_1 \neq (s(i+1))_1$ for $i < n-1$, such that $\mathfrak{A}$ has

(a) $\geqq 0$ C-classes isomorphic to the 2-way infinite fence

$$\ldots \frown s \frown s \frown s \frown \ldots,$$

(b) $<\omega$ C-classes isomorphic to each crown of the form

$$s \frown s \frown \ldots \frown s,$$

(c) $<\omega$ 1-way infinite fences and each of them of the form

$$r \frown s \frown s \frown s \frown \ldots$$

for some $r \in {}^k(\omega \times \{2, 3\})$ such that $(r(k-1))_1 \neq s(0))_1$, or of the form

$$\ldots \frown s \frown s \frown s \frown r$$

for some $r \in {}^k(\omega \times \{2, 3\})$ such that $(r(0))_1 \neq (s(n-1))_1$,

(d) $<\omega$ 2-way infinite fences not isomorphic to

$$\ldots \frown s \frown s \frown s \frown \ldots$$

and each of these of the form

$$\ldots \frown s \frown s \frown s \frown t \frown s \frown s \frown s \frown \ldots$$

for some $t \in {}^k(\omega \times \{2, 3\})$ such that $(t(0))_1 \neq (s(n-1))_1$ and $(t(k-1))_1 \neq (s(0))_1$, or of the form

$$\ldots \frown s^{-1} \frown s^{-1} \frown s^{-1} \frown t \frown s \frown s \frown s \frown \ldots$$

for some $t \in {}^k(\omega \times \{2, 3\})$ such that $(t(0))_1 \neq (s(0))_1$ and $(t(k-1))_1 \neq (s(0))_1$, or of the form

$$\ldots \frown s \frown s \frown s \frown t \frown s^{-1} \frown s^{-1} \frown s^{-1} \frown \ldots$$

for some $t \in {}^k(\omega \times \{2, 3\})$ where $(t(0))_1 \neq (s(n-1))_1$ and $(t(k-1))_1 \neq (s(n-1))_1$,

(e) $<\omega$ other elements.

Note that $\mathfrak{A}$ can be infinite without any copies of $\ldots \frown s \frown s \frown s \frown \ldots$ but has power $\aleph_\alpha$, $\alpha > 0$ if and only if $\mathfrak{A}$ has $\aleph_\alpha$ copies of $\ldots \frown s \frown s \frown s \frown \ldots$.

LEMMA 6.3. *Let $\mathfrak{A}$ be a lattice in $\mathscr{S}$ with a finite bound on the cardinalities of its C-classes. Then $Th(\mathfrak{A})$ is $\aleph_1$-categorical if and only if $\mathfrak{A}$ is type* 1.

Proof. The necessity is clear.

For sufficiency, let $\mathfrak{A}$ be type 1 and suppose $\mathfrak{A}$ has infinite models. So there is a C-class X such that $\mathfrak{A}$ has $\geqq \omega$ C-classes isomorphic to X.

Since there is a finite bound on the cardinalities of C-classes in $\mathfrak{A}$, it follows that there are $N, M \in \omega$ such that

$$\mathfrak{A} \models (\forall v)(\phi_{N,M}{}^{\mathfrak{B},b}(v) \rightarrow \phi_{n,m}{}^{\mathfrak{B},b}(v))$$

for any element b of a lattice $\mathfrak{B}$ in $\mathscr{S}$ and every $n, m \in \omega$. It is now straightforward to show that models of $Th(\mathfrak{A})$ of the same cardinality have the same number of each C-class.

LEMMA 6.4. *Let $\mathfrak{A}$ be a lattice in $\mathcal{S}$ such that there is no finite bound on the numbers of dead ends off individual elements of $\mathfrak{A}$. Then $Th(\mathfrak{A})$ is $\aleph_1$-categorical if and only if $\mathfrak{A}$ is type* 2.

Proof. Suppose $Th(\mathfrak{A})$ is $\aleph_1$-categorical. Clearly there is at most one element with infinitely many dead ends.

Suppose there are no elements in $\mathfrak{A}$ with infinitely many dead ends off them. Then $\mathfrak{A}$ has elements with arbitrarily large finite numbers of dead ends. It follows that the set of sentences

$$\Gamma = Th(\mathfrak{A}) \cup \{c \neq c'\} \cup \{(c \text{ and } c' \text{ have } \geqq n \text{ dead ends}) : n \in \omega\}$$

is consistent. But if $(\mathfrak{B}, c_{\mathfrak{B}}, c_{\mathfrak{B}}') \models \Gamma$, then $\mathfrak{B} \equiv \mathfrak{A}$ and has 2 elements $c_{\mathfrak{B}}$ and $c_{\mathfrak{B}}'$ with infinitely many dead ends. As above, this contradicts $\aleph_1$-categoricity of $Th(\mathfrak{A})$.

So there is exactly one element, say a_0, in $\mathfrak{A}$ with infinitely many dead ends. Suppose there are infinitely many elements of $\mathfrak{A}$ which are not dead ends off a_0. Then the set of sentences

$$\Gamma = Th(\mathfrak{A}, a_0) \cup \{(c_\alpha \text{ is not a dead end off } a_0) : \alpha < \omega_1\}$$

is consistent. We can construct models of Γ of power $\aleph_1$ with different numbers of dead ends off a_0, contradicting $\aleph_1$-categoricity of $Th(\mathfrak{A})$.

So $\mathfrak{A}$ has one element, a_0, with infinitely many dead ends and there are only finitely many elements of $\mathfrak{A}$ which are not dead ends off a_0.

Now let $\mathfrak{A}$ be type 2 and let a_0 be the element in $\mathfrak{A}$ with infinitely many dead ends. Let $\{a_0, \ldots, a_{n-1}\}$ include all elements of $\mathfrak{A}$ which are not dead ends off a_0.

$$Th(\mathfrak{A}) \models (\exists v_0) \ldots (\exists v_{n-1})[\wedge_{i,j<n} (\pm v_i \leqq v_j : \pm a_i \leqq a_j)$$
$$\wedge \ (\forall x)((\bigwedge_{i<n} x \neq v_i) \rightarrow (x \text{ is a dead end off } v_0))].$$

So any models of $Th(\mathfrak{A})$ of the same power are isomorphic.

We need a combinatorial lemma about sequences to deal with the lattices in $\mathcal{S}$ with no finite bound on lengths of connecting sequences.

