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The Carleson Measure Problem Between Analytic Morrey Spaces

Jianfei Wang

Abstract. The purpose of this paper is to characterize positive measure μ on the unit disk such that the analytic Morrey space $\mathcal{AL}_{p,\eta}$ is boundedly and compactly embedded to the tent space

$$\mathcal{T}^{\infty}_{q,1-\frac{q}{p}(1-\eta)}(\mu)$$

for the case $1 \le q \le p < \infty$ respectively. As an application, these results are used to establish the boundedness and compactness of integral operators and multipliers between analytic Morrey spaces.

1 Introduction

Let $\mathbb{D}$ and $\mathbb{T}$ be the unit disk and the unit circle in $\mathbb{C}$, respectively. Let $dA(z)$ stand for the area measure on $\mathbb{D}$. Denote by $\mathcal{H}(\mathbb{D})$ the space of all analytic functions in $\mathbb{D}$.

For $(p-1,\eta) \in [0,\infty) \times [0,\infty)$, let $\mathcal{AL}_{p,\eta}$ be the analytic Campanato space that consists of functions $f \in \mathcal{H}(\mathbb{D})$ obeying

$$\|f\|_{p,\eta} = \sup_{I \subseteq \mathbb{T}} \left(|I|^{-\eta} \int_I |f(\zeta) - f_I|^p \frac{|d\zeta|}{2\pi} \right)^{\frac{1}{p}} < \infty,$$

where the supremum is taken over all sub-arcs $I \subseteq \mathbb{T}$, and

$$|d\zeta| = |de^{i\theta}| = d\theta, \quad |I| = (2\pi)^{-1} \int_I |d\zeta|, \quad f_I = (2\pi|I|)^{-1} \int_I f(\zeta)\,|d\zeta|.$$

Obviously, this value defines a seminorm on $\mathcal{AL}_{p,\eta}$. A complete norm can be equipped with

$$\|f\|_{\mathcal{AL}_{p,\eta}} = |f(0)| + \|f\|_{p,\eta}.$$

Also, the following table can help us to understand the structure of $\mathcal{AL}_{p,\eta}$ (see *e.g.*, [4, 8] and [5, pp. 209–217] for the real counterparts.)

In particular, when $p = 2$, $\mathcal{AL}_{2,\eta}$ is introduced by Wu and Xie in [14]. Recently, some fundamental function and operator-theoretic properties of $\mathcal{AL}_{p,\eta}$ have been investigated including [6, 7, 12, 13, 17].

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Index (p,η)	Analytic Campanato space $\mathcal{AL}_{p,\eta}$
$\eta = 0$	Analytic Hardy space $\mathcal{H}^p$
$\eta \in (0,1)$	Analytic Morrey space $\mathcal{AL}^{p,1-\eta}$
$\eta = 1$	Analytic John-Nirenberg space $\mathcal{BMOA}$
$\eta \in (1, 1+p]$	Analytic Lipschitz space $\mathcal{A}_{(\eta-1)/p}$
$\eta \in (1+p, \infty)$	Space of constants $\mathbb{C}$

Carleson measure plays an important role in complex analysis and harmonic analysis, which was introduced earlier in [2] in connection with the interpolation problem. Inspired by the idea of Xiao [16], it is natural to consider the following Carleson measure problem between the analytic Campanato spaces.

Problem 1.1 *Let μ be a positive Borel measure. What geometric assumption must μ have such that $\mathcal{AL}_{p,\eta}$ embeds boundedly (resp. compactly) into the tent space $\mathcal{T}^{\infty}_{q,\lambda}(\mu)$?*

Here and henceforth, $\mathcal{T}^{\infty}_{q,\lambda}(\mu)$ represents the tent space of all μ-measurable functions f on $\mathbb{D}$ satisfying

$$\|f\|_{\mathcal{T}^{\infty}_{q,\lambda}(\mu)} = \sup_{S(I)\subset\mathbb{D}} \Big(|I|^{-\lambda} \int_{S(I)} |f|^q d\mu \Big)^{\frac{1}{q}} < \infty,$$

where the supremum ranges over all Carleson square

$$S(I) = \big\{ z = re^{i\theta} \in \mathbb{D} : 1 - |I| \le r < 1,\ e^{i\theta} \in I \big\}.$$

The answer of Problem 1.1 to $1 \le q \le p < \infty$ is the following result.

Theorem 1.2 *Let μ be a nonnegative Borel measure on $\mathbb{D}$. If $1 \le q \le p < \infty$ and $\eta \in [0,1)$, then the identity operator $\mathsf{I}\colon \mathcal{AL}_{p,\eta} \mapsto \mathcal{T}^{\infty}_{q,1-\frac{q}{p}(1-\eta)}(\mu)$ is bounded (resp. compact) if and only if*

$$\sup_{S(I)\subset\mathbb{D}} \frac{\mu(S(I))}{|I|} < \infty \quad \Big(\textit{resp.}\ \lim_{|I|\to 0} \frac{\mu(S(I))}{|I|} = 0\Big).$$

Furthermore, we will consider the following Problem 1.3.

