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Modelling of relativistic electron transport with non-relativistic DKES solver

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The paper considers the electron transport in toroidal systems taking into account relativistic effects for electrons. The treatment is based on the relativistic drift-kinetic equation with the thermodynamic equilibrium given by the relativistic Maxwell–Jüttner distribution function. The definition of relativistic fluxes is given in a classic-like form using the same set of thermodynamic forces as in the classical (non-relativistic) approach. Such a formulation allows us to apply the currently used non-relativistic solvers for calculation of relativistic mono-energetic transport coefficients. As an example, the procedure for calculating electron fluxes is proposed, in which relativistic effects are taken into account using the DKES code. The model can be easily implemented in various transport codes, developed for the non-relativistic limit, making them accurate also for hot plasmas with non-negligible relativistic effects.

Keywords: fusion plasma

1. Introduction

Relativistic effects in astrophysical objects and fusion plasmas do not necessarily require extremely high temperatures and energies. They can be non-negligible at electron temperatures T_e of the order of tens of keV, i.e. when $T_e \ll m_e c^2 \simeq 511$ keV. In kinetics and transport physics, these effects appear due to macroscopic features of relativistic thermodynamic equilibrium given by the Maxwell–Jüttner distribution function (Beliaev & Budker 1956; de Groot, van Leewen & van Weert 1980).

In fact, the role of relativistic effects for electron transport in hot plasmas is well studied (Dzhavakhishvili & Tsintsadze 1973; Braams & Karney 1989; Mettens & Balescu 1990; Pike & Rose 2016). At the same time, the direct implementation of these results into transport codes for toroidal plasmas is not a trivial task, especially for those obtained in the covariant formalism.

In fusion devices such as ITER (Doyle *et al.* 2007) and DEMO (Ward 2010), where the expected plasma temperature is in the range of 20–50 keV, relativistic effects for electron

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transport are no longer negligible. Moreover, for D+^{3}He fusion with an optimum plasma temperature of approximately 70 keV (Stott 2005), the relativistic correction makes the thermal energy, $W_e = (\frac{3}{2} + \mathcal{R})n_e T_e$ (Marushchenko, Azarenkov & Marushchenko 2013), where n_e is the electron density and $\mathcal{R}$ is the relativistic correction, approximately 15 % higher than the classical (non-relativistic) definition, $W_e^{(\mathrm{nr})} = \frac{3}{2}n_e T_e$, while for the aneutronic p+^{11}B scheme with $T_e \simeq 300$ keV (Putvinski, Ryutov & Yushmanov 2019) the difference is approximately 45 %.

In addition, the relativistic features of electrons affect the neoclassical radial electron fluxes (Marushchenko *et al.* 2013; Kapper *et al.* 2018), changing them not critically, but noticeably, compared with those calculated in the non-relativistic limit for both tokamaks and stellarators. In particular, it has been shown that the non-relativistic modelling is acceptably accurate only for $T_e < 10$ keV, while for the range of temperatures expected for ITER and DEMO, the classical definitions overestimate electron fluxes of particles, energy and heat. Similar conclusions follow from the relativistic consideration of electron–ion collisional coupling (Marushchenko, Azarenkov & Marushchenko 2012). At the same time, standard models, that are routinely used for predictive simulations of the fusion scenarios, do not account for these effects, like in the tokamak code CRONOS (Artaud *et al.* 2010) and the stellarator code NTSS (Turkin *et al.* 2011).

The main purpose of the present work is to formulate a relativistic transport model for electrons in a form acceptable for implementation in any non-relativistic transport code. The fluxes and transport equations for hot electrons in a toroidal plasmas are formulated in a classic-like form without the covariant 4-vector formalism. This approach is valid when $V_{\mathrm{flow}} \ll v_{\mathrm{Te}}$, where V_{flow} is the characteristic flow velocity. In this case, the relativistic transport equations mathematically have a form similar to the corresponding non-relativistic equations and the relativistic corrections are transparent for physical interpretation. As a practical recipe, it is shown how to make a non-relativistic transport model based on the mono-energetic approach (see, for example, Beidler *et al.* 2011) be valid for arbitrary high temperatures.

It is also known that the mono-energetic approach is not sufficient for the parallel transport, where a parallel momentum conservation in the Coulomb operator is required. In the non-relativistic limit, the problem can be solved by the momentum correction technique (Sugama & Nishimura 2008; Maassberg, Beidler & Turkin 2009). A similar problem was also considered in the relativistic approach (Marushchenko, Beidler & Maassberg 2009). Since in all versions of this technique the corresponding mono-energetic transport coefficient is used, only the mono-energetic approach is considered in this paper.