Definition 6.5. Let A be a set, $s \in {}^{\omega}A$, $s_0 \in {}^{n_0}A$. We say that an element $n \in \omega$ has an s_0-*neighbourhood* (with respect to s) if there exists m_n such that

$$m_n + n_0 \leqq n < m_n + 2n_0$$

and either

$$\langle s(m_n), \ldots, s(m_n + 3n_0 - 1) \rangle = s_0 {}^\frown s_0 {}^\frown s_0$$

or

$$\langle s(m_n), \ldots, s(m_n + 3n_0 - 1) \rangle = s_0^{-1} \frown s_0^{-1} \frown s_0^{-1}.$$

LEMMA 6.6. *Let $s \in {}^\omega A$, $s_0 \in {}^{<\omega}A$. Suppose there is an N such that every $n \geqq N$ has an s_0-neighbourhood (with respect to s). Then s is eventually periodic. Moreover, if s_0 is chosen to have minimal length, then there is a $t \in {}^{<\omega}A$ such that either*

$$s = t \frown s_0 \frown s_0 \frown s_0 \frown \ldots \text{ or } s = t \frown s_0^{-1} \frown s_0^{-1} \frown s_0^{-1} \frown \ldots.$$

Proof. Pick s_0 of minimal length n_0 such that there exists an N as in the lemma. If $n_0 = 1$, then $m_n + n_0 \leqq n < m_n + 2n_0$ implies that $n = m_n + 1$. So for $n \geqq N$, $s(n) = s(m_n + 1) = s_0(0)$ and s is constant after N. So we will assume that $n_0 > 1$. We will also assume there is an m_N such that

$$m_N + n_0 \leqq N < m_N + 2n_0 \quad \text{and}$$
$$\langle s(m_N), \ldots, s(m_N + 3n_0 - 1) \rangle = s_0 \frown s_0 \frown s_0,$$

the other case being dual.

Let $N_k = m_N + kn_0$ for each $k \in \omega$. We will show by induction on k that

$$\langle s(N_k), \ldots, s(N_k + 3n_0 - 1) \rangle = s_0 \frown s_0 \frown s_0.$$

For $k = 0$, this follows from the definitions of m_N and N_0.

Now consider $k + 1$ and suppose

$$\langle s(N_{k+1}), \ldots, s(N_{k+1} + 3n_0 - 1) \rangle \neq s_0 \frown s_0 \frown s_0.$$

Using the induction hypothesis we will derive a contradiction. Note that

$$N_{k+2} \geqq N_2 = m_N + 2n_0 > N.$$

Case A. There is an m such that $m + n_0 \leqq N_{k+2} < m + 2n_0$ and

$$\langle s(m), \ldots, s(m + 3n_0 - 1) \rangle = s_0 \frown s_0 \frown s_0.$$

Then $N_k < m < N_{k+1}$, so let $m = N_k + m'$ where $0 < m' < n_0$. For $0 \leqq i < n_0 - m'$,

$$s_0(i) = s(m + n_0 + i) = s(N_k + m' + n_0 + i) = s_0(m' + i),$$

and for $n_0 - m' \leqq i < n_0 - 1$,

$$\begin{aligned} s_0(i) &= s(m + n_0 + i) = s(N_k + n_0 + m' + i) \\ &= s(N_k + 2n_0 + i - (n_0 - m')) = s_0(i - (n_0 - m')). \end{aligned}$$

So

$$\begin{aligned} &\langle s_0(0), \ldots, s_0(n_0 - m' - 1), s_0(n_0 - m'), \ldots, s_0(n_0 - 1) \rangle \\ &\quad = \langle s_0(m'), \ldots, s_0(n_0 - 1), s_0(0), \ldots, s_0(m' - 1) \rangle. \end{aligned}$$

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Let $s_1 = s_0\restriction_{m'}$. Letting $t = s_0 \frown s_0 \frown s_0 \frown \ldots$, we see that

$$t = s_1 \frown s_0 \frown s_0 \frown s_0 \frown \ldots = s_1 \frown t.$$

By induction on k, we have that for each $k \in \omega$,

$$t\restriction_{k\cdot m'} = s_1 \frown s_1 \frown \ldots \frown s_1 \ (k \text{ times}).$$

And if $t' = s_0^{-1} \frown s_0^{-1} \frown s_0^{-1} \frown \ldots$, then

$$t'\restriction_{k\cdot m'} = s_1^{-1} \frown s_1^{-1} \frown \ldots \frown s_1^{-1} \ (k \text{ times}).$$

(Look at $t\restriction_{lcm(n_0,m')}$.)

Now consider any $n \geqq N$. Pick m_n such that $m_n + n_0 \leqq n < m_n + 2n_0$ and either

(a) $\langle s(m_n), \ldots, s(m_n + 3n_0 - 1)\rangle = s_0 \frown s_0 \frown s_0$, or

(b) $\langle s(m_n), \ldots, s(m_n + 3n_0 - 1)\rangle = s_0^{-1} \frown s_0^{-1} \frown s_0^{-1}$.

Suppose (a) and $n < m_n + 2m'$. Then

$$m_n + m' < m_n + n_0 \leqq n < m_n + 2m'$$

and

$$\langle s(m_n), \ldots, s(m_n + 3m' - 1)\rangle = t\restriction_{3m'} = s_1 \frown s_1 \frown s_1,$$

and so n has an s_1 neighbourhood. Suppose (a) and $n \geqq m_n + 2m'$. Then $n - 2m' \geqq m_n$. If we let k be greatest such that

$$(m_n + 3n_0 - 1) - (k + 1)m' \geqq n,$$

then

$$(m_n + 3n_0 - 1) - ((k + 3)m' - 1) > (m_n + 3n_0 - 1) - (k + 3)m' \geqq n - 2m' \geqq m_n.$$

So

$$\begin{aligned}\langle s(m_n + 3n_0 - 1), \ldots, s((m_n + 3n_0 - 1) - ((k + 3)m' - 1))\rangle &= t'\restriction_{(k+3)m'} \\ &= s_1^{-1} \frown s_1^{-1} \frown \ldots \frown s_1^{-1} \ (k + 3 \text{ times}).\end{aligned}$$

Letting $m_n' = (m_n + 3n_0 - 1) - ((k + 3)m' - 1)$, we have

$$\langle s(m_n'), \ldots, s(m_n' + 3m' - 1)\rangle = s_1 \frown s_1 \frown s_1$$

and

$$m_n' + m' = (m_n + 3n_0 - 1) - (k + 2)m' + 1 \leqq n$$

(by choice of k) and

$$n < m_n' + 2m'$$

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(again by choice of k), so n has an s_1 neighbourhood. If (b), then a similar argument shows that n has an s_1 neighbourhood.