Problem 1.3 *Let $(p-2,\eta) \in [0,\infty) \times [0,1)$. What finite property must g have in order for V_g, U_g and M_g to be respectively bounded (resp. compact) from $\mathcal{AL}_{p,\eta}$ to $\mathcal{AL}_{2,1-\frac{2}{p}(1-\eta)}$?*

As usual, V_g and U_g stand for the Volterra-type operators with the analytic symbol g on $\mathbb{D}$, respectively:

$$\mathsf{V}_g f(z) = \int_0^z f(w)g'(w)\,dw, \quad z \in \mathbb{D},$$
$$\mathsf{U}_g f(z) = \int_0^z f'(w)g(w)\,dw, \quad z \in \mathbb{D}.$$

Then the multiplication operator M_g is given as follows

$$\mathsf{M}_g f(z) = f(z)g(z) = f(0)g(0) + \mathsf{V}_g f(z) + \mathsf{U}_g f(z).$$

Below is the solution to Problem 1.3.

Theorem 1.4 *Suppose* $(p-2,\eta)\in[0,\infty)\times[0,1)$ *and* $g\in\mathcal{H}(\mathbb{D})$*. Let* $d\mu_g(z)=(1-|z|^2)|g'(z)|^2dA(z)$ *and* $\|g\|_\infty=\sup_{z\in\mathbb{D}}|g(z)|$*. Then*

(i) $\mathsf{V}_g:\mathcal{AL}_{p,\eta}\mapsto\mathcal{AL}_{2,1-\frac{2}{p}(1-\eta)}$ *is bounded (resp. compact) if and only if* $g\in\mathcal{BMOA}$ *(resp.* $g\in\mathcal{VMOA}$*).*

(ii) $\mathsf{U}_g:\mathcal{AL}_{p,\eta}\mapsto\mathcal{AL}_{2,1-\frac{2}{p}(1-\eta)}$ *is bounded (resp. compact) if and only if* $g\in\mathcal{H}^\infty$ *(resp.* $g\equiv0$*).*

(iii) $\mathsf{M}_g:\mathcal{AL}_{p,\eta}\mapsto\mathcal{AL}_{2,1-\frac{2}{p}(1-\eta)}$ *is bounded (resp. compact) if and only if* $g\in\mathcal{H}^\infty$ *(resp.* $g\equiv0$*).*

Particularly, Theorem 1.4 generalizes recent results in [6,13] to $\mathcal{AL}_{p,\eta}$.

Notation Let $\mathsf{X}\lesssim\mathsf{Y}$ and $\mathsf{X}\gtrsim\mathsf{Y}$ denote that there exists an absolute constant $C>0$ such that $\mathsf{X}\le C\mathsf{Y}$ and $\mathsf{X}\ge C\mathsf{Y}$, respectively. Thus, $\mathsf{X}\approx\mathsf{Y}$ means that $\mathsf{X}\lesssim\mathsf{Y}$ and $\mathsf{X}\gtrsim\mathsf{Y}$ hold.

2 Some Lemmas

In order to prove Theorems 1.2 and 1.4, we begin with the following lemmas. The first lemma is due to Xiao and Yuan.

Lemma 2.1 ([18, Theorem 1]) *If* $(p-1,\eta)\in[0,\infty)\times[0,2)$*, then the following two statements are equivalent:*

(i) $f\in\mathcal{AL}_{p,\eta}$.

(ii) $\|f\|_{p,\eta,*}=\sup_{a\in\mathbb{D}}(1-|a|^2)^{\frac{1-\eta}{p}}\|f\circ\sigma_a-f(a)\|_p<\infty$*, where* $\sigma_a(z)=\frac{a-z}{1-\bar{a}z}$.

Next, we are present to a useful integral estimate.

Lemma 2.2 *Let* $z,w\in\mathbb{D}$ *and* $t,s>1$*. Then*

$$\int_{\mathbb{T}}\frac{1}{|1-z\bar\zeta|^t|1-w\bar\zeta|^s}|d\zeta|\lesssim\frac{1}{(1-|z|^2)^{t-1}|1-z\overline{w}|^s}+\frac{1}{(1-|w|^2)^{s-1}|1-z\overline{w}|^t}.$$

Proof Let

$$I=\int_{\mathbb{T}}\frac{1}{|1-z\bar\zeta|^t|1-w\bar\zeta|^s}|d\zeta|.$$

Consider the following partition of $\mathbb{T}$:

$$\Omega_1=\{\zeta\in\mathbb{T}:|1-w\bar\zeta|\le|1-z\bar\zeta|\}$$
$$\Omega_2=\mathbb{T}\setminus\Omega_1=\{\zeta\in\mathbb{T}:|1-z\bar\zeta|<|1-w\bar\zeta|\}.$$

