MULTITYPE BRANCHING PROCESSES WITH INHOMOGENEOUS POISSON IMMIGRATION

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Abstract

In this paper we introduce multitype branching processes with inhomogeneous Poisson immigration, and consider in detail the critical Markov case when the local intensity $r(t)$ of the Poisson random measure is a regularly varying function. Various multitype limit distributions (conditional and unconditional) are obtained depending on the rate at which $r(t)$ changes with time. The asymptotic behaviour of the first and second moments, and the probability of nonextinction are investigated.

Keywords: Multitype branching process; immigration; inhomogeneous Poisson process; limit distribution

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1. Introduction

In this paper we introduce multitype branching processes with inhomogeneous Poisson immigration, and our main goal is the investigation of the critical Markov case. Sevastyanov [20] proposed the first branching model with immigration in the single-type Markov case with homogeneous Poisson immigration. Jagers [12] generalized this model to the Bellman–Harris setting, and further extensions were presented in [22] and [23]. On the other hand, the Sevastyanov branching process (see [21]) extends the Bellman–Harris process by allowing the lifespan and offspring of each individual to be dependent. Critical single-type Sevastyanov processes with inhomogeneous Poisson immigration were studied in [8] and [14]. The super- and sub-critical cases were investigated in [9] and [10]. Multitype Markov branching processes with homogeneous Poisson immigration were considered in [16], [18], and [19], and some cases with time-dependent immigration in [3] and [4].

Jagers [13, Chapter 9] illustrates well that branching processes offer useful models of cell population kinetics. In [11] we used multitype branching processes to study stress erythropoiesis; the model included an immigration process to describe the influx of cells into the pool of progenitor cells that generate the red blood cells. This influx of cells was assumed to be time homogeneous, but there are situations in which it depends on time so branching processes with inhomogeneous Poisson immigration should be considered. The scarcity of results for such models motivated this work.

This paper is organized as follows. In Section 2 we introduce a class of multitype branching processes in which individuals evolve independently of each other and arise from an immigration process at time points generated by a Poisson random measure. The probability generating function (PGF) of these processes is obtained in Theorem 1.

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Our investigations then concentrate on multitype critical Markov branching processes with inhomogeneous Poisson immigration. These models are formulated in Section 3. Their asymptotic behaviour is investigated in Sections 4 and 5. Unless stated otherwise, results are established in the case that the local intensity function $r(t)$ of the Poisson random measure is regularly varying (RV). The first and second moments are studied in Section 4.1, and the probability of nonextinction $W(t)$ is investigated in the three theorems of Section 4.2. These theorems provide conditions under which $W(t)$ converges either to 1, or to some positive constant smaller than 1, or to 0 (at different rates) depending on $r(t)$.

Finally, limit distributions are established in the eight theorems of Section 5. Theorems 6 and 7 are respectively interpreted as a law of large numbers and a central limit theorem (with degenerate multivariate normal distribution). Other degenerate multitype limit distributions, in which the marginal random variables are almost surely (a.s.) identical, are obtained depending on the particular asymptotic rates of $r(t)$ (Theorems 8–13). For example, when $r(t)$ converges to some positive constant (Theorem 8), the limiting univariate margins are gamma distributed. This finding generalizes an earlier result due to Sevastyanov in the single-type critical Markov case with homogeneous Poisson immigration [20]. On the other hand, Theorem 13 establishes a result similar to that satisfied by the critical multitype Markov branching process without immigration (see Sevastyanov [21]). We note that the other limit distributions established in this paper have no equivalent among existing results for multitype Markov branching processes with or without homogeneous Poisson immigration. These novel asymptotic behaviours arise from the fact that the immigration processes we consider are time inhomogeneous.

Comments that help with the interpretation of the results are provided in the concluding Section 6 where also we compare our results with those in the extant literature. Note finally that Theorem 1 makes it possible to investigate the properties of sub- and super-critical Markov processes and multitype age-dependent branching processes. Establishing these properties remains an open problem.

2. Poisson random measures and multitype branching processes

We consider a population that consists of d types of cells (individuals, particles), and evolves in accordance with an immigration process and a branching mechanism.

Let $0 < T_1 < T_2 < \cdots$ be random time points arising from a Poisson random measure $\Pi(t) = \sum_{i=1}^{\infty} \mathbf{1}_{\{T_i \le t\}}$, $t \ge 0$, with local intensity $r(t) > 0$ and mean measure $R(t) = \int_0^t r(x)\,\mathrm{d}x$. Then $\mathbb{P}\{\Pi(t) = n\} = \mathrm{e}^{-R(t)} R^n(t)/n!$ for $n = 0, 1, \ldots$. Assume that $\boldsymbol{I}_k = (I_{k1}, \ldots, I_{kd})$, $k = 1, 2, \ldots$ are independent and identically distributed (i.i.d.) nonnegative integer-valued random vectors with PGF

$$g(\boldsymbol{s}) = \mathbb{E}[\boldsymbol{s}^{\boldsymbol{I}_k}] = \sum_{\boldsymbol{\alpha} \in \mathbb{N}^d} \mathbb{P}(\boldsymbol{I}_k = \boldsymbol{\alpha})\boldsymbol{s}^{\boldsymbol{\alpha}}, \qquad \boldsymbol{s} = (s_1, \ldots, s_d),\ |\boldsymbol{s}| \le 1,$$

where $\boldsymbol{s}^{\boldsymbol{\alpha}} = \prod_{i=1}^{d} s_i^{\alpha_i}$ for every $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$. We consider the marked point process $\{(T_k, \boldsymbol{I}_k),\ k = 1, 2, \ldots\}$, interpreting the vector $\boldsymbol{I}_k$ as the number of immigrants joining the population at time T_k.

Let $\boldsymbol{Z} = \{\boldsymbol{Z}_i(t) = (Z_{i1}(t), Z_{i2}(t), \ldots, Z_{id}(t)), i = 1, \ldots, d; t \ge 0\}$ be a multitype branching process, where, for $i, j = 1, \ldots, d$, $Z_{ij}(t)$ denotes the number of type-j cells at time t produced from a single type-i cell born at $t = 0$, and we assume that cells evolve independently of each other.

Introduce next the PGF $F_i(t; \boldsymbol{s}) = \sum_{\boldsymbol{\alpha} \in \mathbb{N}^d} \mathbb{P}\{\boldsymbol{Z}_i(t) = \boldsymbol{\alpha}\}\boldsymbol{s}^{\boldsymbol{\alpha}}$, with $F_i(0, \boldsymbol{s}) = s_i$, and define the vector $\boldsymbol{F}(t; \boldsymbol{s}) = (F_1(t; \boldsymbol{s}), F_2(t; \boldsymbol{s}), \ldots, F_d(t; \boldsymbol{s}))$.

Let $\tilde{\boldsymbol{Z}} = \{\tilde{\boldsymbol{Z}}_k(t) = (\tilde{Z}_{k1}(t), \dots, \tilde{Z}_{kd}(t)); t \geq 0; k = 1, 2, \dots\}$ be i.i.d. copies of $\boldsymbol{Z}$, but with initial conditions $\tilde{\boldsymbol{Z}}_k(0) = \boldsymbol{I}_k$. Then, because the individual evolutions are independent, $\mathbb{E}[\boldsymbol{s}^{\tilde{\boldsymbol{Z}}_k(t)}] = g(\boldsymbol{F}(t; \boldsymbol{s}))$. We assume that the sets $\tilde{\boldsymbol{Z}}$ and $\boldsymbol{\Pi} = \{\Pi(t), t \geq 0\}$ are independent.

Define the process

$$\boldsymbol{Y}(t) = \sum_{k=1}^{\Pi(t)} \tilde{\boldsymbol{Z}}_k(t - T_k)\mathbf{1}_{\{\Pi(t)>0\}}, \quad t \geq 0, \qquad \boldsymbol{Y}(0) = \mathbf{0}. \tag{1}$$

Its first increment occurs when the first batch of $\boldsymbol{I}_1$ immigrants enters the population at time T_1, each of which evolves independently and in accordance with a process $\boldsymbol{Z}$. A second batch of $\boldsymbol{I}_2$ immigrants arrives at time T_2, etc. We refer to $\boldsymbol{Y} = \{\boldsymbol{Y}(t) = (Y_1(t), \dots, Y_d(t)),\ t \geq 0\}$ as a *d-type branching process with inhomogeneous Poisson immigration* (dBPiPI).

Theorem 1. *If* $\Phi(t; \boldsymbol{s}) = \mathbb{E}[\boldsymbol{s}^{\boldsymbol{Y}(t)}]$, *then*

$$\Phi(t; \boldsymbol{s}) = \exp\biggl\{-\int_0^t r(t - x)[1 - g(\boldsymbol{F}(x; \boldsymbol{s}))]\,\mathrm{d}x\biggr\}, \qquad \Phi(0; \boldsymbol{s}) = \mathbf{1}, \tag{2}$$

where $\boldsymbol{F}(t; \boldsymbol{s}) = (F_1(t; \boldsymbol{s}), \dots, F_d(t; \boldsymbol{s}))$ *is defined above.*

Proof. Equation (1) leads directly to $\Phi(t; \boldsymbol{s}) = \sum_{n=0}^{\infty} \mathbb{P}\{\Pi(t) = n\}\Delta_n(t; \boldsymbol{s})$ where

$$\Delta_n(t; \boldsymbol{s}) = \mathbb{E}\{\boldsymbol{s}^{\sum_{k=1}^{\Pi(t)} \tilde{\boldsymbol{Z}}_k(t-T_k)\mathbf{1}_{\{\Pi(t)>0\}}} \mid \Pi(t) = n\}.$$

Let $x_1 \leq \cdots \leq x_n$. Then, using the assumption that individuals evolve independently,

$$\Delta_n(t; \boldsymbol{s}) = \int_0^t \int_{x_1}^t \dots \int_{x_{n-1}}^t \prod_{i=1}^n g(\boldsymbol{F}(t - x_i; \boldsymbol{s}))\,\mathrm{d}\mathbb{P}\{T_1 \leq x_1, \dots, T_n \leq x_n\}.$$

Let $R^{-1}(\cdot)$ denote the inverse function of the mean measure $R(\cdot)$. It is well known (see, e.g. [17, Chapter 4]) that $\eta(t) = \Pi(R^{-1}(t))$ is a Poisson process with constant rate 1, and $\xi_k := R(T_k)$ is gamma distributed with parameters k and 1, $k = 1, 2, \dots$. Hence,

$$\begin{aligned}\mathbb{P}\{T_1 \leq x_1, \dots, T_n \leq x_n\} &= \mathbb{P}\{R^{-1}(\xi_1) \leq x_1, \dots, R^{-1}(\xi_n) \leq x_n\} \\ &= \mathbb{P}\{\xi_1 \leq R(x_1), \dots, \xi_n \leq R(x_n)\}.\end{aligned}$$