2. Relativistic kinetics of electrons

2.1. *Relativistic drift-kinetic equation*

The relativistic equations of motion for the electron guiding centre, $\boldsymbol{X}_{\mathrm{gc}}$, can be written as follows (Littlejohn 1984):

$$\dot{\boldsymbol{X}}_{\mathrm{gc}} = v_\parallel \hat{\boldsymbol{B}} + \boldsymbol{v}_{\mathrm{dr}}, \tag{2.1a}$$

$$\boldsymbol{v}_{\mathrm{dr}} = \frac{c}{B^2}[\boldsymbol{E}\times\boldsymbol{B}] - \frac{m_e c u^2(1+\xi^2)}{2eB^3\gamma}[\boldsymbol{B}\times\nabla B] \equiv \boldsymbol{v}_E + \boldsymbol{v}_{\nabla B}. \tag{2.1b}$$

Here, $u = p_e/m_e = v\gamma$ is the electron momentum per unit mass and $\gamma = \sqrt{1+u^2/c^2}$ is the relativistic factor; $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric and magnetic field; $v_\parallel = u_\parallel/\gamma$ and $u_\parallel = (\boldsymbol{u}\cdot\hat{\boldsymbol{B}})$ with $\hat{\boldsymbol{B}} = \boldsymbol{B}/B$ as the unit vector and $\xi = u_\parallel/u$ the pitch; the drift velocities $\boldsymbol{v}_E$ and $\boldsymbol{v}_{\nabla B}$ correspond in the toroidal plasma to the poloidal precession due to $\boldsymbol{E}\times\boldsymbol{B}$ and the

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vertical drift due to ∇B, respectively. The adiabatic invariance of the magnetic moment, $\lambda = m_e u_\perp^2/2B$, is also taken into account. Here and below, the sign of electron charge is taken into account explicitly, i.e. $e = |e|$. All other notations are standard.

Using the pair of variables (u, ξ), one can write also

$$\dot{u} = -\frac{e\gamma}{m_e u}(\dot{\boldsymbol{X}}_{\rm gc} \cdot \boldsymbol{E}) = -\frac{e\xi}{m_e}E_\parallel + \frac{cu(1+\xi^2)}{2B^3}(\boldsymbol{E} \cdot [\boldsymbol{B} \times \nabla B]), \tag{2.2a}$$

$$\dot{\xi} = (1-\xi^2)\left[-\frac{e}{m_e u}E_\parallel + \frac{c\xi}{2B^3}(\boldsymbol{E} \cdot [\boldsymbol{B} \times \nabla B]) - \frac{u}{2\gamma B^2}(\boldsymbol{B} \cdot \nabla B)\right], \tag{2.2b}$$

where $E_\parallel$ is the inductive electric field. With these variables the relativistic drift-kinetic equation (rDKE) can be written as

$$\frac{\partial f_e}{\partial t} + \nabla \cdot (\dot{\boldsymbol{X}}_{\rm gc} f_e) + \frac{1}{u^2}\frac{\partial}{\partial u}(u^2 \dot{u} f_e) + \frac{\partial}{\partial \xi}(\dot{\xi} f_e) = C_e(f_e), \tag{2.3}$$

which has practically the same form as in the classical approach. The collision operator in the right-hand side, $C_e = C_{\rm ee} + C_{\rm ei}$, describes Coulomb collisions of electrons with themselves and with ions, where ions are considered as non-relativistic particles. The fully relativistic formulation for C_e is taken from Beliaev & Budker (1956) and Braams & Karney (1989). The source and loss terms are omitted since they do not play any role in the present consideration. Below, the case of low characteristic velocities of the electron flows, $V_{\rm flow} \ll v_{\rm Te}$, is considered, while the temperature is assumed to be arbitrarily high.

The distribution function can be assumed to be a perturbed thermodynamic equilibrium, $f_e = f_{e0} + f_{e1}$. The relativistic thermal equilibrium, f_{e0}, is given by the Jüttner distribution function, also called the relativistic Maxwellian (de Groot *et al.* 1980; Braams & Karney 1989)

$$f_{e0} = \frac{n_e m_e}{4\pi c T_e K_2(\mu_e)} \exp(-\mu_e \gamma), \tag{2.4}$$

where $K_n(x)$ is a modified Bessel function of the second kind, and $\mu_e = m_e c^2/T_e$ (typically, $\mu_e \simeq 10$–20 for D+T fusion plasmas, while for a p+^{11}B (Putvinski *et al.* 2019) aneutronic scheme it can be approximately $\mu_e \simeq 1.7$).

It seems more convenient to represent the Maxwell–Jüttner distribution function in classic-like form

$$f_{e0} = C_{\rm MJ}\frac{n_e}{\pi^{3/2}u_{\rm Te}^3}\exp[-\mu_e(\gamma - 1)], \tag{2.5}$$

with $u_{\rm Te} = \sqrt{2T_e/m_e}$. Note that $u_{\rm Te}$ only formally coincides with the classical thermal velocity $v_{\rm Te}$, while its physical meaning is different: it is the thermal momentum per unit mass, $u_{\rm Te} \equiv p_{\rm Te}/m_e$ with $p_{\rm Te} = \sqrt{2m_e T_e}$, which, unlike $v_{\rm Te}$, is not limited by the speed of light. It is also possible to find a simple relation

$$\mu_e(\gamma - 1) = \frac{u^2}{u_{\rm Te}^2}\frac{2}{\gamma + 1}, \tag{2.6}$$

that is particularly useful for numerical calculations.