Then

$$I=\int_{\Omega_1}\frac{1}{|1-z\bar\zeta|^t|1-w\bar\zeta|^s}|d\zeta|+\int_{\Omega_2}\frac{1}{|1-z\bar\zeta|^t|1-w\bar\zeta|^s}|d\zeta|=I_1+I_2.$$

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For $\zeta \in \mathbb{T}$, we have

$$\begin{aligned} |1 - z\overline{w}| &\le |1 - z\overline{\zeta}| + |z\overline{\zeta} - z\overline{w}| \le |1 - z\overline{\zeta}| + |\zeta - w| \\ &= |1 - z\overline{\zeta}| + |1 - w\overline{\zeta}| \le 2|1 - z\overline{\zeta}|. \end{aligned}$$

Also, the following inequality is well known:

$$\int_{\mathbb{T}} \frac{1}{|1 - w\overline{\zeta}|^s} |d\zeta| \lesssim \frac{1}{(1 - |w|^2)^{s-1}}.$$

It shows that

$$I_1 \le \frac{1}{|1 - z\overline{w}|^t} \int_{\mathbb{T}} \frac{1}{|1 - w\overline{\zeta}|^s} |d\zeta| \lesssim \frac{1}{(1 - |w|^2)^{s-1} |1 - z\overline{w}|^t}.$$

Similarly, we have

$$I_2 \le \frac{1}{|1 - z\overline{w}|^s} \int_{\mathbb{T}} \frac{1}{|1 - z\overline{\zeta}|^t} |d\zeta| \lesssim \frac{1}{(1 - |z|^2)^{t-1} |1 - z\overline{w}|^s}.$$

Putting the above estimates for I_1 and I_2 together, we get

$$I \lesssim \frac{1}{(1 - |z|^2)^{t-1} |1 - z\overline{w}|^s} + \frac{1}{(1 - |w|^2)^{s-1} |1 - z\overline{w}|^t},$$

as desired. ■

By Lemma 2.2, we can obtain the test function on Morrey space.

Lemma 2.3 *Let $(p-1, \eta, \lambda) \in [0, \infty) \times [0, 1] \times [\frac{1-\eta}{p}, \infty)$ and $w \in \mathbb{D}$. Then functions*

$$f_w(z) = \frac{(1 - |w|^2)^{\lambda - \frac{1-\eta}{p}}}{(1 - \overline{w}z)^{\lambda}}$$

belong to $\mathcal{AL}_{p,\eta}$. Moreover, f_w is uniformly bounded in $\mathcal{AL}_{p,\eta}$, i.e., $\sup_{w \in \mathbb{D}} \|f_w\|_{p,\eta} \lesssim 1$.

Proof According to Lemma 2.2, we get

$$\begin{aligned} &(1 - |a|^2)^{1-\eta} \|f_w \circ \sigma_a - f(a)\|_p^p \\ &= (1 - |a|^2)^{1-\eta} \int_{\mathbb{T}} |f_w \circ \sigma_a(\zeta) - f(a)|^p \frac{|d\zeta|}{2\pi} \\ &= (1 - |a|^2)^{1-\eta} \int_{\mathbb{T}} |f_w(\zeta) - f_w(a)|^p \frac{1 - |a|^2}{|1 - \langle a, \zeta\rangle|^2} \frac{|d\zeta|}{2\pi} \\ &\lesssim |f_w(a)|^p (1 - |a|^2)^{1-\eta} + (1 - |a|^2)^{1-\eta} \int_{\mathbb{T}} |f_w(\zeta)|^p \frac{1 - |a|^2}{|1 - \langle a, \zeta\rangle|^2} |d\zeta| \\ &\lesssim \frac{(1 - |a|^2)^{1-\eta} (1 - |w|^2)^{p(\lambda - \frac{1-\eta}{p})}}{|1 - \langle a, w\rangle|^{p\lambda}} + \int_{\mathbb{T}} \frac{(1 - |a|^2)^{2-\eta} (1 - |w|^2)^{p(\lambda - \frac{1-\eta}{p})}}{|1 - \langle a, \zeta\rangle|^2 |1 - \langle w, \zeta\rangle|^{p\lambda}} |d\zeta| \end{aligned}$$