The order statistics property of a Poisson process with bounded local intensity (see, e.g. [17, Theorem 4.5.2]) implies the third equality in

$$\begin{aligned}\Delta_n(t; \boldsymbol{s}) &= \frac{n!}{R^n(t)} \int_0^t \int_{x_1}^t \cdots \int_{x_{n-1}}^t \prod_{i=1}^n g(\boldsymbol{F}(t - x_i; \boldsymbol{s}))\,\mathrm{d}R(x_n) \dots \mathrm{d}R(x_1) \\ &= \frac{n!}{R^n(t)} \int_0^t \int_{x_1}^t \cdots \int_{x_{n-1}}^t \prod_{i=1}^n r(x_i) g(\boldsymbol{F}(t - x_i; \boldsymbol{s}))\,\mathrm{d}x_n \dots \mathrm{d}x_1 \\ &= \frac{1}{R^n(t)} \int_0^t \int_0^t \cdots \int_0^t \prod_{i=1}^n r(x_i) g(\boldsymbol{F}(t - x_i; \boldsymbol{s}))\,\mathrm{d}x_1 \dots \mathrm{d}x_n,\end{aligned}$$

because

$$f(x_1, \dots, x_n) = \prod_{i=1}^n r(x_i) g(\boldsymbol{F}(t - x_i; \boldsymbol{s}))$$

is a symmetric function. Hence, $\Delta_n(t; \boldsymbol{s}) = \{\int_0^t r(x) g(\boldsymbol{F}(t - x; \boldsymbol{s}))\,\mathrm{d}x / R(t)\}^n$, and the result follows from the identity $\Phi(t; \boldsymbol{s}) = \exp\{-R(t) + \int_0^t r(x) g(\boldsymbol{F}(t - x; \boldsymbol{s}))\,\mathrm{d}x\}$. □

Remark 1. Equation (2) is valid for a broad class of branching processes in which individuals evolve independently of each other. Such processes include multitype Markov, Bellman–Harris, Sevastyanov, and Crump–Mode–Jagers branching models, as described in various monographs: Harris [7], Sevastyanov [21], Mode [15], Athreya and Ney [2], Jagers [13], and Asmussen and Hering [1].

3. Multitype Markov branching processes with inhomogeneous Poisson immigration

From now on we consider the case that $\boldsymbol{Z}$ is a multitype Markov branching process; that is, the lifespan τ_i and the offspring vector $\boldsymbol{\nu}_i = (\nu_{i1}, \dots, \nu_{id})$ of any type-i cell are independent, $G_i(t) = \mathbb{P}\{\tau_i \le t\} = 1 - \mathrm{e}^{-t/\mu_i}$, and $h_i(\boldsymbol{s}) = \mathbb{E}[\boldsymbol{s}^{\boldsymbol{\nu}_i}] = \sum_{\boldsymbol{\alpha}\in\mathbb{N}^d} p_{\boldsymbol{\alpha}} \boldsymbol{s}^{\boldsymbol{\alpha}}$, $i = 1, \dots, d$. Under these assumptions, the PGFs $F_i(t; \boldsymbol{s}) = \sum_{\boldsymbol{\alpha}\in\mathbb{N}^d} \mathbb{P}\{\boldsymbol{Z}_i(t) = \boldsymbol{\alpha}\}\boldsymbol{s}^{\boldsymbol{\alpha}}$ satisfy the system of differential equations

$$\frac{\partial}{\partial t}\boldsymbol{F}(t;\boldsymbol{s}) = \boldsymbol{f}(\boldsymbol{F}(t;\boldsymbol{s})), \qquad \frac{\partial}{\partial t}\boldsymbol{F}(t;\boldsymbol{s}) = \sum_{i=1}^{d} f_i(\boldsymbol{s})\frac{\partial}{\partial s_i}\boldsymbol{F}(t;\boldsymbol{s}), \qquad \boldsymbol{F}(0;\boldsymbol{s}) = \boldsymbol{s},$$

where $f_i(\boldsymbol{s}) = [h_i(\boldsymbol{s}) - s_i]/\mu_i$ are the infinitesimal generating functions and $\boldsymbol{f}(\boldsymbol{s}) = (f_1(\boldsymbol{s}), \dots, f_d(\boldsymbol{s}))$. Under these assumptions, $\boldsymbol{Y}(t)$ is a d-type Markov branching process with inhomogeneous Poisson immigration (dMBPiPI). For $1 \le i, j \le d$, let

$$A_{ij}(t) = \mathbb{E}[Z_{ij}(t)] = \frac{\partial F_i(t;\boldsymbol{s})}{\partial s_j}\bigg|_{\boldsymbol{s}=\boldsymbol{1}},$$

and introduce the matrix of first infinitesimal characteristics

$$\boldsymbol{a} = (a_{ij})_{1\le i,j\le d}, \quad \text{where} \quad a_{ij} = \frac{\partial f_i(\boldsymbol{s})}{\partial s_j}\bigg|_{\boldsymbol{s}=\boldsymbol{1}}.$$

It is well known that $\boldsymbol{A}(t) = (A_{ij}(t))_{1\le i,j\le d} = \exp(\boldsymbol{a}t) = \sum_{n=0}^{\infty} \boldsymbol{a}^n t^n/n!$. Later we assume that $\boldsymbol{a}$ is an irreducible matrix with Perron–Frobenius root ρ. The associated right and left eigenvectors $\boldsymbol{u} = (u_1, \dots, u_d)$ and $\boldsymbol{v} = (v_1, \dots, v_d)$ can be chosen positive, with $u_1 > 0$ and $v_1 > 0$, and normalized such that $\sum_{i=1}^{d} u_i = 1$ and $\sum_{i=1}^{d} u_i v_i = 1$. Define

$$m_i = \frac{\partial g(\boldsymbol{s})}{\partial s_i}\bigg|_{\boldsymbol{s}=\boldsymbol{1}}, \qquad \beta_{ij} = \frac{\partial^2 g(\boldsymbol{s})}{\partial s_i \partial s_j}\bigg|_{\boldsymbol{s}=\boldsymbol{1}},$$

$$B^i_{jk}(t) = \mathbb{E}[Z_{ij}(t)(Z_{ik}(t) - \delta_{jk})] = \frac{\partial^2 F_i(t;\boldsymbol{s})}{\partial s_j \partial s_k}\bigg|_{\boldsymbol{s}=\boldsymbol{1}},$$

where $\delta_{jk} = 1$ if $j = k$, $= 0$ otherwise, and the second factorial infinitesimal characteristics $b^i_{jk} = \partial^2 f_i(\boldsymbol{s})/\partial s_j \partial s_k|_{\boldsymbol{s}=\boldsymbol{1}}$ for $1 \le i, j, k \le d$. We also define

$$M_i(t) = \mathbb{E}[Y_i(t)], \qquad C_{ij}(t) = \operatorname{cov}(Y_i(t), Y_j(t)), \qquad V_i(t) = C_{ii}(t) = \operatorname{var}(Y_i(t)),$$

and set

$$b = \sum_{i,j,k} v_i b^i_{jk} u_j u_k, \qquad C = \sum_{k=1}^{d} m_k u_k, \qquad \alpha = \frac{2C}{b}. \tag{3}$$

Then

$$M_i(t) = \frac{\partial \log \Phi(t;\boldsymbol{s})}{\partial s_i}\bigg|_{\boldsymbol{s}=\boldsymbol{1}} = \sum_{k=1}^{d} m_k \int_0^t r(t-x) A_{ki}(x)\,\mathrm{d}x \tag{4}$$

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and

$$C_{ij}(t) = \frac{\partial^2 \log \Phi(t;\mathbf{s})}{\partial s_i \partial s_j}\bigg|_{\mathbf{s}=\mathbf{1}}$$
$$= b\sum_{k=1}^{d} m_k \int_0^t r(t-x) B_{ij}^k(x)\,\mathrm{d}x + \sum_{k=1}^{d}\sum_{l=1}^{d} \beta_{kl} \int_0^t r(t-x) A_{ki}(x) A_{lj}(x)\,\mathrm{d}x. \tag{5}$$

4. Asymptotic behaviour of the critical dMBPiPI

In the remainder of this paper we assume that, for $1 \le i, j, k \le d$, a_{ij}, b_{jk}^i, m_i and β_{ij} are all finite, and that the Markov process $\boldsymbol{Z}$ is irreducible and critical, i.e. $\rho = 0$, and $b \in (0, \infty)$. It is well known (see, e.g. [21]) that, as $t \to \infty$,

$$A_{ij}(t) = u_i v_j + o(\mathrm{e}^{-\gamma t}), \qquad \gamma > 0, \tag{6}$$

$$B_{jk}^i(t) \sim u_i v_j v_k bt, \tag{7}$$

$$1 - \mathbb{P}\{\mathbf{Z}_i(t) = \mathbf{0}\} = \mathbf{1} - F_i(t;\mathbf{0}) \sim \frac{2u_i}{bt}. \tag{8}$$

We study asymptotic properties of $\boldsymbol{Y}(t)$ when the function $r(t) = L(t)t^{\theta}$ is RV, bounded on finite intervals, and $L(t)$ is slowly varying (SV) as $t \to \infty$. In addition to b, C, and α at (3) define

$$b^*(t) = \frac{b}{2Cr(t)} = \frac{1}{\alpha r(t)}. \tag{9}$$

4.1. Means, variances, covariances, and correlations

Theorem 2. *Let the function $r(t)$ be RV.*

(i) *Assume that (6) holds and that $m_i < \infty$ for $i = 1, \dots, d$. Then, as $t \to \infty$, for $i = 1, \dots, d$,*

$$M_i(t) \sim C v_i R(t). \tag{10}$$

(ii) *Assume that (6) and (7) hold, and that $\beta_{jk} < \infty$ for $1 \le j, k \le d$. Then, as $t \to \infty$,*

$$C_{ij}(t) \sim bCv_i v_j \int_0^t R(x)\,\mathrm{d}x \quad \textit{for } 1 \le i, j \le d. \tag{11}$$

Proof. (i) We deduce from (4) and (6), for $i = 1, \dots, d$ and $t \to \infty$, that

$$M_i(t) = R(t) v_i \sum_{k=1}^{d} m_k u_k + \sum_{k=1}^{d} m_k \int_0^t r(t-x)\psi(x)\,\mathrm{d}x, \tag{12}$$

where $\psi(t) = o(\mathrm{e}^{-\gamma t})$ with $\gamma > 0$. Let $I(t) = \int_0^t r(t-x)\psi(x)\,\mathrm{d}x = (\int_0^{t/2} + \int_{t/2}^t)(\cdots) =: I_1(t) + I_2(t)$. When $\theta > 0$, the RV function $r(t)$ can be chosen to be asymptotically nondecreasing [5, Theorem 1.5.3]. Then $I_1(t) \le r(t)\int_0^{t/2}\psi(x)\,\mathrm{d}x = O(r(t)) = o(R(t))$ as $t \to \infty$. Similarly, $I_2(t) \le (\sup_{u \ge t/2}\psi(u))\int_0^t r(x)\,\mathrm{d}x = o(R(t))$.