Since the Maxwellian is normalized by density, $n_e = \int f_{e0}\, {\rm d}^3u$, the normalizing factor can accordingly be written as

$$C_{\rm MJ}(\mu_e) = \sqrt{\frac{\pi}{2\mu_e}}\frac{{\rm e}^{-\mu_e}}{K_2(\mu_e)} = 1 - \frac{15}{8\mu_e} + \frac{345}{128\mu_e^2} - \frac{3285}{1024\mu_e^3} + O(\mu_e^{-4}), \tag{2.7}$$

where the asymptotic series for $K_2(x)$ is applied (Abramowitz & Stegun 1972).

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2.2. *Moments of distribution function and fluxes*

Here, the necessary set of moments for f_e is given in the relativistic formulation. As usual, we assume that the first two even moments, density and thermal energy, are given by the zeroth order of the distribution function, f_{e0}, while the particle and heat fluxes come from the linear perturbation, f_{e1}, which must be found by solving the linearized rDKE.

In classical transport theory, the energy balance is usually written for thermal energy as $W = \frac{3}{2}nT$, while in the relativistic approach the use of this definition makes the energy balance equation non-divergent.

In the literature devoted to relativistic kinetics (Dzhavakhishvili & Tsintsadze 1973; de Groot *et al.* 1980; Mettens & Balescu 1990), the total energy is typically used. For the relativistic Maxwellian, it can be written as

$$\mathcal{E}_{\text{total}} = \int m_e c^2 \gamma f_{e0}\, \mathrm{d}^3u = n_e m_e c^2 \frac{K_3(\mu_e)}{K_2(\mu_e)} - n_e T_e. \tag{2.8}$$

Alternatively, it is convenient to represent the thermal energy, $W_e = \mathcal{E}_{\text{total}} - n_e m_e c^2$, in the classic-like form (Marushchenko *et al.* 2013)

$$W_e = (\tfrac{3}{2} + \mathcal{R}) n_e T_e, \tag{2.9}$$

where $\mathcal{R}$ is the relativistic correction

$$\mathcal{R}(\mu_e) = \mu_e \left(\frac{K_3(\mu_e)}{K_2(\mu_e)} - 1 \right) - \frac{5}{2} = \frac{15}{8\mu_e} - \frac{15}{8\mu_e^2} + \frac{135}{128\mu_e^3} + O(\mu_e^{-4}), \tag{2.10}$$

This is a monotonic function increasing with T_e. One can find that, for the range T_e from 25 to 75 keV, the correction term $\mathcal{R}$ varies from 0.09 to 0.24.

As usual in neoclassical transport theory, we assume that even moments defined only by the zeroth order of the distribution function, f_{e0}, give quantities such as n_e and W_e, while odd moments defined by f_{e1} give fluxes

$$\boldsymbol{\Gamma}_e = \int \dot{\boldsymbol{X}}_{\text{gc}} f_{e1}\, \mathrm{d}^3u, \tag{2.11a}$$

$$\boldsymbol{Q}_e = \int \dot{\boldsymbol{X}}_{\text{gc}} m_e c^2 (\gamma - 1) f_{e1}\, \mathrm{d}^3u, \tag{2.11b}$$

where $\boldsymbol{\Gamma}_e$ and $\boldsymbol{Q}_e$ are the fluxes of particles and energy, respectively. It follows that the heat flux can also be written in the classic-like form (Marushchenko *et al.* 2013)

$$\boldsymbol{q}_e = \boldsymbol{Q}_e - (\tfrac{5}{2} + \mathcal{R}) T_e \boldsymbol{\Gamma}_e. \tag{2.12}$$

This expression can also be obtained from the 4-vector formalism (see Appendix A). Again, the difference from the standard non-relativistic definition (Hinton & Hazeltine 1976; Helander & Sigmar 2002) consists of the presence of the relativistic correction term $\mathcal{R}$.

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2.3. *Relativistic transport equations for toroidal plasmas*

To obtain the equations of radial transport in toroidal plasmas, the standard procedure of flux-surface averaging is applied (Hinton & Hazeltine 1976; Helander & Sigmar 2002)

$$\langle F\rangle = \frac{1}{V'}\iint F\sqrt{g}\,\mathrm{d}\theta\,\mathrm{d}\phi, \tag{2.13}$$

with θ and ϕ as the poloidal and toroidal angles, respectively, $V' \equiv \mathrm{d}V/\mathrm{d}\rho$ and $V = \iint \mathrm{d}\theta\,\mathrm{d}\phi \int_0^\rho \sqrt{g}\,\mathrm{d}\rho$ as the volume enclosed within the magnetic surface labelled ρ and where $\sqrt{g} = (\boldsymbol{\nabla}\rho\boldsymbol{\cdot}\boldsymbol{\nabla}\theta\times\boldsymbol{\nabla}\phi)^{-1}$ is the Jacobian.

It seems convenient to interpret the flux-surface label ρ as the effective radius r_{eff}, making $\boldsymbol{\nabla}\rho$ a dimensionless quantity. The geometric factor $\langle|\boldsymbol{\nabla}\rho|\rangle$, which formally should be present in the equations for the averaged moments, plays no role here and is omitted for simplicity.