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$$
\begin{aligned}
&\lesssim \frac{(1-|a|^2)^{1-\eta}(1-|w|^2)^{p(\lambda-\frac{1-\eta}{p})}}{|1-\langle a,w\rangle|^{p\lambda}} + \frac{(1-|a|^2)^{2-\eta}(1-|w|^2)^{p(\lambda-\frac{1-\eta}{p})}}{(1-|w|^2)^{p\lambda-1}|1-\langle a,w\rangle|^2}\\
&\quad + \frac{(1-|a|^2)^{2-\eta}(1-|w|^2)^{p(\lambda-\frac{1-\eta}{p})}}{|1-\langle a,w\rangle|^{p\lambda}(1-|a|^2)}\\
&\lesssim \frac{(1-|a|^2)^{1-\eta}(1-|w|^2)^{p(\lambda-\frac{1-\eta}{p})}}{|1-\langle a,w\rangle|^{p\lambda}} + \frac{(1-|a|^2)^{2-\eta}(1-|w|^2)^{\eta}}{|1-\langle a,w\rangle|^2}\\
&\lesssim \Big(\frac{1-|a|^2}{|1-\langle a,w\rangle|}\Big)^{1-\eta}\Big(\frac{1-|w|^2}{|1-\langle a,w\rangle|}\Big)^{p(\lambda-\frac{1-\eta}{p})} + \Big(\frac{1-|a|^2}{|1-\langle a,w\rangle|}\Big)^{2-\eta}\Big(\frac{1-|w|^2}{|1-\langle a,w\rangle|}\Big)^{\eta}\\
&\lesssim 1.
\end{aligned}
$$

Hence, applying Lemma 2.1,

$$
f_w \in \mathcal{AL}_{p,\eta} \quad \text{and} \quad \sup_{w\in\mathbb{D}} \|f_w\|_{p,\eta,*} \lesssim 1.
$$

Arguing as above and using $\sup_{w\in\mathbb{D}} |f_w(0)| \lesssim 1$, we obtain $\sup_{w\in\mathbb{D}} \|f_w\|_{\mathcal{AL}_{p,\eta}} \lesssim 1$. ∎

Remark If $p>1$ and $\lambda=1$, then Lemma 2.3 reduces to the result of Xiao and Yuan in [18, Lemma 13].

The following lemma can also been found in [18].

Lemma 2.4 *Let $(p-1,\eta)\in[0,\infty)\times[0,1)$. If $f\in\mathcal{AL}_{p,\eta}$, then*

$$
|f'(z)| \lesssim \frac{\|f\|_{p,\eta,*}}{(1-|z|^2)^{1+\frac{1-\eta}{p}}}, \quad \textit{for } z\in\mathbb{D}.
$$

Moreover, the exponent $1+\frac{1-\eta}{p}$ in the above inequality is sharp.

Based on Lemma 2.4, it is easy to obtain the following decay estimate on $\mathcal{AL}_{p,\eta}$, which can be found in [3].

Lemma 2.5 *Let $(p-1,\eta)\in[0,\infty)\times[0,1)$. Then*

$$
|f(z)| \lesssim \frac{\|f\|_{\mathcal{AL}_{p,\eta}}}{(1-|z|^2)^{\frac{1-\eta}{p}}}
$$

holds for all $z\in\mathbb{D}$.

The following lemma, which can be found in [11, Lemma 3.7], plays an important role in proving the compactness of the Carleson measure problem and the integral operators.

Lemma 2.6 *Let X, Y be two Banach spaces of analytic functions on $\mathbb{D}$. If*

(i) *the point evaluation functions on Y are continous;*
(ii) *the closed unit ball of X is a compact subset of X in the topology of uniform convergence on compact sets;*

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(iii) $\mathsf{T}\colon X \to Y$ *is continuous when* X *and* Y *are given the topology of uniform convergence on compact sets,*

then T *is a compact operator if and only if given a bounded sequence* $\{f_j\}$ *in* X *such that* $f_j \to 0$ *uniformly on compact sets. Then the sequence* $\{\mathsf{T}f_j\}$ *converges to zero in the norm of* Y.

3 Proof of Theorem 1.2

Proof The argument is divided into the following two steps.

Step 1: Boundedness. Suppose

$$\mathsf{I}\colon \mathcal{AL}_{p,\eta} \longmapsto \mathcal{T}^{\infty}_{q,1-\frac{q}{p}(1-\eta)}(\mu)$$

is bounded. Then we have to prove μ is a Carleson measure.

In fact, for a given subarc $I \subset \mathbb{T}$, let $w = (1-|I|)\zeta$ and let ζ be the center of I.