When $\theta < 0$, we can choose RV $r(t)$ to be asymptotically nonincreasing, and the proof follows a similar line of argument.

If $\theta = 0$, $r(t)$ is SV. Then $x \in [0, t/2]$ implies that $t - x \in [t/2, t]$. The uniform convergence of SV $r(t)$ [5, Theorem 1.5.3] yields $r(t-x) \sim r(t)$ as $t \to \infty$. Hence,

$$I_1(t) = \int_0^{t/2} r(t-x)\psi(x)\,\mathrm{d}x \sim r(t)\int_0^{t/2}\psi(x)\,\mathrm{d}x = o(R(t)),$$

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and, similarly, $I_2(t) = \int_{t/2}^t r(t-x)\psi(x)\,\mathrm{d}x \le \sup_{t/2\le x\le t}\psi(x)\int_0^t r(y)\,\mathrm{d}y = o(R(t))$. This with (12) proves (i).

(ii) Put $J(t) = \int_0^t r(t-x)B_{ij}^k(x)\,\mathrm{d}x$. Equations (5) and (7) imply that, as $t\to\infty$,

$$J(t) = bu_k v_i v_j\biggl[\int_0^t r(t-x)x\,\mathrm{d}x + \int_0^t r(t-x)\varphi(x)\,\mathrm{d}x\biggr], \qquad \varphi(t) = o(t). \tag{13}$$

Note that $\int_0^t r(t-x)x\,\mathrm{d}x = \int_0^t (t-x)\,\mathrm{d}R(x) = tR(t) - \int_0^t x\,\mathrm{d}R(x) = \int_0^t R(x)\,\mathrm{d}x$. Then, just as in (i), $\int_0^t r(t-x)\psi(x)\,\mathrm{d}x = o(R(t)) = o(\int_0^t R(x)\,\mathrm{d}x)$ as $t\to\infty$. It follows from (13) that $J(t)\sim bu_k v_i v_j\int_0^t R(x)\,\mathrm{d}x$, which together with (5) and (10) yields

$$C_{ij}(t)\sim bCv_i v_j\int_0^t R(x)\,\mathrm{d}x + (C\delta_{ij} + C_1 v_i)v_j R(t)$$

for some finite $C_1>0$. This proves (11) because $R(t) = o(\int_0^t R(x)\,\mathrm{d}x)$. □

Corollary 1. *For $1\le i,j\le d$ with $i\ne j$, define the correlation coefficients $\rho_{ij}(t) = \operatorname{corr}(Y_i(t), Y_j(t))$. Then $\lim_{t\to\infty}\rho_{ij}(t) = 1$.*

Remark 2. When $\theta = 0$ and $R(t)\to R<\infty$ as $t\to\infty$, (11) implies that $C_{ij}(t) = bCv_i v_j Rt[1+o(1)]$, as for the process without immigration. When immigration is time homogeneous (so $R(t) = Rt$), (11) then implies that $C_{ij}(t)\sim\frac{1}{2}bCv_i v_j Rt^2$, as Sevastyanov [20] proved in the single-type case.

4.2. Probability of nonextinction

Define $W(t) = \mathbb{P}\{\boldsymbol{Y}(t)\ne\boldsymbol{0}\} = 1-\Phi(t;\boldsymbol{0}) = 1-\mathrm{e}^{-\Lambda(t)}$, where, using (2), $\Lambda(t) = \int_0^t r(t-x)Q(x)\,\mathrm{d}x$ and where the function $Q(x) = 1-g(\boldsymbol{F}(x;\boldsymbol{0}))$ is nonincreasing in $[0,\infty)$. Then, using (8) and recalling b, C, and α at (3), as $x\to\infty$,

$$Q(x)\sim\frac{2C}{bx} = \frac{\alpha}{x}. \tag{14}$$

Theorem 3. (i) *If $\theta>0$, or $\theta=0$ and $L(t)\log t\to\infty$, then $\lim_{t\to\infty}W(t) = 1$.*

(ii) *If $\theta = 0$ and $L(t)\alpha\log t\to\varkappa\in(0,\infty)$, then $\lim_{t\to\infty}W(t) = 1-\mathrm{e}^{-\varkappa}$.*

(iii) *If $\theta\in(-1,0)$, or $\theta=0$ and $L(t)\log t\to 0$, then $W(t)\sim r(t)\alpha\log t$.*

Proof. First we show in all three cases that, as $t\to\infty$,

$$\Lambda(t) = \int_0^t r(t-x)Q(x)\,\mathrm{d}x\sim r(t)\alpha\log t. \tag{15}$$

For any $\delta\in(0,1)$, let $\Lambda(t) = \int_0^{t\delta} + \int_{t\delta}^t =: \Lambda_1(t)+\Lambda_2(t)$. When $\theta>0$, we can assume without loss of generality as earlier that $r(t)$ is asymptotically nondecreasing. Therefore,

$$r(t(1-\delta))\int_0^{t\delta} Q(x)\,\mathrm{d}x\le\Lambda_1(t)\le r(t)\int_0^{t\delta}Q(x)\,\mathrm{d}x$$

and

$$0\le\Lambda_2(t) = \int_{t\delta}^t r(t-x)Q(x)\,\mathrm{d}x\le Q(t\delta)R(t) = O(r(t)).$$

We deduce from these inequalities and (14) that

$$(1-\delta)^\theta\le\liminf_{t\to\infty}\frac{\Lambda_1(t)+\Lambda_2(t)}{r(t)\alpha\log t}\le\limsup_{t\to\infty}\frac{\Lambda_1(t)+\Lambda_2(t)}{r(t)\alpha\log t}\le 1.$$

Equation (15) now follows because $\delta\in(0,1)$ is arbitrary.

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When $\theta < 0$, we can still assume $r(t)$ to be asymptotically nonincreasing, and, hence,

$$r(t)\int_0^{t\delta} Q(x)\,\mathrm{d}x \le \Lambda_1(t) \le r(t(1-\delta))\int_0^{t\delta} Q(x)\,\mathrm{d}x$$

and

$$0 \le \Lambda_2(t) = \int_{t\delta}^{t} r(t-x)Q(x)\,\mathrm{d}x \le Q(t\delta)R(t) = O(r(t)).$$

These inequalities and (14) imply that

$$1 \le \liminf_{t\to\infty}\frac{\Lambda_1(t)+\Lambda_2(t)}{r(t)\alpha\log t} \le \limsup_{t\to\infty}\frac{\Lambda_1(t)+\Lambda_2(t)}{r(t)\alpha\log t} \le \frac{1}{(1-\delta)^{-\theta}}.$$

Then (15) again follows because $\delta \in (0,1)$ is arbitrary.

Suppose now that $\theta = 0$, so that $r(t)$ is SV. When $x \in [0, t\delta]$, $t-x \in [t(1-\delta), t]$. The uniform convergence of an SV function [5, Theorem 1.5.3] implies that $r(t-x) \sim r(t)$ as $t \to \infty$. Using (14), we deduce that, as $t \to \infty$,

$$\Lambda_1(t) \sim r(t)\int_0^{t\delta} Q(x)\,\mathrm{d}x \sim r(t)\alpha\log t$$

and

$$\Lambda_2(t) \le Q(t\delta)\int_0^t r(x)\,\mathrm{d}x = O(r(t)).$$

These last two relations imply (15).

Finally, in case (i) $\Lambda(t) \to \infty$ implies that $W(t) \to 1$; in case (iii), we have $W(t) = 1-\mathrm{e}^{-\Lambda(t)} \sim r(t)\alpha\log t$; and in case (ii), $\Lambda(t) \to \varkappa$ implies that $W(t) \to 1-\mathrm{e}^{-\varkappa}$. □

Theorem 4. *If $\theta = -1$ then $W(t) \sim L_1(t)\alpha t^{-1}$ as $t \to \infty$, where $L_1(t)$ is SV and $L_1(t) = L(t)\log t + \int_0^t L(x)x^{-1}\,\mathrm{d}x$.*

Proof. As before, we can choose $r(t)$ to be asymptotically nonincreasing. For any $\delta \in (0, \frac{1}{2})$, put $\Lambda(t) = (\int_0^{t\delta} + \int_{t\delta}^{t(1-\delta)} + \int_{t(1-\delta)}^{t})r(t-x)Q(x)\,\mathrm{d}x =: \Lambda_1(t)+\Lambda_2(t)+\Lambda_3(t)$. Then

$$r(t)\int_0^{t\delta} Q(x)\,\mathrm{d}x \le \Lambda_1(t) = \int_0^{t\delta} r(t-x)Q(x)\,\mathrm{d}x \le r(t(1-\delta))\int_0^{t\delta} Q(x)\,\mathrm{d}x.$$

Hence,

$$1 \le \liminf_{t\to\infty}\frac{\Lambda_1(t)}{r(t)\int_0^t Q(x)\,\mathrm{d}x} \le \limsup_{t\to\infty}\frac{\Lambda_1(t)}{r(t)\int_0^t Q(x)\,\mathrm{d}x} \le \frac{1}{1-\delta},$$

where, using (14), $\int_0^t Q(x)\,\mathrm{d}x \sim \alpha\log t$ as $t \to \infty$. Similarly, we obtain

$$1 \le \liminf_{t\to\infty}\frac{\Lambda_3(t)}{Q(t)\int_0^t r(x)\,\mathrm{d}x} \le \limsup_{t\to\infty}\frac{\Lambda_3(t)}{Q(t)\int_0^t r(x)\,\mathrm{d}x} \le \frac{1}{1-\delta}.$$

Now $R(t) = \int_0^t r(x)\,\mathrm{d}x$ is SV and $tr(t) = o(R(t))$ as $t \to \infty$ (see [5, Theorem 1.5.9a]). Finally,

$$\Lambda_2(t) = \int_{t\delta}^{t(1-\delta)} r(t-x)Q(x)\,\mathrm{d}x \le r(t\delta)Q(t\delta)(t(1-2\delta)) = O(r(t)),$$

because, by using (14), $Q(t\delta)t(1-2\delta) \to \alpha(1-2\delta)/\delta$ as $t \to \infty$. Since δ is an arbitrary strictly positive constant, it follows that, as $t \to \infty$,

$$\begin{aligned}\Lambda(t) = \Lambda_1(t) + \Lambda_2(t) + \Lambda_3(t) &\sim r(t)\int_0^t Q(x)\,\mathrm{d}x + Q(t)\int_0^t r(x)\,\mathrm{d}x \\ &\sim \frac{\alpha}{t}\bigg[L(t)\log t + \int_0^t L(x)x^{-1}\,\mathrm{d}x\bigg] \\ &\to 0.\end{aligned}$$

Therefore, $1 - \mathrm{e}^{-\Lambda(t)} \sim \Lambda(t)$, completing the proof. □

Remark 3. The next theorem does not require $r(t)$ to be RV.