In order to obtain the continuity equation, one needs to integrate (2.3) with averaging over the magnetic surface

$$\frac{\partial n_e}{\partial t} + \frac{1}{V'}\frac{\partial}{\partial\rho}(V'\varGamma_e^\rho) = 0, \tag{2.14}$$

where the radial flux is defined as

$$\varGamma_e^\rho = \left\langle \int (\boldsymbol{v}_{\nabla B}\boldsymbol{\cdot}\boldsymbol{\nabla}\rho) f_{e1}\,\mathrm{d}^3u \right\rangle. \tag{2.15}$$

Although the continuity equation has the same form as in the non-relativistic limit, the radial flux contains the relativistic corrections (Marushchenko *et al.* 2013; Kapper *et al.* 2018).

The next equation is the power balance, which can be obtained by integrating (2.3) with weight $m_ec^2(\gamma-1)$ and using (2.9)

$$\frac{\partial}{\partial t}\left[\left(\frac{3}{2}+\mathcal{R}\right)n_eT_e\right] + \frac{1}{V'}\frac{\partial}{\partial\rho}(V'Q_e^\rho) = P_{\mathrm{ei}} + P_E, \tag{2.16}$$

where the radial energy flux is

$$Q_e^\rho = \left\langle \int (\boldsymbol{v}_{\nabla B}\boldsymbol{\cdot}\boldsymbol{\nabla}\rho) m_ec^2(\gamma-1) f_{e1}\,\mathrm{d}^3u \right\rangle = q_e^\rho + \left(\frac{5}{2}+\mathcal{R}\right)T_e\varGamma_e^\rho. \tag{2.17}$$

On the right-hand side of (2.16), P_{ei} is the rate of heat exchange between electrons and ions (Marushchenko *et al.* 2012)

$$P_{\mathrm{ei}} = \left\langle \int m_ec^2(\gamma-1)C_{\mathrm{ei}}(f_{e0},f_{i0})\,\mathrm{d}^3u \right\rangle = C_{\mathrm{MJ}}(\mu_e)\left(1+\frac{2}{\mu_e}+\frac{2}{\mu_e^2}\right)P_{\mathrm{ei}}^{(\mathrm{nr})}, \tag{2.18}$$

where both electrons and ions are assumed to be Maxwellian (relativistic and classical, respectively) and $P_{\mathrm{ei}}^{(\mathrm{nr})}$ is the non-relativistic expression (Braginskii 1965)

$$P_{\mathrm{ei}}^{(\mathrm{nr})} = -3\frac{m_e}{M_i}\frac{n_e}{\tau_{\mathrm{ei}}}(T_e - T_i), \tag{2.19}$$

with $\tau_{\mathrm{ei}} = (3\sqrt{\pi}/4)(n_e/n_iZ_i^2)(\ln\varLambda^{e/e}/\ln\varLambda^{e/i})\nu_{e0}^{-1}$ and $\nu_{e0} = 4\pi n_e e^4\ln\varLambda^{e/e}/m_e^2u_{\mathrm{Te}}^3$ as the characteristic e/i collisional time and the simplest electron collision frequency,

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respectively. The formulation remains valid as long as the Landau approximation for small scattering angles is valid.

The next term, P_E, represents the work of the electric field

$$P_E = -\left\langle \int m_e c^2 (\gamma - 1) \frac{1}{u^2} \frac{\partial}{\partial u} (u^2 \dot{u} f_e) \, \mathrm{d}^3 u \right\rangle = P_{E_\parallel} + P_{E_\rho}, \tag{2.20}$$

where $\dot{u}$ is given by (2.2a). The terms $P_{E_\parallel}$ and P_{E_ρ} correspond to the work of the parallel (inductive) and radial (ambipolar) components of the electric field, $E_\parallel \hat{\boldsymbol{B}}$ and $E_\rho \nabla \rho$, respectively. Given that $-e\Gamma_{e\parallel} = j_\parallel$ (ion current is not considered here) with $j_\parallel / B = \mathrm{const}$ on the magnetic surface, these quantities can be written as

$$P_{E_\parallel} = \langle j_\parallel B \rangle \frac{\langle E_\parallel B \rangle}{\langle B^2 \rangle} \quad \text{and} \quad P_{E_\rho} = -e\Gamma_e^\rho E_\rho, \tag{2.21a,b}$$

with $E_\rho = -\Phi'$ and $\Phi(\rho)$ as plasma potential.

To close the (2.14) and (2.16), it is necessary to define the fluxes, Γ_e^ρ, q_e^ρ and $j_\parallel$, as functions of the plasma parameters, their gradients and the electric field.

3. Fluxes and transport coefficients in a mono-energetic approach

The fluxes induced by gradients of plasma parameters and electric fields depend on f_{e1}, and to determine them it is necessary to solve the linearized kinetic equation

$$\mathcal{V}(f_{e1}) - C_e(f_{e1}) = -\boldsymbol{v}_{\nabla B} \cdot \nabla f_{e0} + \frac{1}{u^2} \frac{\partial}{\partial u} (u^2 \dot{u} f_{e0}), \tag{3.1}$$

where the last term on the right-hand side should be taken only with $\dot{u} = -(e\xi/m_e)E_\parallel$. Using the standard mono-energetic approach (Hirshman *et al.* 1986; Beidler *et al.* 2011), the operator $\mathcal{V}$, traditionally called the Vlasov operator, can be approximated as follows:

$$\mathcal{V} = v \left(\xi \hat{\boldsymbol{B}} + \frac{cE_\rho}{v \langle B^2 \rangle} \nabla \rho \times \boldsymbol{B} \right) \cdot \nabla - \frac{v}{2} (1 - \xi^2) (\hat{\boldsymbol{B}} \cdot \nabla \ln B) \frac{\partial}{\partial \xi}, \tag{3.2}$$

i.e. in the same form as in the non-relativistic approach.