But such an s_1 contradicts the minimal choice of n_0, so case A cannot occur.

Case B. There is no m as in case A.

Since $N_{k+2} \geqq N$, we can pick m_1 such that $m_1 + n_0 \leqq N_{k+2} < m_1 + 2n_0$ and

$$\langle s(m), \ldots, s(m + 3n_0 - 1) \rangle = s_0^{-1} \frown s_0^{-1} \frown s_0^{-1}.$$

Now choose m_2 such that $m_2 + n_0 \leqq N_{k+3} < m_2 + 2n_0$ and either

(a) $\langle s(m_2), \ldots, s(m_2 + 3n_0 - 1) \rangle = s_0 \frown s_0 \frown s_0$, or

(b) $\langle s(m_2), \ldots, s(m_2 + 3n_0 - 1) \rangle = s_0^{-1} \frown s_0^{-1} \frown s_0^{-1}$.

An argument similar to the one above leads to the same contradiction. So case B cannot occur.

We conclude that

$$\langle s(N_{k+1}), \ldots, s(N_{k+1} + 3n_0 - 1) \rangle = s_0 \frown s_0 \frown s_0$$

after all, and the induction is complete. It follows easily that

$$s = s\restriction_{N_0} \frown s_0 \frown s_0 \frown s_0 \frown \ldots.$$

COROLLARY 6.7. (to the proof of Lemma 6.6). *Let $s \in {}^{\omega}A$, $s_0 \in {}^{<\omega}A$, s_0 of minimal length n_0 such that there is an N as in the lemma. Suppose $s = s_0 \frown s_0 \frown s_0 \frown s'$ (respectively $s_0^{-1} \frown s_0^{-1} \frown s_0^{-1} \frown s'$) and every $n \geqq 2n_0$ has an s_0-neighbourhood (with respect to s). Then $s = s_0 \frown s_0 \frown s_0 \frown \ldots$ (respectively $s_0^{-1} \frown s_0^{-1} \frown s_0^{-1} \frown \ldots$).*

LEMMA 6.8. *Let $\mathfrak{A}$ be a lattice in $\mathscr{S}$ and suppose there is no finite bound on the lengths of connecting sequences in $\mathfrak{A}$. Then $Th(\mathfrak{A})$ is $\aleph_1$-categorical if and only if $\mathfrak{A}$ is type 3.*

Proof. Suppose there is no finite bound on the lengths of connecting sequences in $\mathfrak{A}$ and $Th(\mathfrak{A})$ is $\aleph_1$-categorical. Note that by Lemma 6.4 there is a finite bound M such that no element in $\mathfrak{A}$ has $>M$ dead ends.

Claim 1. There is a model $\mathfrak{B} \equiv \mathfrak{A}$ and an $s \in {}^{<\omega}(\{0, \ldots, M\} \times \{2, 3\})$, with length(s) even and $(s(i))_1 \neq (s(i + 1))_1$ for $i <$ length(s) $- 1$, such that $\mathfrak{B}$ has a C-class isomorphic to $\ldots \frown s \frown s \frown s \frown \ldots$.

Proof of Claim 1. Let

$$\Gamma_0(v) = Th(\mathfrak{A}_A) \cup \{(v \text{ is not connected to } a \text{ in } n \text{ steps}): a \in A, n \in \omega\}$$
$$\cup \{(\exists v_{-n}) \ldots (\exists v_0) \ldots (\exists v_n)[(v = v_0) \wedge \bigwedge_{i \neq j} (v_i \neq v_j)$$
$$\wedge \bigwedge_{-n \leqq i \leqq n} (v_i \notin \{0, 1\})$$
$$\wedge \bigwedge_{-n \leqq i \leqq n} (v_i < v_{i+1} \vee v_i > v_{i+1})]: n \in \omega\}.$$

There is no finite bound on the lengths of connecting sequences in $\mathfrak{A}$, so by the compactness theorem there is a model $\mathfrak{B} > \mathfrak{A}$ and an element $b \in B$ such that $\mathfrak{B} \models \Gamma_0 [b]$. It follows that b is a non dead end on a 2-way infinite fence. We will show that $[b] = \dots \frown s \frown s \frown s \frown \dots$ for some s.

Define a set of formulae

$$\begin{aligned}\Gamma_1(v_k: k \in \omega) = Th(\mathfrak{A}_A) & \\ \cup \{ (v_i \text{ is not connected to } v_j \text{ in } n \text{ steps}) &: i, j, n \in \omega, i \neq j\} \\ \cup \{\phi_{n,M}{}^{\mathfrak{B},b} (v_i): i, n \in \omega\}. &\end{aligned}$$

Since $b \in B - A$ and $\mathfrak{B} > \mathfrak{A}$, for each $n, m, k \in \omega$,

$$\mathfrak{A} \models (\forall v_0) \dots (\forall v_{k-1}) (\exists v) (\phi_{n,M}{}^{\mathfrak{B},b} (v)$$
$$\wedge \bigwedge_{i<k} (v_i \text{ is not connected to } v \text{ in } \leqq m \text{ steps})).$$

It follows that $\mathfrak{A}$ realizes every finite subset of Γ_1, so by compactness there is a model $\mathfrak{B}_1$ with elements b_k, $k \in \omega$, such that

$$\mathfrak{B}_1 \models \Gamma_1 [b_k: k \in \omega].$$

Now for each n and k,

$$\mathfrak{B}_1 \models \phi_{n,M}{}^{\mathfrak{B},b} [b_k].$$

Since M is a bound on the numbers of dead ends off elements of $\mathfrak{A}$ and therefore of $\mathfrak{B}_1$,

$$\mathfrak{B}_1 \models \phi_{n,m}{}^{\mathfrak{B},b} [b_k] \text{ for all } n, m, k \in \omega.$$