We choose

$$f_w(z) = \frac{(1-|w|^2)^{\frac{2}{q}-\frac{1-\eta}{p}}}{(1-\overline{w}z)^{\frac{2}{q}}}.$$

From Lemma 2.3, we obtain that

$$f_w \in \mathcal{AL}_{p,\eta} \quad \text{and} \quad \sup_{w\in\mathbb{D}} \|f_w\|_{\mathcal{AL}_{p,\eta}} \lesssim 1.$$

Notice that

$$|1-\overline{w}z| \approx 1-|w|^2 \approx |I|, \quad z \in S(I). \tag{3.1}$$

In terms of (3.1), we have

$$1 \gtrsim \|f_w\|^q_{\mathcal{T}^{\infty}_{q,1-\frac{q(1-\eta)}{p}}(\mu)} \geq \frac{1}{|I|^{1-\frac{q(1-\eta)}{p}}} \int_{S(I)} \frac{(1-|w|^2)^{2-\frac{q(1-\eta)}{p}}}{|1-\overline{w}z|^2} d\mu(z) \approx \frac{\mu(S(I))}{|I|}.$$

Accordingly,

$$\|\mu\|_{\mathcal{CM}} = \sup_{S(I)\subset\mathbb{D}} \frac{\mu(S(I))}{|I|} < \infty. \tag{3.2}$$

Conversely, if (3.2) holds and $f \in \mathcal{AL}_{p,\eta}$, then we need to prove that

$$\|f\|_{\mathcal{T}^{\infty}_{q,1-\frac{q(1-\eta)}{p}}(\mu)} \lesssim \|f\|_{\mathcal{AL}_{p,\eta}}.$$

Suppose $I \subset \mathbb{T}$ is an arc and ζ is the center. Let $a = (1-|I|)\zeta$. Then

$$\begin{aligned}
&\frac{1}{|I|^{1-\frac{q(1-\eta)}{p}}} \int_{S(I)} |f(z)|^q d\mu(z) \\
&\quad\lesssim \frac{1}{|I|^{1-\frac{q(1-\eta)}{p}}} \int_{S(I)} |f(z)-f(a)|^q d\mu(z) + \frac{\mu(S(I))}{|I|^{1-\frac{q(1-\eta)}{p}}} |f(a)|^q \\
&\quad= \mathrm{Int}_1 + \mathrm{Int}_2.
\end{aligned}$$

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We first estimate the Int_1. A simple calculation shows that

$$1-|\sigma_a(z)|^2=\frac{(1-|a|^2)(1-|z|^2)}{|1-\overline{a}z|^2}\approx\frac{1-|z|^2}{|I|}$$

holds for all $z\in S(I)$.

Since $\|\mu\|_{\mathcal{CM}}<\infty$, we have μ is a Carleson measure on $\mathbb{D}$.

The condition $1\le q\le p<\infty$ shows that

$$\mathcal{AL}_{p,\eta}\subset\mathcal{AL}_{q,1-\frac{q(1-\eta)}{p}}.$$

Using the Carleson theorem and Lemma 2.1 we calculate

$$\begin{aligned}
\mathrm{Int}_1&=\frac{1}{|I|^{1-\frac{q(1-\eta)}{p}}}\int_{S(I)}|f(z)-f(a)|^q\,d\mu(z)\\
&\approx(1-|a|^2)^{\frac{q(1-\eta)}{p}}\int_{S(I)}|f(z)-f(a)|^q\frac{1-|a|^2}{|1-\overline{a}z|^2}\,d\mu(z)\\
&\lesssim(1-|a|^2)^{\frac{q(1-\eta)}{p}}\int_{\mathbb{D}}|f(z)-f(a)|^q\frac{1-|a|^2}{|1-\overline{a}z|^2}\,d\mu(z)\\
&\approx(1-|a|^2)^{\frac{q(1-\eta)}{p}}\int_{\mathbb{D}}|f\circ\sigma_a(z)-f(a)|^q\,d\mu(z)\\
&\lesssim\|\mu\|_{\mathcal{CM}}(1-|a|^2)^{\frac{q(1-\eta)}{p}}\int_{\mathbb{T}}|f\circ\sigma_a(\xi)-f(a)|^q|d\xi|\\
&\lesssim\|\mu\|_{\mathcal{CM}}\|f\|^q_{q,1-\frac{q(1-\eta)}{p},*}\lesssim\|\mu\|_{\mathcal{CM}}\|f\|^q_{\mathcal{AL}_{p,\eta}}.
\end{aligned}$$

For Int_2, the decay estimate gives

$$|f(a)|\lesssim|I|^{\frac{\eta-1}{p}}\|f\|_{\mathcal{AL}_{q,1-\frac{q(1-\eta)}{p}}}.$$

Hence,

$$\mathrm{Int}_2\lesssim\frac{\mu(S(I))}{|I|}\|f\|^q_{\mathcal{AL}_{q,1-\frac{q(1-\eta)}{p}}}\lesssim\|\mu\|_{\mathcal{CM}}\|f\|^q_{\mathcal{AL}_{q,1-\frac{q(1-\eta)}{p}}}.$$

Putting the estimates of Int_1 and Int_2 together, we get

$$\|f\|_{\mathcal{T}^\infty_{q,1-\frac{q(1-\eta)}{p}}(\mu)}\lesssim\|\mu\|^{\frac{1}{q}}_{\mathcal{CM}}\|f\|_{\mathcal{AL}_{p,\eta}}.$$

Thus, the identity operator $\mathsf{I}:\mathcal{AL}_{p,\eta}\mapsto\mathcal{T}^\infty_{q,1-\frac{q}{p}(1-\eta)}(\mu)$ is bounded.