Theorem 5. *If $R = \int_0^\infty r(x)\,\mathrm{d}x < \infty$ and $r(t) = o((t\log t)^{-1})$ as $t \to \infty$, $W(t) \sim \alpha R/t$.*

Proof. Assume that $\delta \in (0,1)$, and let $\Lambda(t) = \int_0^{t\delta} + \int_{t\delta}^t =: \Lambda_1(t) + \Lambda_2(t)$. Then, since $Q(t)R(\delta t) \le \Lambda_1(t) = \int_0^{t\delta} Q(t-x)\,\mathrm{d}R(x) \le Q(t(1-\delta))R(t\delta)$, it follows that

$$1 \le \liminf_{t\to\infty}\frac{\Lambda_1(t)}{Q(t)R(t\delta)} \le \limsup_{t\to\infty}\frac{\Lambda_1(t)}{Q(t)R(t\delta)} \le \frac{1}{1-\delta}.$$

Therefore, $\Lambda_1(t) \sim RQ(t) = \alpha R/t$ as $t \to \infty$ because δ can be set small enough. Now, for this δ and large enough t, we obtain, by the mean value theorem,

$$\Lambda_2(t) = \int_{t\delta}^t Q(t-x)r(x)\,\mathrm{d}x = r(t^*)\int_0^{t(1-\delta)} Q(x)\,\mathrm{d}x = r(t^*)O(\log t) = o\bigg(\frac{1}{t}\bigg),$$

where $t\delta \le t^* \le t$. This completes the proof because we now have $\Lambda(t) \sim \Lambda_1(t)$ and, thus, $W(t) = 1 - \mathrm{e}^{-\Lambda(t)} \sim \alpha R/t$. □

5. Limit distributions

Throughout this section, let $D(\boldsymbol{x}) = \mathbb{P}\{\xi_1 \le x_1, \dots, \xi_d \le x_d\}$, $\boldsymbol{x} = (x_1, \dots, x_d)$, denote the degenerate distribution of any random vector $\boldsymbol{\xi} = (\xi_1, \dots, \xi_d)$ such that $\xi_1 = \cdots = \xi_d$ a.s. Put $\bar{\lambda} = \sum_{l=1}^d \lambda_l$ and $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_d)$.

Theorem 6. *For $k = 1, \dots, d$, let $\eta_k(t) = Y_k(t)/M_k(t)$.*

(i) *If $\theta > 0$, or $\theta = 0$ and $L(t) \to \infty$ as $t \to \infty$, then $\eta_k(t) \to 1$ in probability.*

(ii) *If $\theta > 1$, or $\theta = 1$ and $\int_0^\infty [xL(x)]^{-1}\,\mathrm{d}x < \infty$, then $\eta_k(t) \to 1$ a.s.*

Proof. Note that $\mathbb{E}[\eta_k(t)] = 1$. Using Chebyshev's inequality, for every $\varepsilon > 0$,

$$\mathbb{P}\{|\eta_k(t) - 1| > \varepsilon\} \le \varepsilon^{-2}\operatorname{var}(\eta_k(t)).$$

Equations (10) and (11) imply that $\operatorname{var}(\eta_k(t)) = V_k(t)/M_k^2(t) \sim b(\theta+1)/[C(\theta+2)L(t)t^\theta]$. Hence, $\operatorname{var}(\eta_k(t)) \to 0$ under conditions (i), proving the first assertion. Assertion (ii) follows from $\int_0^\infty \operatorname{var}(\eta_k(t))\,\mathrm{d}t < \infty$ and [7, Theorem 21.1]. □

Theorem 7. *Assume that $\theta > 0$, or that $\theta = 0$ and $L(t) \to \infty$ as $t \to \infty$. Let $\boldsymbol{X}(t) = (X_1(t), \dots, X_d(t))$, where $X_k(t) = [Y_k(t) - M_k(t)]/\sqrt{V_k(t)}$ for every $k = 1, \dots, d$. Then $\lim_{t\to\infty}\mathbb{P}\{\boldsymbol{X}(t) \le \boldsymbol{x}\} = D(\boldsymbol{x})$ and ξ_1 is standard normal.*

Proof. Define the d-variate characteristic function $\varphi(t;\boldsymbol{\lambda}) = \mathbb{E}[\mathrm{e}^{i\boldsymbol{\lambda}\boldsymbol{X}^{\top}(t)}]$. Then (2) implies that

$$\log \varphi(t;\boldsymbol{\lambda}) = -i\sum_{k=1}^{d}\frac{M_k(t)}{\sqrt{V_k(t)}}\lambda_k + \int_0^t r(t-x)[1-g(\boldsymbol{F}(x;\boldsymbol{\lambda}_t))]\,\mathrm{d}x,$$

where $\bar{\boldsymbol{\lambda}}_t = (\mathrm{e}^{i\lambda_1/\sqrt{V_1(t)}}\ldots,\mathrm{e}^{i\lambda_d/\sqrt{V_d(t)}})$. Define $Q(x;\boldsymbol{s}) = 1-g(\boldsymbol{F}(x;\boldsymbol{s}))$. Then, using second-order Taylor expansions of $1-g(\boldsymbol{s})$ and $1-F_k(x;\boldsymbol{s})$ around $\boldsymbol{s}=\mathbf{1}$, we deduce that

$$\begin{aligned}Q(x;\boldsymbol{s}) \sim{}& \sum_{j,k=1}^{d} m_k A_{kj}(x)(1-s_j) + \frac{1}{2}\sum_{j,k,l=1}^{d} m_k B_{lj}^{k}(x)(1-s_l)(1-s_j)\\ &+\frac{1}{2}\sum_{j,k,l,m=1}^{d}\beta_{kj}A_{kl}(x)A_{jm}(x)(1-s_l)(1-s_m).\end{aligned}$$

Now using (11), we obtain $V_k(t) \sim bCv_k^2L(t)t^{\theta+2}/(\theta+1)(\theta+2) \to \infty$ as $t\to\infty$, and, moreover, $1-\mathrm{e}^{i\lambda_k/\sqrt{V_k(t)}} \sim -i\lambda_k/\sqrt{V_k(t)}$ for $k=1,\ldots,d$. Hence,

$$\begin{aligned}Q(x;\bar{\boldsymbol{\lambda}}_t) \sim{}& -i\sum_{j,k=1}^{d}\frac{m_kA_{kj}(x)}{\sqrt{V_j(t)}}\lambda_j + \frac{1}{2}\sum_{j,k,l=1}^{d}\frac{m_kB_{lj}^{k}(x)}{\sqrt{V_j(t)V_l(t)}}\lambda_j\lambda_l\\ &+\frac{1}{2}\sum_{j,k,l,m=1}^{d}\frac{\beta_{kj}A_{kl}(x)A_{jl}(x)}{\sqrt{V_l(t)V_m(t)}}\lambda_l\lambda_m.\end{aligned}$$

Changing the order of summation, integrating, and using (4) and (5), gives

$$\begin{aligned}\int_0^t r(t-x)Q(x;\bar{\boldsymbol{\lambda}}_t)\,\mathrm{d}x \sim{}& -i\sum_{j,k=1}^{d}\frac{\lambda_j}{\sqrt{V_j(t)}}m_k\int_0^t r(t-x)A_{kj}(x)\,\mathrm{d}x\\ &+\frac{1}{2}\sum_{j,k,l=1}^{d}\frac{\lambda_j\lambda_k}{\sqrt{V_j(t)V_k(t)}}m_l\int_0^t r(t-x)B_{jk}^{l}(x)\,\mathrm{d}x\\ &+\frac{1}{2}\sum_{j,k,l,m=1}^{d}\frac{\lambda_l\lambda_m}{\sqrt{V_l(t)V_m(t)}}\beta_{kj}\int_0^t r(t-x)A_{kl}(x)A_{jl}(x)\,\mathrm{d}x\\ \sim{}& -i\sum_{j=1}^{d}\frac{\lambda_jM_j(t)}{\sqrt{V_j(t)}} + \frac{1}{2}\sum_{j,k=1}^{d}\frac{\lambda_j\lambda_k}{\sqrt{V_j(t)V_k(t)}}[C_{jk}(t)-\delta_{jk}M_j(t)].\end{aligned}$$

From this last relation we deduce that

$$\log\varphi(t;\boldsymbol{\lambda}) \sim -\frac{1}{2}\sum_{j,k=1}^{d}\lambda_j\lambda_k\left[\rho_{jk}(t) - \frac{\delta_{jk}M_j(t)}{\sqrt{V_j(t)V_k(t)}}\right],$$

where, as $t\to\infty$, $\rho_{jk}(t)\to 1$ by Corollary 1 and $M_j(t)/V_j(t) \sim (\theta+2)/bv_jt \to 0$. Hence,

$$\lim_{t\to\infty}\varphi(t;\boldsymbol{\lambda}) = \exp\left[-\frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}\lambda_i\lambda_j\right] = \exp\left[-\frac{1}{2}\boldsymbol{\lambda}\boldsymbol{I}\boldsymbol{\lambda}^{\top}\right] =: \varphi(\boldsymbol{\lambda}),$$

where $\boldsymbol{I}$ denotes the $d\times d$ matrix with all entries equal to 1. Finally, noting that $\varphi(\boldsymbol{\lambda})$ is the characteristic function of the distribution $D(\boldsymbol{x})$ completes the proof. $\square$

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Corollary 2. *Theorems 2 and 7 imply the asymptotic normality property*

$$\frac{Y_k(t)}{C v_k(\theta+1)^{-1}L(t)t^{\theta+1}} \sim N\left(1, \frac{2(\theta+1)}{\alpha(\theta+2)L(t)t^{\theta}}\right) \quad \text{for } k=1,\dots,d.$$

Theorem 8. *Assume that $\theta=0$ and $r(t)=L(t)\to r^*\in(0,\infty)$ as $t\to\infty$. Then*

$$\lim_{t\to\infty}\mathbb{P}\left\{\frac{Y_1(t)}{v_1 r^* t}\le x_1,\dots,\frac{Y_d(t)}{v_d r^* t}\le x_d\right\} = D(\boldsymbol{x}),$$

where $\mathbb{P}\{\xi_1\le x\} = (1/\beta^{\alpha}\Gamma(\alpha))\int_0^x y^{\alpha-1}\mathrm{e}^{-y/\beta}\,\mathrm{d}y$ and $\beta := \frac{1}{2}b/r^*$.