By Lemma 3.7, for each $k \in \omega$,

$$\langle [b_k], \leqq \rangle \equiv_{\infty,\omega} \langle [b], \leqq \rangle$$

and since these are countable

$$\langle [b_k], \leqq \rangle \simeq \langle [b], \leqq \rangle.$$

Let $\{c_z: z \in Z\}$ list the non dead ends of $[b_0]$ so that $c_z < c_{z+1}$ for even z and $c_z > c_{z+1}$ for odd z. Let

$$\mathfrak{C}_1 = \{0, 1\} \cup \langle [b_0], \leqq \rangle.$$

Suppose there exist $i, j \in Z$ and a formula $\phi_0(v_i, v_j)$ such that $\mathfrak{C}_1 \models \phi_0[c_i, c_j]$ and for all $z, z' \in Z$, if $\{z, z'\} \not\subset \{i, j\}$ then

$$\mathfrak{C}_1 \models \sim \phi_0 [c_z, c_{z'}].$$

Let

$$\begin{aligned}\Gamma_2(v_z: z \in Z) = Th(\mathfrak{C}_{1C_1}) &\cup \{ (v_z \neq v_{z'}): z, z' \in Z, z \neq z'\} \\ &\cup \{ (v_z \neq c_{z'}): z, z' \in Z\} \\ &\cup \{ (v_z < v_{z+1} \vee v_z > v_{z+1}): z \in Z\} \\ &\cup \{\sim\phi_0(v_z, v_{z'}): z, z' \in Z\}.\end{aligned}$$

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Any finite subset of Γ_2 is realized by a subset of $\{c_z: z \in Z - \{i, j\}\}$ in $\mathfrak{C}_1$, so by compactness there is a model $\mathfrak{C}_2 \succ \mathfrak{C}_1$ with elements c_z', $z \in Z$, such that

$$\mathfrak{C}_{2C_1} \models \Gamma_2 [c_z': z \in Z].$$

Note that $[c_0'] \not\simeq [c_0]$, for otherwise there would be an automorphism f on $\mathfrak{C}_2$ such that $f([c_0]) = [c_0']$, and it would follow that

$$\mathfrak{C}_2 \models \phi_0 [f(c_i), f(c_j)]$$

contradicting Γ_2.

Now form $\mathfrak{B}_2$ from $\mathfrak{B}_1$ by exchanging $\mathfrak{C}_1$ for $\mathfrak{C}_2$. It is easy to check that $\mathfrak{B}_2 \succ \mathfrak{B}_1$. But $c_0' \in B_2 - B_1$, so the set of formulae

$$\begin{aligned}\Gamma_3(v_k: k \in \omega) = Th(\mathfrak{B}_{1B_1}) &\\ \cup \{(v_i \text{ is not connected to } v_j \text{ in } n \text{ steps}):&\\ i, j, n \in \omega, i \neq j\}&\\ \cup \{\phi_{n,M}{}^{\mathfrak{B}_2, c_0'}(v_k): n, k \in \omega\}&\end{aligned}$$

is consistent just as Γ_1 was consistent. So there is a model $\mathfrak{B}_3 \succ \mathfrak{B}_2$ with elements d_k, $k \in \omega$, such that

$$\mathfrak{B}_3 \models \Gamma_3 [d_k: k \in \omega].$$

Again it follows that $(d_k] \simeq [c_0']$ for each k. So we have $\mathfrak{B}_3 \equiv \mathfrak{A}$, $\mathfrak{B}_3$ has infinitely many C-classes isomorphic to $[b]$ and infinitely many isomorphic to $[c_0']$ but $[b] \not\simeq [c_0']$. As in the proof of Lemma 6.3, it follows that $Th(\mathfrak{A})$ is not $\aleph_1$-categorical, a contradiction. Therefore,

(1) for every formula $\phi(v_i, v_j)$, if $\mathfrak{C}_1 \models \phi [c_i, c_j]$ then there are $z, z' \in Z$ such that $\{z, z'\} \not\subset \{i, j\}$ and $\mathfrak{C}_1 \models \phi [c_z, c_{z'}]$.

Suppose for every $z \in Z - \{0\}$, there is an $n \in \omega$ such that

$$\mathfrak{C}_1 \models \sim \phi_{n,M}{}^{\mathfrak{B},b} [c_z].$$

Let

$$\psi(v_0) = \{\phi_{n,M}{}^{\mathfrak{B},b}(v_0): n \in \omega\}.$$

We will show that $\mathfrak{C}_1$ locally omits ψ. By (1) (with $i = j = 0$), if $\phi(v_0)$ is any formula consistent with $Th(\mathfrak{C}_1)$ then either

(a) $\mathfrak{C}_1 \models \phi [c]$ for some $c \notin \{c_z: z \in Z\}$ or

(b) $\mathfrak{C}_1 \models \phi [c_z]$ for some $z \neq 0$.

Assume (a). Then c is either 0, 1 or a dead end. It follows that

$$\mathfrak{C}_1 \models \sim \phi_{0,M}{}^{\mathfrak{B},b} [c].$$

Now assume (b). We have supposed that for each $z \neq 0$, there is an $n \in \omega$ such that

$$\mathfrak{C}_1 \models \sim \phi_{n,M}{}^{\mathfrak{B},b} [c_z].$$

So in either case $\phi \wedge \sim \phi_{n,M}{}^{\mathfrak{B},b}(v)$ is consistent with $Th(\mathfrak{C}_1)$ for some $n \in \omega$. Hence $Th(\mathfrak{C}_1)$ locally omits $\psi(v_0)$ and by the Omitting Types Theorem there is a countable model $\mathfrak{C}_3 \equiv \mathfrak{C}_1$ which omits $\psi(v_0)$. Now form $\mathfrak{B}_4$ from $\mathfrak{B}_1$ by adding ω_1 new C-classes, each isomorphic to $[b]$. Since $\mathfrak{B}_1$ has infinitely many C-classes isomorphic to $[b]$, it is easy to see that $\mathfrak{B}_4 \equiv \mathfrak{B}_1$. Form $\mathfrak{B}_5$ from $\mathfrak{B}_4$ by replacing every C-class isomorphic to $[b]$ by a copy of $\mathfrak{C}_3 - \{0, 1\}$. Using games, we can easily show that $\mathfrak{B}_5 \equiv \mathfrak{B}_4$. Also $\mathfrak{B}_4$ and $\mathfrak{B}_5$ are of the same uncountable cardinality, since $\mathfrak{C}_3$ is countable. Both $\mathfrak{B}_4$ and $\mathfrak{B}_5$ are models of $Th(\mathfrak{A})$, but $\mathfrak{B}_5$ has no C-classes isomorphic to $[b]$ and hence $\mathfrak{B}_4 \not\cong \mathfrak{B}_6$. By Morley's Theorem [**26**], this contradicts the $\aleph_1$-categoricity of $Th(\mathfrak{A})$.