In general case, the Coulomb operator C_e should be taken with parallel momentum conservation. In particular, this is necessary for calculation of parallel fluxes, where the momentum correction technique is usually applied (Sugama & Nishimura 2008; Maassberg *et al.* 2009). However, the mono-energetic transport coefficients are also required for this technique. For this purpose, the Coulomb operator can be taken in the Lorentz form

$$C_e(f_e) \simeq \nu_D^e(u) \mathcal{L}(f_e) = \nu_D^e(u) \frac{1}{2} \frac{\partial}{\partial \xi} (1 - \xi^2) \frac{\partial f_e}{\partial \xi}, \tag{3.3}$$

where $\nu_D^e(u) = \nu_D^{\mathrm{ee}}(u) + \nu_D^{\mathrm{ei}}(u)$ is the collision (deflection) frequency of electrons, accounting for relativistic effects. Explicit expressions for $\nu_D^{\mathrm{ee}}(u)$ and $\nu_D^{\mathrm{ei}}(u)$, valid for arbitrary high temperature, are given in Appendix B.

Thus, the linearized mono-energetic rDKE can be written as follows:

$$\mathcal{V}(f_{e1}) - \nu_D^e(u) \mathcal{L}(f_{e1}) = -\dot{\rho}[A_1 + (\kappa - \tfrac{5}{2} - \mathcal{R})A_2] f_{e0} - b v_\parallel A_3 f_{e0}, \tag{3.4}$$

with $\kappa \equiv \mu_e(\gamma - 1)$ as the kinetic energy normalized by temperature, T_e.

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On the right-hand side of (3.1), the radial drift velocity is represented as

$$\dot{\rho} \equiv (\boldsymbol{v}_{\nabla B} \cdot \nabla \rho) = v_d(u) R_0 (1 + \xi^2) \frac{1}{b} [\hat{\boldsymbol{B}} \times \nabla \ln B] \cdot \nabla \rho, \tag{3.5}$$

where $b = B/B_0$, R_0 and B_0 are the reference radius and magnetic field, respectively, and

$$v_d(u) = -\frac{m_e c u^2}{2 e \gamma R_0 B_0}, \tag{3.6}$$

is the normalized drift velocity.

Expressions for thermodynamic forces, A_1, A_2 and A_3, are the same as in the non-relativistic approximation (Helander & Sigmar 2002; Sugama & Nishimura 2008)

$$A_1 = \frac{\mathrm{d} \ln p_e}{\mathrm{d}\rho} + \frac{e E_\rho}{T_e}, \tag{3.7a}$$

$$A_2 = \frac{\mathrm{d} \ln T_e}{\mathrm{d}\rho}, \tag{3.7b}$$

$$A_3 = \frac{e}{T_e} B_0 \frac{\langle E_\parallel B \rangle}{\langle B^2 \rangle}, \tag{3.7c}$$

with $p_e = n_e T_e$ as the electron pressure. Please note that the use of ∇n_e and ∇T_e instead of ∇p_e, similar to Hirshman *et al.* (1986), Maassberg *et al.* (2009) and Beidler *et al.* (2011), automatically leads to appearance of the relativistic correction term in A_1 (Marushchenko *et al.* 2013; Kapper *et al.* 2018)

$$A_1^* = \frac{\mathrm{d} \ln n_e}{\mathrm{d}\rho} - \left(\frac{3}{2} + \mathcal{R} \right) \frac{\mathrm{d} \ln T_e}{\mathrm{d}\rho} + \frac{e E_\rho}{T_e}. \tag{3.8}$$

This choice of thermodynamic force with explicit dependence on the absolute value of temperature seems less natural.

Following Beidler *et al.* (2011), we seek the solution of rDKE as a decomposition

$$f_{e1} = \frac{v_d R_0}{v} \left[A_1 + \left(\kappa - \frac{5}{2} - \mathcal{R} \right) A_2 \right] f_{e0} \chi_\rho + R_0 A_3 f_{e0} \chi_\parallel, \tag{3.9}$$

where χ_ρ and $\chi_\parallel$ refer to the radial and parallel fluxes, respectively, and R_0 is the reference radius.

Then, (3.1) can be separated into two independent dimensionless equations

$$\frac{R_0}{v} \mathcal{V}(\chi_\rho) - \frac{\nu_D^e(u) R_0}{v} \mathcal{L}(\chi_\rho) = -\frac{\dot{\rho}}{v_d}, \tag{3.10a}$$

$$\frac{R_0}{v} \mathcal{V}(\chi_\parallel) - \frac{\nu_D^e(u) R_0}{v} \mathcal{L}(\chi_\parallel) = -b\xi, \tag{3.10b}$$

where the ratio $\dot{\rho}/v_d$ depends only on the pitch and geometry; see (3.5).