Proof. Let

$$\psi(t;\boldsymbol{\lambda}) = \mathbb{E}\left[\exp\left(-\sum_{j=1}^{d}\frac{\lambda_j Y_j(t)}{v_j r^* t}\right)\right]$$

be the d-dimensional Laplace transform (LT), and let $\tilde{\boldsymbol{\lambda}}_t = (\mathrm{e}^{-\lambda_1/v_1 r^* t},\dots,\mathrm{e}^{-\lambda_d/v_d r^* t})$. Then $\psi(t;\boldsymbol{\lambda}) = \Phi(t;\tilde{\boldsymbol{\lambda}}_t)$ and $\log\psi(t;\boldsymbol{\lambda}) = -\int_0^t r(t-x)Q(x;\tilde{\boldsymbol{\lambda}}_t)\,\mathrm{d}x = -\Lambda(t;\tilde{\boldsymbol{\lambda}}_t)$, invoking (2). Sevastyanov [21, Chapter 6.3] shows that, as $t\to\infty$,

$$1-F_k(t;\boldsymbol{s}) \sim \frac{u_k\sum_{l=1}^d v_l(1-s_l)}{1+\frac{1}{2}bt\sum_{m=1}^d v_m(1-s_m)} \quad \text{for } k=1,\dots,d,$$

the approximation holding uniformly for $\boldsymbol{s}\in[\boldsymbol{0},\boldsymbol{1}]\backslash\{\boldsymbol{1}\}$. Now, use a first-order Taylor expansion of $1-g(\boldsymbol{s})$ around $\boldsymbol{s}=\boldsymbol{1}$ so that, for $Q(x,\boldsymbol{s}) = 1-g(\boldsymbol{F}(x;\boldsymbol{s}))$, again uniformly for $\boldsymbol{s}\in[\boldsymbol{0},\boldsymbol{1}]\setminus\{\boldsymbol{1}\}$, as $t\to\infty$,

$$Q(t;\boldsymbol{s}) \sim \frac{C\sum_{l=1}^d v_l(1-s_l)}{1+\frac{1}{2}bt\sum_{m=1}^d v_m(1-s_m)}. \tag{16}$$

Since $1-\mathrm{e}^{-\lambda_i/v_i r^* t}\sim\lambda_i/v_i r^* t$ as $t\to\infty$, we deduce from (16) that $Q(x;\tilde{\boldsymbol{\lambda}}_t)\sim C\bar{\lambda}/(r^*t+\frac{1}{2}bx\bar{\lambda})$ for $x\le t$ and $x\to\infty$. For every $\varepsilon>0$, we can choose $T\in(0;t]$ large enough such that, for $T\le x\le t$,

$$r(t)\lesseqgtr(1\pm\varepsilon)r^* \quad\text{and}\quad Q(x;\tilde{\boldsymbol{\lambda}}_t)\lesseqgtr(1\pm\varepsilon)\frac{C\bar{\lambda}}{r^*t+bx\bar{\lambda}/2}. \tag{17}$$

Let $\Lambda(t;\tilde{\boldsymbol{\lambda}}_t) = (\int_0^T+\int_T^{t-T}+\int_{t-T}^t)r(t-x)Q(x,\tilde{\boldsymbol{\lambda}}_t)\,\mathrm{d}x = \Lambda_1(t;\tilde{\boldsymbol{\lambda}}_t)+\Lambda_2(t;\tilde{\boldsymbol{\lambda}}_t)+\Lambda_3(t;\tilde{\boldsymbol{\lambda}}_t)$. We first deduce from (17) that, as $t\to\infty$,

$$\begin{aligned}\Lambda_2(t;\tilde{\boldsymbol{\lambda}}_t) &\gtreqless (1\pm\varepsilon)^2 r^*\int_T^t\frac{C\bar{\lambda}}{r^*t+bx\bar{\lambda}/2}\,\mathrm{d}x\\ &= (1\pm\varepsilon)^2\alpha r^*\log\left(\frac{2r^*t+bt\bar{\lambda}}{2r^*t+bT\bar{\lambda}}\right)\\ &\to (1\pm\varepsilon)^2\alpha r^*\log\left(1+\frac{b\bar{\lambda}}{2r^*}\right).\end{aligned}$$

Since $\tilde{\boldsymbol{\lambda}}_t\to\boldsymbol{1}$,

$$\Lambda_1(t;\tilde{\boldsymbol{\lambda}}_t)\le K_1\int_0^T Q(x;\tilde{\boldsymbol{\lambda}}_t)\,\mathrm{d}x\le K_1 T\,Q(0;\tilde{\boldsymbol{\lambda}}_t)\to 0.$$

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Note that $Q(t-T; \boldsymbol{s}) \leq Q(t-T)$ and, for $t \to \infty$, $\Lambda_3(t; \tilde{\boldsymbol{\lambda}}_t) \leq K_1 T Q(t-T) \to 0$. Hence, $\lim_{t\to\infty} \Lambda(t; \boldsymbol{\lambda}_t) = \alpha \log(1 + b\bar{\lambda}/2r^*)$. Setting $\beta = \frac{1}{2}b/r^*$ gives $\lim_{t\to\infty} \psi(t; \boldsymbol{\lambda}) = \lim_{t\to\infty} \mathrm{e}^{-\Lambda(t;\tilde{\lambda}_t)} = (1 + \beta\bar{\lambda})^{-\alpha}$, which is the LT of a degenerate d-variate gamma-distributed random variable. □

Theorem 9. *Let $\theta = 0$, $L(t) \to 0$, and $L(t)\log t \to \infty$, and recall $b^*(t)$ from (9). Then, as $t \to \infty$,*

$$\mathbb{P}\bigg\{\bigg[\frac{2Y_1(t)}{v_1 bt}\bigg]^{1/b^*(t)} \leq x_1, \dots, \bigg[\frac{2Y_d(t)}{v_d bt}\bigg]^{1/b^*(t)} \leq x_d\bigg\} \to D(\boldsymbol{x}),$$

where ξ_1 has a uniform distribution on $[0, 1]$.

Proof. Consider the LT $\psi(t; \boldsymbol{\lambda}) = \mathbb{E}[\exp(-2\sum_{j=1}^d \lambda_j Y_j(t)/btv_j x_j^{b^*(t)})]$ and set $\tilde{\boldsymbol{\lambda}}_t(\boldsymbol{x}) = (\exp[-2\lambda_1/btv_1 x_1^{b^*(t)}], \dots, \exp[-2\lambda_d/btv_d x_d^{b^*(t)}])$, where $x_i \in (0, 1)$ for every $i = 1, \dots, d$. Then $\log\psi(t; \boldsymbol{\lambda}) = \log\Phi(t; \tilde{\boldsymbol{\lambda}}_t(\boldsymbol{x})) = -\Lambda(t; \tilde{\boldsymbol{\lambda}}_t(\boldsymbol{x}))$, and, thus, as $t \to \infty$,

$$1 - \exp\bigg[-\frac{2\lambda_i}{btv_i x_i^{b^*(t)}}\bigg] \sim \frac{2\lambda_i}{btv_i x_i^{b^*(t)}}$$

since then $tx_i^{1/r(t)} \to \infty$. We deduce from (16), as $y \to \infty$ with $y \leq t$, that

$$Q(y; \tilde{\boldsymbol{\lambda}}_t(\boldsymbol{x})) \sim \frac{2C\sum_{i=1}^d \lambda_i x_i^{-b^*(t)}}{b(t + y\sum_{i=1}^d \lambda_i x_i^{-b^*(t)})}. \tag{18}$$

Now $r(t)$ is SV, so there exists SV $\delta(\cdot)$ for which $\delta(t) \to \infty$ and such that $r(t/\delta^*(t))/r(t) \to 1$ as $t \to \infty$ for every function $\delta^*(t)$ satisfying $1 \leq \delta^*(t) \leq \delta(t)$. Then

$$\Lambda(t; \tilde{\boldsymbol{\lambda}}_t(\boldsymbol{x})) = \bigg[\int_0^{t/\delta(t)} + \int_{t/\delta(t)}^{t(1-x_{i_0}^{b^*(t)})} + \int_{t(1-x_{i_0}^{b^*(t)})}^{t}\bigg](\cdots) =: \Lambda_1(t) + \Lambda_2(t) + \Lambda_3(t),$$

where $x_{i_0} = \min_{1\leq i\leq d} x_i \in (0, 1)$. Then (see [6, p. 114])

$$\Lambda_2(t) = \bar{r}(t)\int_{t/\delta(t)}^{t(1-x_{i_0}^{b^*(t)})} Q(t-y; \tilde{\boldsymbol{\lambda}}_t(\boldsymbol{x}))\,\mathrm{d}y = \bar{r}(t)\int_{tx_{i_0}^{b^*(t)}}^{t(1-1/\delta(t))} Q(y; \tilde{\boldsymbol{\lambda}}_t(\boldsymbol{x}))\,\mathrm{d}y,$$

where

$$\min_{t/\delta(t)\leq u\leq t(1-x^{b^*(t)})} r(u) \leq \bar{r}(t) \leq \max_{t/\delta(t)\leq u\leq t(1-x^{b^*(t)})} r(u).$$

The properties of $\delta(t)$ imply that $\bar{r}(t) \sim r(t)$, so, for large enough t and using (18),

$$\begin{aligned}\Lambda_2(t) &\sim \bar{r}(t)\int_{tx_{i_0}^{b^*(t)}}^{t(1-1/\delta(t))} \frac{\alpha\sum_{i=1}^d \lambda_i x_i^{-b^*(t)}}{t + y\sum_{i=1}^d \lambda_i x_i^{-b^*(t)}}\,\mathrm{d}y \\ &= \alpha\bar{r}(t)\log\bigg(\frac{1 + (\sum_{i=1}^d \lambda_i x_i^{-b^*(t)})(1 - 1/\delta(t))}{1 + (\sum_{i=1}^d \lambda_i x_i^{-b^*(t)})x_{i_0}^{b^*(t)}}\bigg) \\ &= \alpha\bar{r}(t)\log\bigg(x_{i_0}^{-b^*(t)}\frac{x_{i_0}^{b^*(t)} + (\sum_{i=1}^d \lambda_i (x_{i_0}/x_i)^{b^*(t)})(1 - 1/\delta(t))}{1 + \sum_{i=1}^d \lambda_i (x_{i_0}/x_i)^{b^*(t)}}\bigg).\end{aligned}$$