So there is a $z_0 \in Z - \{0\}$ such that

$$\mathfrak{C}_1 \models \phi_{n,M}{}^{\mathfrak{B},b}\,[c_{z_0}] \text{ for every } n \in \omega.$$

Suppose for every $z \in Z - \{0, z_0\}$ there is an n such that $\mathfrak{C}_1 \models \sim \phi_{n,M}{}^{\mathfrak{B},b}\,[c_z]$.
Let

$$\Phi(v_0, v_{z_0}) = \psi(v_0) \cup \psi(v_{z_0}) \cup \{(v_0 \neq v_{z_0})\}.$$

A similar argument applying the Omitting Types Theorem to Φ again leads to a contradiction.

So there is a $z_1 \in Z - \{0, z_0\}$ such that

$$\mathfrak{C}_1 \models \phi_{n,M}{}^{\mathfrak{B},b}\,[c_{z_1}] \text{ for all } n \in \omega.$$

We can assume now, by relabelling, that $z_0 < 0 < z_1$. By Lemma 2.7 there are automorphisms f and g of $[c_0]$ such that $f(c_{z_0}) = c_0$ and $g(c_0) = c_{z_1}$. Let $\#c_z$ denote the number of dead ends off c_z. Of course $\#c_z = \#f(c_z) = \#g(c_z)$ for all $z \in Z$. Clearly either

(i) $f(c_{z_0+1}) = c_1$ or

(ii) $f(c_{z_0+1}) = c_{-1}$.

If (i), then $f(c_{z0+z}) = c_z$ for all $z \in Z$, and if (ii) then $f(c_{z_0+z}) = c_{-z}$ for all $z \in Z$. Similarly either

(i′) $g(c_z) = c_{z_1+z}$ for all $z \in Z$ or

(ii′) $g(c_z) = c_{z_1-z}$ for all $z \in Z$.

But if (ii) and (ii′) then

$$g^{-1}\cdot f(c_{z_0+z}) = g^{-1}(c_{-z}) = c_{z_1+z} \text{ for all } z \in Z.$$

So either by (i) or (i′), or by (ii) and (ii′), we can assume there is a $z' > 0$ and an automorphism h of $[c_0]$ such that $h(c_z) = c_{z'+z}$ for all $z \in Z$. It follows that for all $n \in \omega$ and all i, $0 \leqq i < z'$,

$$\#c_{nz'+i} = \#h^n\,(c_i) = \#c_i \quad\text{and}$$
$$\#c_{-nz'+i} = \#h^{-n}\,(c_i) = \#c_i.$$

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So for each n,

$$\langle \#c_{nz'}, \ldots, \#c_{nz'+(z'-1)} \rangle = \langle \#c_0, \ldots, \#c_{z'-1} \rangle.$$

Letting

$$s = \langle \langle \#c_0, 2 \rangle, \langle \#c_1, 3 \rangle, \ldots, \langle \#c_{z'-1}, 3 \rangle \rangle$$

we have $[b] = \ldots {}^\frown s {}^\frown s {}^\frown s {}^\frown \ldots$ proving claim 1.

Now let s be of minimal length n_0 such that there is a $\mathfrak{B}$ as in claim 1 and let $\mathfrak{B} \equiv \mathfrak{A}$, $b \in B$ such that $[b] = \ldots {}^\frown s {}^\frown s {}^\frown s {}^\frown \ldots$. We can assume $\mathfrak{B}$ is countable (by taking a countable elementary substructure containing $[b]$). We will say that an element $x \in B$ has an *s-neighbourhood* if $\mathfrak{B} \models \phi_s[x]$, where

$$\begin{aligned}\phi_s(v) \equiv (\exists v_0) \ldots (\exists v_{3n_0-1})[\textstyle\bigvee_{n_0 \leqq i < 2n_0} (v \leqq v_i \vee v \geqq v_i) \\ \wedge (v \notin \{0, 1\}) \\ \wedge \textstyle\bigwedge_{0 \leqq i < 3n_0 - 1} (v_i < v_{i+1} \vee v_i > v_{i+1}) \wedge \textstyle\bigwedge_{i<j<3n_0} (v_i \neq v_j) \\ \wedge \textstyle\bigwedge_{i<n} (v_i, v_{n_0+i}, v_{2n_0+i} \text{ are level } (s(i))_1 \text{ and have exactly} \\ (s(i))_0 \text{ dead ends})].\end{aligned}$$

There should be no confusion with the s-neighbourhoods of Definition 6.5.

Claim 2. All but finitely many elements of $\mathfrak{B}$ have s-neighbourhoods.

Proof of Claim 2. This is similar to the proof of Lemma 6.4.

By Claim 2, for some $k \in \omega$,

$$Th(\mathfrak{A}) \models (\exists ! v_0, \ldots, v_{k-1})(\textstyle\bigwedge_{i<k} \sim \phi_s(v_i)).$$

We now show that almost all elements of $\mathfrak{A}$ have C-classes of one of the forms (a)-(d) in the definition of type 3.

Remark 3. Note that if n_0 is minimal such that $Th(\mathfrak{A})$ has a model $\mathfrak{B}$ with a C-class $[b] \simeq \ldots {}^\frown s {}^\frown s {}^\frown s {}^\frown \ldots$, then for any infinite fence X in a model of $Th(\mathfrak{A})$, n_0 is the minimal n such that for some s of length n almost all elements of X have s-neighbourhoods. For if there was a minimal $n_1 < n_0$ and an s_1 of length n_1 for X, then by Lemma 6.6,

$$X \simeq t^{-1} {}^\frown s_1 {}^\frown s_1 {}^\frown s_1 {}^\frown \ldots \text{ or } t^{-1} {}^\frown s_1{}^{-1} {}^\frown s_1{}^{-1} {}^\frown s_1{}^{-1} {}^\frown \ldots$$

for some $t \in {}^{\leqq\omega}(\{0, \ldots, M\} \times \{2, 3\})$. It follows by compactness that $Th(\mathfrak{A})$ has a model $\mathfrak{B}'$ with a C-class isomorphic to $\ldots {}^\frown s_1 {}^\frown s_1 {}^\frown s_1 {}^\frown \ldots$, a contradiction.