Step 2: Compactness. Let us prove the sufficiency part. Suppose

$$\lim_{|I|\to0}\frac{\mu(S(I))}{|I|}=0.$$

Then we need to show that the operator

$$\mathsf{I}:\mathcal{AL}_{p,\eta}\longmapsto\mathcal{T}^\infty_{q,1-\frac{q}{p}(1-\eta)}(\mu)$$

is compact. First, Lemma 2.6(i) holds by setting

$$e_z(f)=f(z):\mathcal{AL}_{p,\eta}\mapsto\mathcal{T}^\infty_{q,1-\frac{q}{p}(1-\eta)}(\mu).$$

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Second, the following estimate

$$|f(z)| \lesssim \frac{\|f\|_{\mathcal{AL}_{p,\eta}}}{(1-|z|^2)^{\frac{1-\eta}{p}}}$$

shows that Lemma 2.6(ii) holds. Hence, we only need to verify assumption (iii), *i.e.*, if any bounded sequence $\{f_j\} \subset \mathcal{AL}_{p,\eta}$ with $f_j \to 0$ converges uniformly to zero on compact sets of $\mathbb{D}$, then $\|f_j\|_{\mathcal{T}^\infty_{q,1-\frac{q}{p}(1-\eta)}(\mu)} \to 0$.

In fact, when $r \in (0,1)$, we choose the cut-off measure $d\mu_r = \chi_{\{z\in\mathbb{D}:|z|>r\}} d\mu$, where χ_E means the characteristic function of a set $E \subset \mathbb{D}$. Then

$$\sup_{S(I)\subset\mathbb{D}} \frac{\mu_r(S(I))}{|I|} \lesssim \sup_{|I|\le 1-r} \frac{\mu(S(I))}{|I|} \longrightarrow 0 \quad \text{if } r \to 1. \tag{3.3}$$

Now, assume $\|f_j\|_{\mathcal{AL}_{p,\eta}} \lesssim 1$ and f_j converges uniformly to zero on compact sets of $\mathbb{D}$, then from (3.3) and Lemma 2.5 we get

$$\begin{aligned}
&\frac{1}{|I|^{1-\frac{q}{p}(1-\eta)}} \int_{S(I)} |f_j|^q d\mu \\
&\quad= \frac{1}{|I|^{1-\frac{q}{p}(1-\eta)}} \int_{S(I)} |f_j|^q \chi_{\{z\in\mathbb{D}:|z|\le r\}} d\mu + \frac{1}{|I|^{1-\frac{q}{p}(1-\eta)}} \int_{S(I)} |f_j|^q \chi_{\{z\in\mathbb{D}:|z|>r\}} d\mu \\
&\quad\lesssim \frac{1}{|I|^{1-\frac{q}{p}(1-\eta)}} \int_{S(I)} |f_j|^q \chi_{\{z\in\mathbb{D}:|z|\le r\}} d\mu + \frac{\mu_r(S(I))}{|I|} \|f_j\|_{\mathcal{AL}_{p,\eta}} \\
&\quad\to 0.
\end{aligned}$$

Hence, the sufficiency part of Theorem 1.2 is proved.

On the other hand, we will verify the necessity. Let the operator

$$\mathsf{I}\colon \mathcal{AL}_{p,\eta} \mapsto \mathcal{T}^\infty_{q,1-\frac{q}{p}(1-\eta)}(\mu)$$

be compact. Suppose I_j is a sequence of subarcs of $\mathbb{T}$ obeying $|I_j| \to 0$. Let ζ_j be the center of I_j and $w_j = (1-|I_j|)\zeta_j$.

Let

$$f_j(z) = \frac{(1-|w_j|^2)^{\frac{2p+\eta-1}{p}}}{(1-\overline{w_j}z)^2}.$$

Then $\|f_j\|_{\mathcal{AL}_{p,\eta}} \lesssim 1$, and $f_j \to 0$ uniformly on compact sets of $\mathbb{D}$.

Applying Lemma 2.6, we have

$$0 \leftarrow \|f_j\|^q_{\mathcal{T}^\infty_{q,1-\frac{q}{p}(1-\eta)}(\mu)} \ge \frac{1}{|I_j|^{1-\frac{q}{p}(1-\eta)}} \int_{S(I_j)} |f_j|^q d\mu \ge \frac{\mu(S(I_j))}{|I_j|}.$$

Thus, the desired vanishing condition holds. ■

4 Proof of Theorem 1.4

Before we give the proof of Theorem 1.4, let us recall some basic facts on η-Carleson measure associated with Morrey space $\mathcal{AL}_{2,\eta}$.