It can be seen that these equations have the same form as the non-relativistic ones. However, due to the relativistic effects, electrons from the tail of the distribution function become more collisional as the temperature increases. This can be seen in figure 1, where the normalized collisionality, $\nu_D^e(u)/v \cdot (u_{\mathrm{Te}}/\nu_{e0})$, is plotted for a set of temperatures. It

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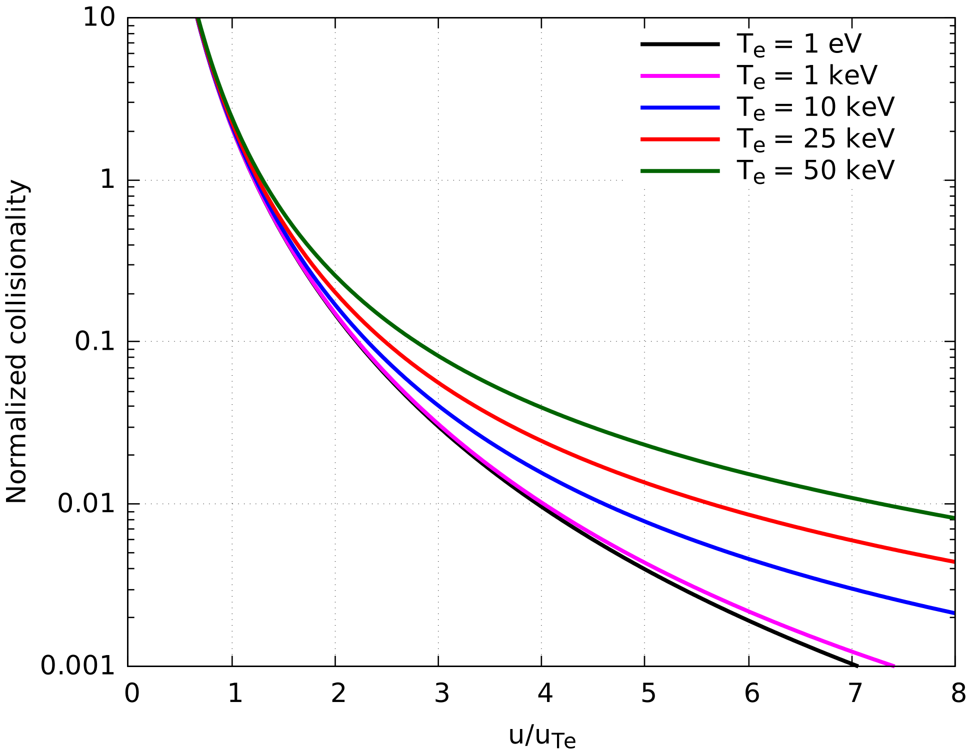

FIGURE 1. (Colour online) The normalized collisionality, $\nu_D^e(u)/v \cdot (u_{\mathrm{Te}}/\nu_{e0})$, is plotted for different temperatures as a function of the normalized momentum, u/u_{Te}. The case with $T_e = 1$ eV, which corresponds to the non-relativistic limit, is shown as the reference line. Calculations were performed for $Z_{\mathrm{eff}} = 1.5$.

is shown, in particular, that the non-relativistic limit is valid only for temperatures of the order of several keV.

Using the definition of fluxes as a linear response to thermodynamic forces

$$I_i = -n_e \sum_{j=1}^{3} L_{ij} A_j, \tag{3.11}$$

consider the particles and heat fluxes obtained above. Following (2.15), the radial particle flux is

$$I_1 = \varGamma_e^\rho = \left\langle \int \dot{\rho} f_{e1} \, \mathrm{d}^3 u \right\rangle, \tag{3.12}$$

the radial heat flux, from (2.17), is

$$I_2 = \frac{q_e^\rho}{T_e} = \left\langle \int \left(\kappa - \frac{5}{2} - \mathcal{R} \right) \dot{\rho} f_{e1} \, \mathrm{d}^3 u \right\rangle, \tag{3.13}$$

and the parallel flux is

$$I_3 = \frac{\langle j_\parallel B \rangle}{e B_0} = \left\langle b \int v_\parallel f_{e1} \, \mathrm{d}^3 u \right\rangle. \tag{3.14}$$

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Assuming that the mono-energetic solutions χ_ρ and $\chi_\parallel$ are known, the matrix transport coefficients can be written as follows:

$$L_{ij} = C_{\mathrm{MJ}} \frac{2}{\sqrt{\pi}} \int_0^\infty \left(1 + \frac{\kappa}{\mu_e}\right) \left(1 + \frac{\kappa}{2\mu_e}\right)^{1/2} h_i h_j \sqrt{\kappa}\, \mathrm{e}^{-\kappa} D_{ij}(\kappa)\, \mathrm{d}\kappa, \tag{3.15}$$

with $h_1 = h_3 = 1$ and $h_2 = \kappa - 5/2 - \mathcal{R}$. Again, for the non-relativistic limit, with $\mu_e \to \infty$ and $\kappa \to m_e v^2/2T_e$, (3.15) coincides exactly with the classical expression; see Beidler *et al.* (2011).