Since there are only finitely many elements in $\mathfrak{A}$ without s-neighbourhoods, it follows from Lemma 6.6 and the above remark that there are finitely many 1-way infinite fences and each is of one of the forms in (c).

All 2-way infinite fences are of one of the forms in (a) or (d). Now suppose every element in X has an s-neighbourhood and

$$X \simeq \ldots \frown s \frown s \frown s \frown t \text{ or } \ldots \frown s^{-1} \frown s^{-1} \frown s^{-1} \frown t$$

where $t \in {}^{\omega}(\{0, \ldots, M\} \times \{2, 3\})$. Consider the sequences $s \frown s \frown s \frown t$ and $s^{-1} \frown s^{-1} \frown s^{-1} \frown t$. By Corollary 6.7,

$$s \frown s \frown s \frown t = s \frown s \frown s \frown \ldots \text{ and}$$
$$s^{-1} \frown s^{-1} \frown s^{-1} \frown t = s^{-1} \frown s^{-1} \frown s^{-1} \frown \ldots .$$

So in fact $X \simeq \ldots \frown s \frown s \frown s \frown \ldots$. It follows that all but finitely many 2-way infinite fences are of the form $\ldots \frown s \frown s \frown s \frown \ldots$.

Now consider the finite C-classes of $\mathfrak{A}$.

Claim 4. There is no finite C-class X such that $\mathfrak{A}$ contains infinitely many C-classes isomorphic to X.

Proof of Claim 4. Suppose $\mathfrak{A}$ contains infinitely many copies of the finite class X. But we can easily construct a model $\mathfrak{A}' > \mathfrak{A}$ containing infinitely many copies of $\ldots \frown s \frown s \frown s \frown \ldots$, as well as X. This contradicts $\aleph_1$-categoricity of $Th(\mathfrak{A})$. Claim 4 is proved.

Claim 5. For almost all finite C-classes X in $\mathfrak{A}$, s is of minimal length such that every element in X has an s-neighbourhood.

Proof of Claim 5. Suppose X_i, $i \in \omega$, are finite C-classes in $\mathfrak{A}$ such that for each i, s is not of minimal length such that every element of X_i has an s-neighbourhood. By Claim 4, we can assume that $X_i \not\simeq X_j$ for $i \neq j$. Since almost all elements of $\mathfrak{A}$ have s-neighbourhoods, we can assume that for each i there is an s_i of minimal length $< n_0$, such that every element of X_i has an s_i-neighbourhood. Since ${}^{<n_0}\{0, \ldots, M\}$ is finite, we can assume that for each $i \in \omega$, s_1 is of minimal length $< n_0$, such that every element of X_i has an s_1-neighbourhood. Again from the bound M on the numbers of dead ends off elements in X_i, we can assume that for $i < j$, X_i has fewer non dead ends than X_j. Now consider the set of formulae,

$$\begin{aligned}\Gamma(v_z : z \in Z) = \{(v_z \notin \{0, 1\} \wedge \phi_{s_1}(v_z)) : z \in Z\} \\ \cup \{(v_z < v_{z+1}) : z \in Z, z \text{ even}\} \\ \cup \{(v_z > v_{z+1}) : z \in Z, z \text{ odd}\} \\ \cup \{(v_z \neq v_{z'}) : z, z' \in Z, z \neq z'\}.\end{aligned}$$

Every finite subset of Γ is realized in $\mathfrak{A}$, so by compactness, there is a model $\mathfrak{A}' \equiv \mathfrak{A}$ with elements a_z, $z \in Z$, such that $\mathfrak{A}' \models \Gamma\,[a_z : z \in Z]$. By Remark 3, this is a contradiction, so Claim 5 is proved.

Now let X be a finite C-class in which every element has an s-neighbourhood, s of minimal length for X. Clearly X is a crown. Let

$x_0, \ldots, x_{k-1}$ list the non dead ends of X such that $x_i < x_{i+1}$ for even i, $x_i > x_{i+1}$ for odd i and $x_{k-1} > x_0$. Let

$$t^+(n) = \# x_{n \,(\mathrm{mod}\ k)} \text{ for each } n \in \omega.$$

By Lemma 6.6, there is a t' such that

$$t^+ = t' \frown s \frown s \frown s \frown \ldots \text{ or } t^+ = t' \frown s^{-1} \frown s^{-1} \frown s^{-1} \frown \ldots.$$

Assume $t^+ = t' \frown s \frown s \frown s \frown \ldots$, the other case being dual. Let

$$\mathrm{length}(t') = m \text{ and } r(n) = \# x_{m+n \,(\mathrm{mod}\ k)},$$

so that $r = s \frown s \frown s \frown \ldots$.

Claim 6. $n_0 \mid k$.

Proof of Claim 6. This is similar to the arguments used in the proof of Lemma 6.6.

Since $n_0 \mid k$ it is easy to see that $X \simeq \overline{s \frown s \frown \ldots \frown s}$ (k/n_0 times). So if X is a finite C-class for which s is of minimal length such that every element of X has an s-neighbourhood, then $X \simeq \overline{s \frown s \frown \ldots \frown s}$ (k times) for some k. By Claim 4 and Claim 5, this completes the proof that $\mathfrak{A}$ is type 3.

Now let $\mathfrak{A}$ be type 3 and let s be of minimal length n_0, such that $\mathfrak{A}$ satisfies (a)-(e) with s. First we prove

Claim 7. If $\mathfrak{B} \equiv \mathfrak{A}$ and X is an infinite C-class of $\mathfrak{B}$, then s is of minimal length such that almost every element of X has an s-neighbourhood.