Since $x_{i_0} \in (0, 1)$, $x_{i_0}/x_i \in (0, 1)$, and $b^*(t) \to \infty$, the fraction in parentheses converges to $\lambda_{i_0}/(1 + \lambda_{i_0})$. Hence, $\lim_{t\to\infty} \Lambda_2(t) = -\log x_{i_0}$. On the other hand,

$$\begin{aligned}\Lambda_1(t) &= \int_0^{t/\delta(t)} r(y)Q(t-y; \tilde{\boldsymbol{\lambda}}_t(\boldsymbol{x}))\,\mathrm{d}y \\ &\le \int_0^{t/\delta(t)} r(y)Q(t-y)\,\mathrm{d}y \\ &\le Q\biggl(t\biggl[1-\frac{1}{\delta(t)}\biggr]\biggr)\int_0^t r(y)\,\mathrm{d}y \\ &\sim \frac{1}{b^*(t)} \\ &\to 0 \quad \text{as } t\to\infty.\end{aligned}$$

The mean value theorem implies that

$$\Lambda_3(t) = \int_{t(1-x_{i_0}^{b^*(t)})}^{t} r(y)Q(t-y; \tilde{\boldsymbol{\lambda}}_t(\boldsymbol{x}))\,\mathrm{d}y = r(t_2^*)\int_0^{tx_{i_0}^{b^*(t)}} Q(y; \tilde{\boldsymbol{\lambda}}_t(\boldsymbol{x}))\,\mathrm{d}y,$$

where $t(1 - x_{i_0}^{b^*(t)}) \le t_2^* \le t$. Furthermore, from (6) we have

$$Q\bigl(y; \tilde{\boldsymbol{\lambda}}_t(\boldsymbol{x})\bigr) \le \frac{2x_{i_0}^{b^*(t)}}{bt}\sum_{j=1}^{d}\frac{\lambda_j}{v_j}\sum_{k=1}^{d} m_k A_{kj}(y) \le \frac{C^* x_{i_0}^{b^*(t)}}{t} \quad \text{for some } C^* > 0.$$

Hence, $\Lambda_3(t) \le r(t_2^*)[C^* x_{i_0}^{b^*(t)}/t]t x_{i_0}^{b^*(t)} = r(t_2^*)C^* x_{i_0}^{2b^*(t)} \to 0$ as $t \to \infty$. We proved that $\lim_{t\to\infty} \Lambda(t; \tilde{\boldsymbol{\lambda}}_t(\boldsymbol{x})) = -\log x_{i_0}$ and $\lim_{t\to\infty} \psi(t; \boldsymbol{\lambda}) = x_{i_0}$. The limit is the Laplace transform of a d-variate degenerate distribution with mass $\min_{1\le i\le d} x_i$ at $\mathbf{0}$ and mass $1 - \min_{1\le i\le d} x_i$ at infinity. Thus, for every vector $(y_1, \ldots, y_d)$ with nonnegative entries,

$$\lim_{t\to\infty} \mathbb{P}\biggl\{\frac{2Y_1(t)}{btv_1x_1^{b^*(t)}} \le y_1, \ldots, \frac{2Y_d(t)}{btv_dx_d^{b^*(t)}} \le y_d\biggr\} = x_{i_0}.$$

Setting $y_1 = \cdots = y_d = 1$ and using some simple algebra completes the proof. □

Theorem 10. *If* $\theta = 0$ *and* $L(t)\alpha\log t \to \varkappa \in (0, \infty)$ *as* $t \to \infty$, *then*

(i) *(unconditional limit distribution)*

$$\mathbb{P}\biggl\{\frac{\log(Y_1(t)/v_1)}{\log t} \le x_1, \ldots, \frac{\log(Y_d(t)/v_d)}{\log t} \le x_d\biggr\} \to D(\boldsymbol{x}),$$

where $\mathbb{P}\{\xi_1 \le y\} = \mathrm{e}^{-\varkappa(1-y)}$ *for* $0 \le y \le 1$; *and*

(ii) *(conditional limit distribution)*

$$\mathbb{P}\biggl\{1-\frac{\log(Y_1(t)/v_1)}{\log t} \le x_1, \ldots, 1-\frac{\log(Y_d(t)/v_d)}{\log t} \le x_d \;\biggm|\; \boldsymbol{Y}(t) \ne \mathbf{0}\biggr\} \to D(\boldsymbol{x}), \quad (19)$$

where $\mathbb{P}\{\xi_1 \le y\} = (1 - \mathrm{e}^{-\varkappa y})/(1 - \mathrm{e}^{-\varkappa})$ *for* $0 \le y \le 1$.

Proof. (i) Define the d-variate LT $\psi(t;\boldsymbol{\lambda})=\mathbb{E}[\exp(-\sum_{i=1}^{d}\lambda_i Y_i(t)/v_i t^{x_i})]$ and introduce $\tilde{\boldsymbol{\lambda}}_t(\boldsymbol{x})=(\mathrm{e}^{-\lambda_1/v_1 t^{x_1}},\dots,\mathrm{e}^{-\lambda_d/v_d t^{x_d}})$, where $x_i\in(0,1)$ for $i=1,\dots,d$. Then $\log\psi(t;\boldsymbol{\lambda})=\log\Phi(t;\tilde{\boldsymbol{\lambda}}_t(\boldsymbol{x}))=-\Lambda(t;\tilde{\boldsymbol{\lambda}}_t(\boldsymbol{x}))$. Let $\delta\in(0,1)$, and define the decomposition $\Lambda(t;\tilde{\boldsymbol{\lambda}}_t(\boldsymbol{x}))=(\int_0^{t\delta}+\int_{t\delta}^{t})r(t-x)Q(x,\tilde{\boldsymbol{\lambda}}_t(\boldsymbol{x}))\,\mathrm{d}x=:\Lambda_1(t;\boldsymbol{x})+\Lambda_2(t;\boldsymbol{x})$. For $\varepsilon>0$ and large enough t, $(1-\varepsilon)\varkappa/(\alpha\log t)\le r(t)\le(1+\varepsilon)\varkappa/(\alpha\log t)$. Then

$$\frac{(1-\varepsilon)\varkappa}{\alpha\log t}\int_0^{t\delta}Q(u;\tilde{\boldsymbol{\lambda}}_t(\boldsymbol{x}))\,\mathrm{d}u\le\Lambda_1(t;\boldsymbol{x})\le\frac{(1+\varepsilon)\varkappa}{\alpha\log t}\int_0^{t\delta}Q(u;\tilde{\boldsymbol{\lambda}}_t(\boldsymbol{x}))\,\mathrm{d}u.$$

Since $s_i=1-\mathrm{e}^{-\lambda_i/v_i t^{x_i}}\sim\lambda_i/v_i t^{x_i}$, it follows from (16), as $y\to\infty$ and $y\le t$, that

$$Q(y;\tilde{\boldsymbol{\lambda}}_t(\boldsymbol{x}))\sim\frac{C\sum_{i=1}^{d}\lambda_i t^{-x_i}}{1+\frac{1}{2}by\sum_{i=1}^{d}\lambda_i t^{-x_i}}.$$

Define $J(t;\boldsymbol{x})=\int_0^{t\delta}Q(y;\tilde{\boldsymbol{\lambda}}_t(\boldsymbol{x}))\,\mathrm{d}y=\int_0^{T}+\int_T^{t\delta}=:J_1(t;\boldsymbol{x})+J_2(t;\boldsymbol{x})$. If $T\to\infty$ in such a way that $T\le t\delta$ and $T=o(\log t)$, then

$$\begin{aligned}J_2(t;\boldsymbol{x})&=\int_T^{t\delta}Q(y;\tilde{\boldsymbol{\lambda}}_t(\boldsymbol{x}))\,\mathrm{d}y\\&\sim\int_T^{t\delta}\frac{C\sum_{i=1}^{d}\lambda_i t^{-x_i}}{1+\frac{1}{2}by\sum_{i=1}^{d}\lambda_i t^{-x_i}}\,\mathrm{d}y\\&=\alpha\log\biggl(\frac{1+\frac{1}{2}bt\delta\sum_{i=1}^{d}\lambda_i t^{-x_i}}{1+\frac{1}{2}bT\sum_{i=1}^{d}\lambda_i t^{-x_i}}\biggr)\\&\sim\alpha(1-x_{i_0})\log t,\end{aligned}\tag{20}$$

where $x_{i_0}=\min_{1\le i\le d}x_i$. Moreover, $J_1(t;\boldsymbol{x})=\int_0^{T}Q(y;\tilde{\boldsymbol{\lambda}}_t(\boldsymbol{x}))\,\mathrm{d}y\le T=o(\log t)$. Hence, $\lim_{t\to\infty}\Lambda_1(t;\boldsymbol{x})=\varkappa(1-x_{i_0})$. Using (14) and the fact that $r(t)$ is SV yields $\Lambda_2(t;\boldsymbol{x})=\int_{t\delta}^{t}r(t-y)Q(y;\tilde{\boldsymbol{\lambda}}_t(\boldsymbol{x}))\,\mathrm{d}y\le Q(t\delta)\int_0^{t}r(y)\,\mathrm{d}y=O(r(t))$. Therefore, $\lim_{t\to\infty}\Lambda(t;\tilde{\boldsymbol{\lambda}}_t(\boldsymbol{x}))=\varkappa(1-x_{i_0})$ and $\lim_{t\to\infty}\psi(t;\boldsymbol{\lambda})=\mathrm{e}^{-\varkappa(1-x_{i_0})}$. This limit is the LT of a degenerate d-variate distribution with mass $\mathrm{e}^{-\varkappa(1-x_{i_0})}$ at $\mathbf{0}$ and mass $1-\mathrm{e}^{-\varkappa(1-x_{i_0})}$ at infinity. We conclude from this result that

$$\lim_{t\to\infty}\mathbb{P}\biggl\{\frac{Y_1(t)}{v_1t^{x_1}}\le y_1,\dots,\frac{Y_d(t)}{v_dt^{x_d}}\le y_d\biggr\}=\mathrm{e}^{-\varkappa(1-x_{i_0})}$$

for any vector $(y_1,\dots,y_d)$ with nonnegative entries. This proves case (i).

(ii) Let $\Psi(t;\boldsymbol{\lambda})=\mathbb{E}[\exp(-\sum_{j=1}^{d}\lambda_jY_j(t)v_j^{-1}t^{-x_j}]\mid\boldsymbol{Y}(t)\ne\mathbf{0}]$ be the conditional LT. Then

$$\Psi(t;\boldsymbol{\lambda})=1-\frac{1-\Phi(t;\tilde{\boldsymbol{\lambda}}_t(\boldsymbol{x}))}{1-\Phi(t;\mathbf{0})}=1-\frac{1-\psi(t;\boldsymbol{\lambda})}{W(t)}$$

by invoking (2). We deduce that $\lim_{t\to\infty}\Psi(t;\boldsymbol{\lambda})=(\mathrm{e}^{-\varkappa(1-x_{i_0})}-\mathrm{e}^{-\varkappa})/(1-\mathrm{e}^{-\varkappa})$ by using case (i) and Theorem 3(ii). Now reasoning similar to that used in (i) completes the proof. □

Theorem 11. *If* $\theta\in(-1,0)$, *or* $\theta=0$ *and* $L(t)=o(1/\log t)$, *then, as* $t\to\infty$,

$$\mathbb{P}\biggl\{\frac{\log(Y_1(t)/v_1)}{\log t}\le x_1,\dots,\frac{\log(Y_d(t)/v_d)}{\log t}\le x_d\biggm|\boldsymbol{Y}(t)\ne\mathbf{0}\biggr\}\to D(\boldsymbol{x}),$$

where ξ_1 *for* $D(\cdot)$ *is uniformly distributed on* $(0,1)$.