The relativistic mono-energetic transport coefficients, D_{ij}, are also represented in the classic-like form

$$D_{11} = D_{12} = D_{21} = D_{22} = -\frac{v_d^2(u)R_0}{2v} \left\langle \int_{-1}^{1} \frac{\dot{\rho}}{v_d} \chi_\rho \,\mathrm{d}\xi \right\rangle, \tag{3.16a}$$

$$D_{13} = D_{23} = -\frac{v_d(u)R_0}{2} \left\langle \int_{-1}^{1} \frac{\dot{\rho}}{v_d} \chi_\parallel \,\mathrm{d}\xi \right\rangle, \tag{3.16b}$$

$$D_{31} = D_{32} = -\frac{v_d(u)R_0}{2} \left\langle b \int_{-1}^{1} \xi \chi_\rho \,\mathrm{d}\xi \right\rangle, \tag{3.16c}$$

$$D_{33} = -\frac{vR_0}{2} \left\langle b \int_{-1}^{1} \xi \chi_\parallel \,\mathrm{d}\xi \right\rangle. \tag{3.16d}$$

To integrate over κ in (3.15), one can use the following relations: $\gamma = 1 + \kappa/\mu_e$, $u^2/u_{\mathrm{Te}}^2 = \kappa(1 + \kappa/2\mu_e)$, which are related to the drift velocity $v_d(u)$ and the parameters for χ_ρ and $\chi_\parallel$ – the collisionality, $\nu_D^e(u)/v$, and the normalized electric field, E_ρ/v. It is also useful to emphasize here that the relation between the normalized energy, κ, and velocity, $v = u/\gamma$, is different from the simple classical relation.

4. Summary

The paper considers electron transport in toroidal plasmas with relativistic effects taken into account. The approach is based on the rDKE for electrons with the thermodynamic equilibrium given by the relativistic Maxwell–Jüttner distribution function. The relativistic fluxes correspond to the set of relativistic transport coefficients, which are defined as a proper convolution of the relativistic mono-energetic transport coefficients. In turn, the relativistic mono-energetic transport coefficients can be calculated using the non-relativistic solvers.

As an example, the procedure for calculating relativistic electron fluxes is described for the code DKES (Hirshman *et al.* 1986) and other similar mono-energetic transport codes. Similar to the classical approach, the solutions of the rDKE (3.10) are defined by two parameters, the collisionality, $\nu_D^e(u)/v$, and E_ρ/v, which differ noticeably from the classical values for temperatures above several keV. As the temperature rises, electrons from the tail of the distribution function become more collisional than in the non-relativistic limit.

Finally, the classic-like definitions of the fluxes as well as of the thermal energy can be used for solving the system of radial transport equations for the arbitrarily high temperatures expected in a fusion reactor.

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Declaration of interest

The authors report no conflict of interest.

Appendix A. Relativistic heat flux in covariant formulation

It is shown here that the definition of the relativistic heat flux introduced above in (2.12) agrees perfectly with the standard definition adopted in relativistic kinetics in the covariant formalism. Below we use the definitions from the book de Groot *et al.* (1980).

In the relativistic covariant formalism with 4-vector momentum, $p^\alpha = (p^0, \boldsymbol{p})$ with $\alpha = 0, 1, 2, 3$ and $p^0 = \sqrt{m^2c^2 + p^2}$, all quantities needed for the transport equations (such as fluxes of particles, energy and heat) are usually defined in terms of the 4-vector particle flux

$$N^\alpha = c\int \frac{\mathrm{d}^3p}{p^0} p^\alpha f(\boldsymbol{x}, \boldsymbol{p}), \tag{A1}$$

and the 4-tensor of energy-momentum

$$T^{\alpha\beta} = c\int \frac{\mathrm{d}^3p}{p^0} p^\alpha p^\beta f(\boldsymbol{x}, \boldsymbol{p}). \tag{A2}$$

(We do not need higher-order moments here.) Note that $T^{\alpha\beta}$ naturally contains the rest energy and its flux, while the transport equations do not require these quantities. Since $\boldsymbol{v} = c\boldsymbol{p}/p^0$ (by definition), the spatial flux of particles has the form $N^i \equiv \Gamma^i$ with $i = 1, 2, 3$, and the time-like flux is related to the density as $N^0 = cn$.

For the relativistic Maxwellian, a zeroth component of the energy–momentum tensor can be written as

$$T^{00} = nT\left(\mu\frac{K_3(\mu)}{K_2(\mu)} - 1\right), \tag{A3}$$

with $K_n(\mu)$ as a modified Bessel function of the second kind and $\mu = mc^2/T$. It follows that introducing

$$\mathcal{R}(\mu) = \mu\left(\frac{K_3}{K_2} - 1\right) - \frac{5}{2}, \tag{A4}$$

one can easily express the internal thermal energy in classic-like form

$$W = T^{00} - nmc^2 = (\tfrac{3}{2} + \mathcal{R})nT. \tag{A5}$$

The energy and heat fluxes, Q^i and q^i, respectively, can be defined in the same way