Proof of Claim 7. Suppose not. For some $k \in \omega$,

$$Th(\mathfrak{A}) \models (\exists !v_0 \ldots v_{k-1})(\textstyle\bigwedge_{i<k} \sim \phi_s(v_i)),$$

so exactly k elements of $\mathfrak{B}$ have no s-neighbourhood and there is an s_1 of minimal length $n_1 < n_0$ such that all but finitely many elements of X have an s_1 neighbourhood. It follows from Lemma 6.6 that

$$X \simeq t^{-1} \frown s_1 \frown s_1 \frown s_1 \frown \ldots \text{ or } t^{-1} \frown s_1^{-1} \frown s_1^{-1} \frown s_1^{-1} \frown \ldots$$

for some $t \in {}^{\leqq\omega}(\{0, \ldots, M\} \times \{2, 3\})$. Now for some $k' \in \omega$,

$$Th(\mathfrak{A}) \models (\exists !v_0 \ldots v_{k'-1})(\textstyle\bigwedge_{i<k'} \sim \phi_{s^{n_1}}(v_i)),$$

where $s^{n_1} = s \frown s \frown \ldots \frown s$ (n_1 times). It follows that for

$$\bar{s}_1 = \langle s_1(m), \ldots, s_1(n_1 - 1), s_1(0), \ldots, s_1(m-1) \rangle \text{ or}$$
$$\bar{s}_1 = \langle s_1(m-1), \ldots, s_1(0), s_1(n_1 - 1), \ldots, s_1(m) \rangle$$

for some m, $0 \leqq m \leqq n_1$, $\bar{s}_1^{n_0} = s^{n_1}$. So s is not of minimal length for $\mathfrak{A}$, a contradiction. Claim 7 is proved.

For any finite C-class X, $\mathfrak{A}$ has only finitely many C-classes isomorphic to X. As in the proof of Lemma 6.3, it is easy to show that all models of $Th(\mathfrak{A})$ have the same finite number of C-classes isomorphic to X.

For any infinite C-class X of a lattice $\mathfrak{B} \equiv \mathfrak{A}$ it follows from Claim 7 and Lemma 6.6 that X has form (a), (c) or (d). Suppose an element $b \in X$ has no s-neighbourhood. Since $\mathfrak{A}$ has only finitely many C-classes of form (c) or (d), there is an $N \in \omega$ such that

$$Th(\mathfrak{A}) \vDash (\forall v)[\sim\phi_s(v) \to (\forall v') \\ ((v' \text{ connected to } v \text{ in } n \text{ steps}) \to \phi_s(v'))]$$

for every $n \geqq N$. By Claim 7 and Corollary 6.7, if

$$\mathfrak{C} \equiv \mathfrak{A} \text{ and } \mathfrak{C} \vDash \phi^{\mathfrak{B},b}_{N+2n_0,M}\,[c]$$

then $\langle [c], \leqq \rangle \simeq \langle [b], \leqq \rangle$. It follows easily that all models of $Th(\mathfrak{A})$ have the same finite number of C-classes isomorphic to X.

Note that we have shown that any model of $Th(\mathfrak{A})$ is type 3 (with s). It follows that any model of $Th(\mathfrak{A})$ of power ω_1 has exactly ω_1 C-classes isomorphic to $\ldots {}^\frown s {}^\frown s {}^\frown s {}^\frown \ldots$.

This completes the proof that all models of $Th(\mathfrak{A})$ of power ω_1 are isomorphic.

THEOREM 6.9. *Let $\mathfrak{A}$ be a lattice in $\mathscr{S}$. $Th(\mathfrak{A})$ is $\aleph_1$-categorical if and only if $\mathfrak{A}$ is type* 1, *type* 2, *or type* 3.

Proof. Suppose $Th(\mathfrak{A})$ is $\aleph_1$-categorical. If there is no finite bound on the lengths of connecting sequences in $\mathfrak{A}$, then $\mathfrak{A}$ is type 3 by Lemma 6.8. If there is no finite bound on the numbers of dead ends off elements of $\mathfrak{A}$, then $\mathfrak{A}$ is type 2 by Lemma 6.4. Otherwise there is a bound on the cardinalities of the C-classes of $\mathfrak{A}$, and $\mathfrak{A}$ is type 1 by Lemma 6.3.

The other direction is immediate from Lemmas 6.3, 6.4 and 6.8.

COROLLARY 6.10. *Let $\mathfrak{A}$ be a lattice in $\mathscr{S}$. $Th(\mathfrak{A})$ is totally categorical if and only if $\mathfrak{A}$ is type* 1 *or type* 2.

7. Lattices of finite height. In this section we state a generalization of the superstability result for the class $\mathscr{S}$.

The proof of Theorem 3.4 ("every lattice in $\mathscr{S}$ has a superstable theory") made use of the structure theorem for lattices in $\mathscr{S}$. The key ideas of that proof can be applied to a much larger class of lattices, without the benefit of such nice structure. As a result, we can prove superstability for a class of lattices which will include all dimension $\leqq 2$ finite height lattices.

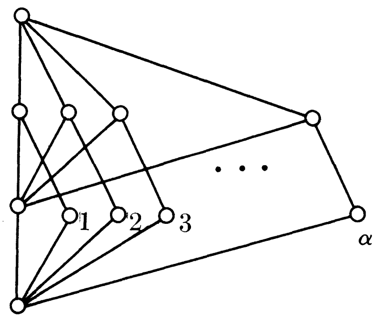
The first step in the proof of Theorem 3.4 was the observation that elements added to a lattice $\mathfrak{A}$ of $\mathscr{S}$ in an elementary extension could only be added as dead ends or as members of C-classes disjoint from $\mathfrak{A}$ (Lemma

3.1). For the class of all finite height lattices which omit both B_ω and $B_\omega{}^d$ (regarding B as B_3), there is a similar restriction on the elements added in elementary extensions.

A second important fact about $\mathcal{S}$ was the existence of the "approximating formulae", $\phi_{n,m}{}^{\mathfrak{A},a}(v)$, and their role in the construction of partial isomorphisms (in the proof of Lemma 3.3). Omitting B_ω and $B_\omega{}^d$ in finite height lattices turns out to be sufficient to construct "approximating formulae". In the general case, however, we cannot give formulae to determine the structure of C-classes as completely as in $\mathcal{S}$. We do get enough structural information in our formulae to classify C-classes up to elementary equivalence, but must use different techniques to prove completeness of types.

The theorems of this section follow from a more general result in [**24**] (by regarding finite height lattices as directed graphs), so we omit the details of our original proofs here. However, as in the height 4 case, the original proofs provide detailed analyses of the structures of the lattices involved. We feel that this structural information may have application to other model theoretic aspects of these lattices (such as categoricity) and refer the interested reader to [**36**] for details.

Definition 7.1. For each $\alpha \leqq \omega$, let B_α be the lattice



and let $B_\alpha{}^d$ be the dual of B_α. Note that $B_3 \simeq B$ (as defined in Section 2).

LEMMA 7.2. *Let T be a theory of lattices. Every model of T omits B_ω if and only if for some $k \in \omega$, every model of T omits B_k. Dually for $B_\omega{}^d$ and $B_k{}^d$.*

Proof. If for each $k \in \omega$, there is a model of T which does not omit B_k, then by a straightforward compactness argument there is a model of T which does not omit $B\omega$. The other direction is obvious.