For $\eta \in (0,1]$, we write $\mathcal{CM}_\eta$ for the class of all η-Carleson measures on $\mathbb{D}$. A nonnegative measure μ on $\mathbb{D}$ is called a *bounded* or *compact η-Carleson measure* provided

$$\|\mu\|_{\mathcal{CM}_\eta} = \sup_{I \subseteq \mathbb{T}} \frac{\mu(S(I))}{|I|^\eta} < \infty \quad \text{or} \quad \lim_{I \subseteq \mathbb{T},\ |I| \to 0} \frac{\mu(S(I))}{|I|^\eta} = 0,$$

where

$$S(I) = \{z = re^{i\theta} \in \mathbb{D} : 1 - |I| \le r < 1,\ e^{i\theta} \in I\}$$

is the Carleson square based on a subarc $I \subseteq \mathbb{T}$.

Assume the following:

$$(\eta, a, z) \in (0,1] \times \mathbb{D} \times \mathbb{D}, \qquad \sigma_a(z) = \frac{a - z}{1 - \bar{a}z},$$
$$E(f, a) = (1 - |a|^2)^{1-\eta} \int_{\mathbb{D}} |f'(z)|^2 (1 - |\sigma_a(z)|) \, dA(z).$$

According to [15, 17], the following statements are equivalent:

- $f \in \mathcal{AL}_{2,\eta}$.
- $|f'(z)|^2 (1 - |z|^2) dA(z)$ is a bounded η-Carleson measure.
- $\sup_{a \in \mathbb{D}} E(f, a) < \infty$.

Now we give the proof of Theorem 1.4.

Proof We split the argument into three parts.

(i) The operator V_g. Let $\lambda = 1 - 2/p(1 - \eta)$. Notice that $f \in \mathcal{AL}_{2,\lambda}$ is equivalent to $|f'(z)|^2(1 - |z|^2) dA(z)$ is a bounded λ-Carleson measure. Thus, Theorem 1.2 and $(\mathsf{V}_g f)'(z) = f(z) g'(z)$ guarantee that V_g is bounded. The corresponding compactness can be obtained similarly.

(ii) The operator U_g. First, we consider the boundedness. Suppose $\|g\|_\infty < \infty$. Since $\mathsf{U}_g(f)(0) = 0$, using Lemma 2.1 and the Littlewood–Paley equality, we obtain that

$$\begin{aligned}
&\|\mathsf{U}_g f\|_{\mathcal{AL}_{2,1-\frac{2}{p}(1-\eta)}} \\
&\quad\lesssim |\mathsf{U}_g(f)(0)| + \sup_{a \in \mathbb{D}} \Big((1 - |a|^2)^{\frac{2(1-\eta)}{p}} \int_{\mathbb{D}} |f'(z)|^2 |g(z)|^2 (1 - |\sigma_a(z)|^2) \, dA(z) \Big)^{\frac{1}{2}} \\
&\quad\lesssim \|g\|_\infty \sup_{a \in \mathbb{D}} (1 - |a|^2)^{\frac{1-\eta}{p}} \|f \circ \sigma_a - f(a)\|_2 \\
&\quad\lesssim \|g\|_\infty \sup_{a \in \mathbb{D}} (1 - |a|^2)^{\frac{1-\eta}{p}} \|f \circ \sigma_a - f(a)\|_p \\
&\quad\lesssim \|g\|_\infty \|f\|_{\mathcal{AL}_{p,\eta}}.
\end{aligned}$$

This implies that $\|\mathsf{U}_g\| \lesssim \|g\|_\infty$. Hence, $\mathsf{U}_g : \mathcal{AL}_{p,\eta} \mapsto \mathcal{AL}_{2,1-\frac{2}{p}(1-\eta)}$ is bounded.

Conversely, if $\mathsf{U}_g : \mathcal{AL}_{p,\eta} \mapsto \mathcal{AL}_{2,1-\frac{2}{p}(1-\eta)}$ is bounded, we then need to prove that $g \in \mathcal{H}^\infty$ and $\|g\|_\infty \lesssim \|\mathsf{U}_g\|$. To do so, given a nonzero point w and $\frac{1}{2} \le |w| < 1$, we choose a Carleson square $S(I)$ such that

$$\{z \in \mathbb{D} : |\sigma_w(z)| < \tfrac{1}{2}\} \subset S(I) \quad \text{and} \quad 1 - |w|^2 \approx |I|.$$

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Let

$$f_w(z) = \frac{(1-|w|^2)^{\frac{p+\eta-1}{p}}}{\overline{w}(1-\overline{w}z)}.$$

Then

$$f_w \in \mathcal{AL}_{p,\eta} \quad \text{and} \quad \|f_w\|_{\mathcal{AL}_{p,\eta}} \lesssim 1.$$