$$Q^i = cT^{0i} - mc^2N^i, \tag{A6a}$$

$$q^i = cT^{0i} - hN^i, \tag{A6b}$$

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where cT^{0i} is the total energy flux, $N^i \equiv \Gamma^i$ is the particle flux and $h = T^{00}/n + T$ is the enthalpy, which for the relativistic Maxwellian is given by $h = mc^2 K_3/K_2$. Rewriting the heat flux (A6*b*) as

$$q^i = Q^i - \mu \left(\frac{K_3}{K_2} - 1 \right) T\Gamma^i, \tag{A7}$$

and, again, using the definition for $\mathcal{R}$, one can obtain an expression for the heat flux in classic-like form

$$q^i = Q^i - (\tfrac{5}{2} + \mathcal{R}) T\Gamma^i, \tag{A8}$$

which is identical to (2.12).

Appendix B. Coulomb deflection frequency for relativistic electrons

Explicit expressions for the Coulomb deflection frequency for relativistic electrons are reproduced here for convenience. These expressions are valid for arbitrary high temperature. The general definitions for pitch-scattering diffusion coefficients (Braams & Karney 1989) are applied, $\nu_D^{ab}(u) = (2/u^2) D_{\vartheta\vartheta}^{ab}(u)$, where $\vartheta = \arccos(u_\parallel/u)$ is the pitch angle with respect to the direction of the magnetic field.

Substituting into the expressions for $D_{\vartheta\vartheta}^{\mathrm{ee}}$ the relativistic Maxwellian in the form (2.5), one obtains the following:

$$\begin{aligned} \nu_D^{\mathrm{ee}}(u) &= \nu_{e0} C_{\mathrm{JM}}(\mu_e) \frac{4}{\sqrt{\pi}} \times \\ &\left(\frac{\gamma}{x^3} \int_0^x \left[\left(\gamma' - \frac{x'^2}{3x^2} \tilde{j}'_{0[2]02} \right) - \frac{2}{3\mu} \frac{x'^2}{\gamma^2} \left(\tilde{j}'_{0[2]02} - \frac{x'^2}{5x^2} \tilde{j}'_{0[3]022} \right) \right] \frac{x'^2}{\gamma'} \mathrm{e}^{-\kappa'} \, \mathrm{d}x' \right. \\ &\left. + \frac{1}{x^2} \int_x^\infty \left[\left(\gamma'^2 - \frac{\gamma}{3} \tilde{j}_{0[2]02} \right) - \frac{2}{3\mu} \frac{x^2}{\gamma} \left(\frac{x'^2}{x^2} \tilde{j}_{0[2]02} - \frac{x^2}{5} \tilde{j}_{0[3]022} \right) \right] \frac{x'}{\gamma'} \mathrm{e}^{-\kappa'} \, \mathrm{d}x' \right), \end{aligned} \tag{B1}$$

where $x = u/u_{\mathrm{Te}}$, $\gamma = \sqrt{1 + z^2}$ with $z = u/c$ and $\kappa = x^2(2/(\gamma + 1))$. The specific functions $\tilde{j}_{l[k]*}(z)$ (Braams & Karney 1989) are given by

$$\tilde{j}_{0[2]02}(z) = 3(z\gamma - \sigma)/2z^3 = 1 - 3z^2/10 + O(z^4), \tag{B2a}$$

$$\tilde{j}_{0[3]022}(z) = 15[-3z\gamma + (3 + 2z^2)\sigma]/4z^5 = 1 - 9z^2/14 + O(z^4), \tag{B2b}$$

where $\sigma(z) = \ln(z + \gamma)$ and $\gamma = \sqrt{1 + z^2}$ with $z = u/c$. Here, $\nu_{e0} = 4\pi n_e e^4 \ln \Lambda^{e/e}/m_e^2 u_{\mathrm{Te}}^3$ is the simplest electron collision frequency.

From here the first two terms of the expansion by μ_e^{-1} can be obtained

$$\begin{aligned} \nu_D^{\mathrm{ee}}(x) \simeq \nu_{e0} \frac{1}{2x^3} &\left\{ \left[\left(2 - \frac{1}{x^2} \right) \mathrm{erf}(x) + \frac{\mathrm{erf}'(x)}{x} \right] \right. \\ &\left. + \frac{1}{\mu_e} \left[\left(\frac{5}{2x^2} - 3 + 2x^2 \right) \mathrm{erf}(x) + (3x^2 - 5) \frac{\mathrm{erf}'(x)}{2x} \right] \right\}, \end{aligned} \tag{B3}$$

where the first term corresponds to the non-relativistic limit (Hinton & Hazeltine 1976; Helander & Sigmar 2002).

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For ν_D^{ei}, the ion background can be assumed to be a non-relativistic Maxwellian. Taking into account that $M_i/m_e \gg 1$, for $x \gg v_{\mathrm{Ti}}/u_{\mathrm{Te}}$ it is sufficient to apply the high-speed-limit

$$\nu_D^{\mathrm{ei}}(x) = \nu_{e0} Z_{\mathrm{eff}} \frac{\gamma}{x^3}. \tag{B4}$$

Again, the non-relativistic limit for this expression coincides exactly with the well-known classical approximation.

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