On the other hand, it is easy to see that for each $k \in \omega$, the class of lattices which omit B_k and $B_k{}^d$ is elementary. Hence, to apply the key facts mentioned above to arbitrary models of complete theories, we will require that, for some $k \in \omega$, those models omit B_k and $B_k{}^d$.

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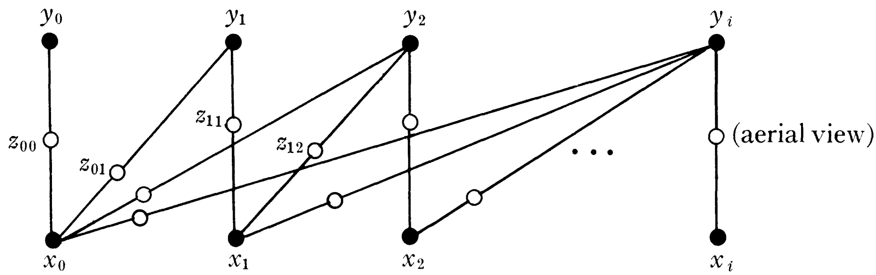
THEOREM 7.3. *Let $\mathfrak{A}$ be a lattice of finite height which for some $k \in \omega$ omits both B_k and $B_k{}^d$. Then $Th(\mathfrak{A})$ is superstable.*

There are finite height lattices which omit both B_ω and $B_\omega{}^d$ that are unstable.

Example 7.4. Recall the unstable lattice $\mathscr{U}$ of Example 1.2. For each $i \in \omega$, let X_i be a C-class isomorphic to



and let

$$\mathfrak{B} = \{0, 1\} \cup \bigcup (X_i: i \in \omega),$$

where $X_i \cap X_j = \phi$ if $i \neq j$.

$$T = Th(\mathfrak{B}) \cup \Delta_U \cup \{\sim(\exists z)(x_i < z < 1 \wedge y_j < z): j < i < \omega\}.$$

Every finite subset of T is satisfiable in an expansion of $\mathfrak{B}$, so there is a model $\mathfrak{C} \equiv \mathfrak{B}$ containing an isomorphic copy of $\mathscr{U}$ such that

$$\mathfrak{C} \models (\exists z)(x_i < z < 1 \wedge y_j < z) \text{ if and only if } i \leqq j.$$

As in the case of $\mathscr{U}$, it follows that $Th(\mathfrak{C}) = Th(\mathfrak{B})$ is unstable.

8. Conclusion. Mekler and the author [**24**] have generalized the superstability results of this thesis to a class of (directed) graphs containing all planar graphs. This is a substantial strengthening of a result in [**30**] which showed that these graphs have stable theories.

A study of the stability of lattices turns out to be a logical starting point for a study of stability of a variety of semigroups called bands. The varieties of left zero bands, right zero bands and semilattices are the atoms of the lattice of equational classes, or varieties, of bands. (See [**17**], page 124.) The model theory of left and right zero bands is very simple and in regard to stability it turns out that studying lattices and semilattices amounts to the same thing.

Besides being the atoms in the lattice of equational classes of bands, left and right zero bands and semilattices are important to the study of bands for another reason. It is known that every rectangular band is isomorphic to a direct product of a left zero and a right zero band. Furthermore, every band is a semilattice of rectangular bands. So one might hope that the results of this paper could be applied to help classify stability in the class of all bands. However, P. Olin and the author in [**28**] determine rather strong necessary and sufficient conditions for the preservation of stability by this construction.

Alan Mekler noted that the analysis of Section 3 (in particular the proof of Lemma 3.3) can also be applied to show that the theory of $\mathscr{S}$ has a primitive recursive decision procedure. We have not checked the details but suspect that this and a similar result for $\mathscr{T}$ can be extracted from our proofs. (One might compare our neighbourhoods to the spheres of Hanf in [**15**].)

John T. Baldwin has pointed out to us that a class of pseudoplanes can be regarded as height four lattices. Lachlan in [**22**] has shown that if there are no $\aleph_0$-categorical pseudoplanes then: (1) If T is stable and $\aleph_0$-categorical then T is ω-stable; and (2) If T is ω-stable and $\aleph_0$-categorical then T has finite Morley rank. ((1) and (2) are two unproven conjectures relating stability and $\aleph_0$-categoricity. We refer the reader to [**4**], Chapter 7 for the definition of Morley rank and to [**22**] for the definition of pseudoplane.) So it is of interest whether or not the height four lattices which are pseudoplanes are $\aleph_0$-categorical.

It can be shown that the $\aleph_0$-categorical theories in $\mathscr{S}$ (see Theorem 6.1) have finite Morley rank, in fact $\leqq 2$. This supports (1) and (2). None of the height 4 lattices which are pseudoplanes fall into the class $\mathscr{S}$, so they are not covered by Theorem 6.1. There are pseudoplanes in the class $\mathscr{T}$, but none of these are $\aleph_0$-categorical. Whether or not there exist $\aleph_0$-categorical height 4 lattices outside of $\mathscr{S} \cup \mathscr{T}$ which are pseudoplanes is an interesting question which we leave open.

Many other questions about the stability of lattices remain open.

We have shown that a dimension $\leqq 2$ lattice is superstable if and only if it is stable if and only if it has finite height.

Problem 1. Characterize ω-stability and $\aleph_1$-categoricity in the class of all finite height dimension $\leqq 2$ lattices.

In the case of $\aleph_0$-categoricity the infinite height lattices are not ruled out.

Problem 2. Characterize $\aleph_0$-categoricity in the class of all dimension $\leqq 2$ lattices.

We have demonstrated that there is no simple relationship between stability and dimension in the class of lattices of finite dimension $\geqq 4$. But almost nothing is known about the stability of lattices of dimension 3 or $\geqq \omega$.

Problem 3. Characterize κ-stability, $\aleph_0$ and $\aleph_1$-categoricity in the class of dimension 3 lattices. In particular, is there a lattice of height 4 and dimension 3 with an unstable theory?

Question 4. Do there exist finite height lattices of infinite dimension? How stable are they?

Properties other than dimension may yield interesting stability results as well, but a complete classification of lattices by their stability properties, even of height 4 lattices, is unlikely.

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