Proof. Consider the conditional LT given in case (ii) of Theorem 10 as well as the notation $\boldsymbol{\lambda}$, $\boldsymbol{x}$, and $\tilde{\boldsymbol{\lambda}}_t(\boldsymbol{x})$ used in its proof. Assume first that $\theta < 0$ such that $r(t)$ can be chosen asymptotically nonincreasing. Then

$$r(t)\int_0^{t\delta} Q(y;\tilde{\boldsymbol{\lambda}}_t(\boldsymbol{x}))\,\mathrm{d}y \le \Lambda_1(t;\tilde{\boldsymbol{\lambda}}_t(\boldsymbol{x})) \le r(t(1-\delta))\int_0^{t\delta} Q(y;\tilde{\boldsymbol{\lambda}}_t(\boldsymbol{x}))\,\mathrm{d}y.$$

Now from (20), $J(t,\boldsymbol{x}) = \int_0^{t\delta} Q(y;\tilde{\boldsymbol{\lambda}}_t(\boldsymbol{x}))\,\mathrm{d}y \sim (1-x_{i_0})\alpha\log t$. Therefore,

$$1 \le \liminf_{t\to\infty}\frac{\Lambda_1(t;\tilde{\boldsymbol{\lambda}}_t(\boldsymbol{x}))}{(1-x_{i_0})r(t)\alpha\log t} \le \limsup_{t\to\infty}\frac{\Lambda_1(t;\tilde{\boldsymbol{\lambda}}_t(\boldsymbol{x}))}{(1-x_{i_0})r(t)\alpha\log t} \le \frac{1}{(1-\delta)^{-\theta}}.$$

We also have $\Lambda_2(t;\tilde{\boldsymbol{\lambda}}_t(\boldsymbol{x})) \le Q(t\delta)\int_0^t r(x)\,\mathrm{d}x = o(r(t)\log t)$, and deduce that $\Lambda(t;\tilde{\boldsymbol{\lambda}}_t(\boldsymbol{x})) \sim (1-x_{i_0})r(t)\alpha\log t$. The same result holds true when $\theta = 0$ because the uniform convergence of $r(\cdot)$ which is then SV ensures that $r(t-y)\sim r(t)$ for every $t-y\in[t(1-\delta),t]$. Hence, $\Lambda_1(t;\tilde{\boldsymbol{\lambda}}_t(\boldsymbol{x})) \sim r(t)J(t;\boldsymbol{x})$ as $t\to\infty$, and the rest is as above. Thus, $1-\mathrm{e}^{-\Lambda(t;\tilde{\boldsymbol{\lambda}}_t(\boldsymbol{x}))} \sim \Lambda(t;\tilde{\boldsymbol{\lambda}}_t(\boldsymbol{x})) \sim (1-x_{i_0})r(t)\alpha\log t$, which with Theorem 3(iii) gives

$$\lim_{t\to\infty}\mathbb{P}\bigg\{\frac{Y_1(t)}{v_1t^{x_1}}\le y_1,\dots,\frac{Y_d(t)}{v_dt^{x_d}}\le y_d \,\bigg|\, \boldsymbol{Y}(t)\ne\boldsymbol{0}\bigg\} = x_{i_0}$$

for any $(y_1,\dots,y_d)$ with nonnegative entries. This completes the proof. □

Theorem 12. *Let* $\theta=-1$, $\tilde{L}(t)=\int_0^t(L(x)/x)\,\mathrm{d}x$, *and let* $\hat{L}(t)=L(t)\log t$. *If as* $t\to\infty$, $\tilde{L}(t)/\hat{L}(t)\to q\in(0,\infty)$ *then also*

$$\text{(i)}\qquad \mathbb{P}\bigg\{\frac{\log[Y_1(t)/v_1]}{\log t}\le x_1,\dots,\frac{\log[Y_d(t)/v_d]}{\log t}\le x_d \,\bigg|\, \boldsymbol{Y}(t)\ne\boldsymbol{0}\bigg\}\to D(\boldsymbol{x}),$$

where

$$\mathbb{P}\{\xi_1\le y\} = \frac{y}{1+q}\mathbf{1}_{\{0\le y\le 1\}} + \frac{1}{1+q}\mathbf{1}_{\{y\ge 1\}},$$

$$\text{(ii)}\qquad \mathbb{P}\bigg\{\frac{2Y_1(t)}{v_1bt}\le x_1,\dots,\frac{2Y_d(t)}{v_dbt}\le x_d \,\bigg|\, \boldsymbol{Y}(t)\ne\boldsymbol{0}\bigg\}\to D(\boldsymbol{x}),$$

where

$$\mathbb{P}\{\xi_1\le y\} = \frac{1}{1+q}+\frac{q}{1+q}(1-\mathrm{e}^{-y}),\qquad y\ge 0.$$

Proof. (i) Consider again the conditional LT used in case (ii) of Theorem 10, and recall the notation $\boldsymbol{x}$ and $\tilde{\boldsymbol{\lambda}}_t(\boldsymbol{x})$ defined in its proof. Let $\delta\in(0,\frac12)$, and write $\Lambda(t;\tilde{\boldsymbol{\lambda}}_t(\boldsymbol{x})) = (\int_0^{t\delta}+\int_{t\delta}^{t(1-\delta)}+\int_{t(1-\delta)}^t)(\cdots) =: \Lambda_1(t;\boldsymbol{x})+\Lambda_2(t;\boldsymbol{x})+\Lambda_3(t;\boldsymbol{x})$. We can choose $r(t)$ to be asymptotically nonincreasing. It follows from the monotonicity of $r(t)$ and $Q(t;\boldsymbol{s})$ (see also (14)) that, as $t\to\infty$,

$$\begin{aligned}\Lambda_2(t;\boldsymbol{x}) &= \int_{t\delta}^{t(1-\delta)} r(t-y)Q(y;\tilde{\boldsymbol{\lambda}}_t(\boldsymbol{x}))\,\mathrm{d}y\\ &\le \int_{t\delta}^{t(1-\delta)} r(t-y)Q(y)\,\mathrm{d}y\\ &\le r(t\delta)\,Q(t\delta)\,t(1-2\delta)\\ &= o(W(t)).\end{aligned}$$

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Since $1-\mathrm{e}^{-\lambda_i t^{-x_i}} \sim \lambda_i t^{-x_i} \to 0$, $\sum_{i=1}^d \lambda_i t^{-x_i} \to 0$ also. Then, using (16) and the fact that δ is arbitrary, it follows that, for $t(1-\delta) \le y \le t$ and $t \to \infty$,

$$Q(y;\tilde{\boldsymbol{\lambda}}_t(\boldsymbol{x})) \sim \frac{C\sum_{i=1}^d \lambda_i t^{-x_i}}{1+\frac{1}{2}by\sum_{i=1}^d \lambda_i t^{-x_i}} \sim \frac{\alpha}{t}.$$

Therefore, $\Lambda_3(t;\boldsymbol{x}) = \int_{t(1-\delta)}^t r(t-y)Q(y;\tilde{\boldsymbol{\lambda}}_t(\boldsymbol{x}))\,\mathrm{d}y \sim (\alpha/t)\int_0^{t\delta} r(y)\,\mathrm{d}y$. Recalling that $\int_0^t r(y)\,\mathrm{d}y$ is SV, we prove that $\Lambda_3(t;\boldsymbol{x}) \sim (\alpha/t)\int_0^t r(y)\,\mathrm{d}y$ as $t\to\infty$.

For every $\varepsilon > 0$, there exists $T > 0$ such that, for $T < y < t$,

$$Q(y;\tilde{\boldsymbol{\lambda}}_t(\boldsymbol{x})) \lesseqqgtr (1\pm\varepsilon)\frac{C\sum_{i=1}^d \lambda_i t^{-x_i}}{1+\frac{1}{2}by\sum_{i=1}^d \lambda_i t^{-x_i}}.$$

Then $\Lambda_1(t;\boldsymbol{x}) = \int_0^{t\delta} r(t-y)Q(y;\tilde{\boldsymbol{\lambda}}_t(\boldsymbol{x}))\,\mathrm{d}y = (\int_0^T + \int_T^{t\delta})(\cdots) =: \Lambda_{11}(t;\boldsymbol{x}) + \Lambda_{12}(t;\boldsymbol{x})$. Here, $\Lambda_{11}(t;\boldsymbol{x}) \le r(t-T)T = O(r(t))$, while

$$\begin{aligned}\Lambda_{12}(t;\boldsymbol{x}) &\lesseqqgtr (1\pm\varepsilon)\int_T^{t\delta} r(t-y)\frac{C\sum_{i=1}^d \lambda_i t^{-x_i}}{1+\frac{1}{2}by\sum_{i=1}^d \lambda_i t^{-x_i}}\,\mathrm{d}y\\ &\sim (1\pm\varepsilon)r(t)\int_T^{t\delta} \frac{C\sum_{i=1}^d \lambda_i t^{-x_i}}{1+\frac{1}{2}by\sum_{i=1}^d \lambda_i t^{-x_i}}\,\mathrm{d}y\\ &\sim (1\pm\varepsilon)r(t)\alpha\log\biggl(\frac{2+bt\delta\sum_{i=1}^d \lambda_i t^{-x_i}}{2+bT\sum_{i=1}^d \lambda_i t^{-x_i}}\biggr)\\ &\sim (1\pm\varepsilon)r(t)(1-x_{i_0})\alpha\log t.\end{aligned}$$

Therefore, $\Lambda_1(t;\boldsymbol{x}) \sim r(t)(1-x_{i_0})\alpha\log t$, which proves that, as $t\to\infty$, $\Lambda(t;\boldsymbol{x}) \sim r(t)(1-x_{i_0})\alpha\log t + (\alpha/t)\int_0^t r(y)\,\mathrm{d}y \to 0$. Hence,

$$1-\Phi(t;\tilde{\boldsymbol{\lambda}}_t(\boldsymbol{x})) \sim r(t)(1-x_{i_0})\alpha\log t + \frac{\alpha}{t}\int_0^t r(y)\,\mathrm{d}y \sim \biggl(\frac{\alpha}{t}\biggr)\{(1-x_{i_0})\hat{L}(t) + \tilde{L}(t)\}.$$

Then $\lim_{t\to\infty}(1-\Phi(t;\tilde{\boldsymbol{\lambda}}_t(\boldsymbol{x})))/W(t) = (1-x_{i_0}+q)/(1+q)$, and we finally obtain

$$\lim_{t\to\infty}\mathbb{E}\biggl[\exp\biggl(-\sum_{j=1}^d \lambda_j Y_j(t)v_j^{-1}t^{-x_j}\biggr)\,\bigg|\,\boldsymbol{Y}(t)\neq\boldsymbol{0}\biggr] = 1-\frac{1-x_{i_0}+q}{1+q} = \frac{1}{1+q}x_{i_0},$$

which proves the first limit distribution, much as in Theorem 11.

(ii) We use (16) with $\boldsymbol{s} = \tilde{\boldsymbol{\lambda}}_t = (\mathrm{e}^{-2\lambda_1/v_1bt},\dots,\mathrm{e}^{-2\lambda_d/v_dbt})$, where, for every $i=1,\dots,d$, $1-\mathrm{e}^{-2\lambda_i/v_ibt} \sim 2\lambda_i/v_ibt$. Let $\delta\in(0,1)$, and consider the decomposition

$$\Lambda(t;\tilde{\boldsymbol{\lambda}}_t) = -\log\Phi(t;\tilde{\boldsymbol{\lambda}}_t) = \int_0^{t\delta} + \int_{t\delta}^t =: \Lambda_1(t)+\Lambda_2(t).$$

Then, as $t\to\infty$ with $t\delta\le y\le t$, (16) implies that $Q(y;\tilde{\boldsymbol{\lambda}}_t) \sim (\alpha/t)(\bar{\lambda}/(1+\bar{\lambda}))$. Therefore,

$$\Lambda_2(t) = \int_{t\delta}^t r(t-y)Q(y;\tilde{\boldsymbol{\lambda}}_t)\,\mathrm{d}y \sim \frac{\alpha}{t}\frac{\bar{\lambda}}{1+\bar{\lambda}}\int_0^{t(1-\delta)} r(y)\,\mathrm{d}y \sim \frac{\alpha}{t}\frac{\bar{\lambda}}{1+\bar{\lambda}}\tilde{L}(t).$$

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Hence, as $t \to \infty$,

$$\begin{aligned}\Lambda_1(t) &= \int_0^{t\delta} r(t-y) Q(y; \tilde{\boldsymbol{\lambda}}_t)\, \mathrm{d}y \\ &\le r(t(1-\delta))\left[T + \int_T^{t\delta} Q(y; \tilde{\boldsymbol{\lambda}}_t)\, \mathrm{d}y\right] \\ &\sim r(t(1-\delta))[T + \alpha \log(1+\delta\bar{\lambda})] \\ &= o(\Lambda_2(t))\end{aligned}$$

because $r(t)$ is asymptotically nonincreasing. Thus,

$$\Lambda(t; \tilde{\boldsymbol{\lambda}}_t) \sim \frac{q\bar{\lambda}\, W(t)}{(1+q)(\bar{\lambda}+1)} \quad \text{and} \quad 1 - \Phi(t; \tilde{\boldsymbol{\lambda}}_t) = 1 - \mathrm{e}^{-\Lambda(t; \tilde{\boldsymbol{\lambda}}_t)} \sim \frac{q\bar{\lambda}\, W(t)}{(1+q)(\bar{\lambda}+1)},$$

and we finally obtain

$$\lim_{t\to\infty} \Psi(t; \boldsymbol{\lambda}) = 1 - \lim_{t\to\infty} \frac{1 - \Phi(t; \tilde{\boldsymbol{\lambda}}_t)}{W(t)} = 1 - \frac{q\bar{\lambda}}{(1+q)(\bar{\lambda}+1)} = \frac{1}{1+q} + \frac{q}{1+q}\frac{1}{1+\bar{\lambda}},$$

which is the LT of a degenerate multivariate exponential distribution concentrated on the main diagonal with an atom at zero. The second assertion follows. □

Remark 4. The two singular limit distributions obtained in Theorem 12 allow us to classify the nonextinct sample paths into two distinct groups based on their growth rates: (i) with probability $1 - 1/(1+q)$, the growth is linear with an exponentially distributed slope, else (ii) with probability $1/(1+q)$, the growth is parabolic with a power that is uniformly distributed on $(0, 1)$. For example, if $L(t) = (\log t)^\alpha$, then $q = 1/(\alpha+1)$ if $\alpha > -1$, and $q = \infty$ if $\alpha \le -1$.

Remark 5. The next theorem does not require $r(t)$ to be RV.

Theorem 13. *If $R = \int_0^\infty r(x)\, \mathrm{d}x < \infty$ and $r(t) = o([t \log t]^{-1})$, then as $t \to \infty$*

$$\mathbb{P}\{Y_1(t) v_1^{-1} W(t) \le x_1, \dots, Y_d(t) v_d^{-1} W(t) \le x_d \mid \boldsymbol{Y}(t) \ne \mathbf{0}\} \to D(\boldsymbol{x}),$$

where $\mathbb{P}\{\xi_1 \le x\} = 1 - \mathrm{e}^{-x/RC}$, $x \ge 0$.

Proof. Define the conditional d-variate LT $\Psi(t; \boldsymbol{\lambda}) = 1 - [1 - \Phi(t; \tilde{\boldsymbol{\lambda}}_t)]/W(t)$, and put $\tilde{\boldsymbol{\lambda}}_t = (\mathrm{e}^{-\lambda_1 v_1^{-1} W(t)}, \dots, \mathrm{e}^{-\lambda_d v_d^{-1} W(t)})$. Let $\delta \in (0, 1)$, and define the function

$$\Lambda(t) = \int_0^t Q(t-y; \tilde{\boldsymbol{\lambda}}_t) r(y)\, \mathrm{d}y =: \left(\int_0^{t\delta} + \int_{t\delta}^t\right)(\cdots) =: \Lambda_1(t) + \Lambda_2(t).$$

The mean value theorem shows that $\Lambda_1(t) = Q(t_1^*; \tilde{\boldsymbol{\lambda}}_t) \int_0^{t\delta} r(x)\, \mathrm{d}x$ for $t(1-\delta) \le t_1^* \le t$. Using (16) with $\boldsymbol{s} = \tilde{\boldsymbol{\lambda}}_t$, we deduce that $Q(t_1^*; \tilde{\boldsymbol{\lambda}}_t) \sim W(t) C\bar{\lambda}/[1 + \frac{1}{2} b t_1^* W(t)\bar{\lambda}]$ and

$$\Lambda_1(t) \sim \frac{W(t) RC\bar{\lambda}}{1 + \frac{1}{2} b t_1^* W(t) \bar{\lambda}} \sim \frac{W(t) RC\bar{\lambda}}{1 + RC\bar{\lambda}} \quad \text{as } t \to \infty,$$

because $t_1^* W(t) \to \alpha R$. Again applying the mean value theorem, we obtain

$$\Lambda_2(t) = r(t_2^*) \int_0^{t(1-\delta)} Q(y; \tilde{\boldsymbol{\lambda}}_t)\, \mathrm{d}y, \qquad t\delta \le t_2^* \le t.$$

Then we deduce from (14) that $\Lambda_2(t) \le r(t_2^*) \int_0^t Q(y)\, \mathrm{d}y = r(t_2^*) O(\log t) = o(1/t)$, from which we find that $\Lambda_2(t) = o(\Lambda_1(t))$ and $\Lambda(t) \sim W(t) RC\bar{\lambda}/(1 + RC\bar{\lambda})$ as $t \to \infty$.

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This implies that $1 - \Phi(t; \tilde{\boldsymbol{\lambda}}_t) = 1 - \mathrm{e}^{-\Lambda(t)} \sim W(t)RC\bar{\lambda}/(1 + RC\bar{\lambda})$ as $t \to \infty$. Thus, $\lim_{t\to\infty} \Psi(t; \boldsymbol{\lambda}) = 1/(1 + RC\bar{\lambda})$, the LT of a degenerate d-variate exponential distribution concentrated on the main diagonal. □

6. Concluding remarks

One aim of this paper was to introduce multitype branching processes initiated from the immigration of individuals at time points generated by a Poisson random measure with mean measure $R(t) = \int_0^t r(x)\,\mathrm{d}x$. Theorem 1 gives a formula for the PGF of such processes when individuals evolve independently of each other, which includes the multitype Markov, Bellman–Harris, Sevastyanov, and Crump–Mode–Jagers branching models. Using Theorem 1, the asymptotic behaviour of the multitype critical Markov branching processes with inhomogeneous Poisson immigration is investigated next.

In the single-type case with homogeneous Poisson immigration, Sevastyanov [20] proved that the mean increases linearly, the variance grows quadratically, and the population size normalized by time converges to a gamma-distributed random variable. Sevastyanov's results are particular cases of Theorems 1 and 8 which hold when $r(x) \to r^* < \infty$, since, when $r(x) \equiv r^*$, we have homogeneous Poisson immigration. The results of Theorems 1, 5, and 13 obtained when $R(t) \to R < \infty$, generalize the classical case for critical multitype Markov branching processes without immigration, but differ substantially from those that hold with homogeneous immigration (see [16], [18], and [19]). Note that in [19] the offspring distributions are assumed to be stable with stability parameter in (0, 1], whereas in [18] the distributions of the offspring and of the number of immigrants are assumed to be stable with parameters in (1, 2] and (0, 1), respectively. For these cases, we do not know of results for processes with inhomogeneous Poisson immigration.

The other new results obtained in this paper have no equivalent among those for multitype Markov branching processes with homogeneous Poisson immigration. For instance, when $r(t) \to 0$ in such a way that $W(t) \to 1$, and under a specific normalization, the limiting univariate margins of the number of cells of each type are uniformly distributed on [0, 1] (Theorem 9). Theorem 10 deals with the case that $W(t)$ converges to some positive constant, and provides conditional and unconditional limit distributions under logarithmic normalization. The most interesting results are given in Theorem 12 where we present two singular limit distributions under identical conditions but different normalizations. Results obtained in the single-type case in [14] are particular cases of this theorem.

The time-inhomogeneity of the immigration process leads to other novel results which can be interpreted as a law of large numbers and central limit theorem (Theorems 6 and 7). They generalize results previously obtained in the single-type case [8], [14]. Different models of time-dependent immigration are considered in [3] and [4] but using other methods. Some results similar to Theorems 9 and 11 are obtained in [3], while a discrete-time process is studied in [4].

Based on our investigations, the local intensity $r(t)$ can be interpreted as a control function, the asymptotic behaviour of which leads to distinct limit distributions of the process. These results have important biological applications in the context of sequence evolution. As noted in the introduction, our work opens up new research problems.

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