Using [19, Lemma 4.12] and the boundednenss of U_g, we get

$$\begin{aligned}
\|\mathsf{U}_g\|^2 &\gtrsim \|\mathsf{U}_g f_w\|^2_{\mathcal{AL}_{2,1-\frac{2}{p}(1-\eta)}} \\
&\gtrsim \frac{1}{|I|^{1-\frac{2}{p}(1-\eta)}} \int_{S(I)} |(\mathsf{U}_g f_w)'(z)|^2 (1-|z|^2)\, dA(z) \\
&= \frac{1}{|I|^{1-\frac{2}{p}(1-\eta)}} \int_{S(I)} |f_w'(z)|^2 |g(z)|^2 (1-|z|^2)\, dA(z) \\
&= \frac{1}{|I|^{1-\frac{2}{p}(1-\eta)}} \int_{S(I)} \frac{(1-|w|^2)^{\frac{2(p+\eta-1)}{p}}}{|1-\overline{w}z|^4} |g(z)|^2 (1-|z|^2)\, dA(z) \\
&\approx \frac{1}{|I|^3} \int_{S(I)} |g(z)|^2 (1-|z|^2)\, dA(z) \\
&\gtrsim \frac{1}{|I|^3} \int_{\{z\in\mathbb{D}: |\sigma_w(z)|<\frac{1}{2}\}} |g(z)|^2 (1-|z|^2)\, dA(z) \\
&\gtrsim \frac{1}{|I|^3} |g(w)|^2 (1-|w|^2)^3 \approx |g(w)|^2.
\end{aligned}$$

The maximal principle is used to deduce

$$\|g\|_\infty = \sup_{\frac{1}{2}\le|w|<1} |g(w)| \lesssim \|\mathsf{U}_g\|.$$

Second, we prove the compactness of U_g. It is enough to show that if $\mathsf{U}_g\colon \mathcal{AL}_{p,\eta} \mapsto \mathcal{AL}_{2,1-\frac{2}{p}(1-\eta)}$ is compact, then $g \equiv 0$. By Theorem 1.4, $g \in \mathcal{H}^\infty$. Suppose g is not identically equal to zero. Then the maximum principle tells us that $g|_{\mathbb{T}}$ cannot be identically zero. It means that there exists a positive constant $\epsilon > 0$ and a sequence $\{w_j\} \subset \mathbb{D}$ such that $w_j \to w_0 \in \mathbb{T}$ and $|g(w_j)| > \epsilon$. Applying the classical Schwarz's lemma for $g/\|g\|$, we have

$$|g(z_1) - g(z_2)| \le 2\|g\|_\infty |\sigma_{z_1}(z_2)|,$$

where $\sigma_{z_1}(z_2) = \frac{z_1-z_2}{1-\overline{z}_1 z_2}$ and $z_1, z_2 \in \mathbb{D}$.

The above inequality shows that there is a sufficiently small number $\delta > 0$ such that $|g(z)| \ge \frac{\epsilon}{2}$ holds for all j and z with $|\sigma_{w_j}(z)| < \delta$. It is easy to see that each pseudo-hyperbolic ball $\{z \in \mathbb{D} : |\sigma_{w_j}(z)| < \delta\}$ is a subset of a Carleson box $S(I_j)$ with $|I_j| \approx 1-|w_j|^2$. Hence, if $\eta \in [0,1)$, we take

$$f_j(z) = \frac{(1-|w_j|^2)^{\frac{p+\eta-1}{p}}}{1-\overline{w_j}z} \quad \text{and} \quad f_0(z) \equiv 0.$$

Then $f_j - f_0 \to 0$ uniformly holds on any compact set of $\mathbb{D}$.

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Accordingly,

$$\begin{aligned}
&\|\mathsf{U}_g(f_j)\|^2_{\mathcal{AL}_{2,1-\frac{2}{p}(1-\eta)}}\\
&\gtrsim \frac{1}{|I_j|^{1-\frac{2}{p}(1-\eta)}}\int_{S(I_j)}\frac{(1-|w_j|^2)^{\frac{2(p+\eta-1)}{p}}}{|1-\overline{w_j}z|^4}|g(z)|^2(1-|z|^2)\,dA(z)\\
&\gtrsim \frac{1}{|I_j|^3}\int_{S(I_j)}|g(z)|^2(1-|z|^2)\,dA(z)\\
&\gtrsim \frac{\epsilon^2}{|I_j|^3}\int_{\{z\in\mathbb{D}:|\sigma_{w_j}(z)|<\delta\}}(1-|z|^2)\,dA(z)\gtrsim\epsilon^2.
\end{aligned}$$

The compactness of U_g gives that $\|\mathsf{U}_g(f_j)\|_{\mathcal{AL}_{p,\eta}}\to 0$. It is a contradiction with $\epsilon>0$. Thus, $g\equiv 0$.

(iii). The operator M_g. We will first prove that the boundedness of M_g. The above statementa (i) and (ii) show that the "if" part holds. To prove the "only part", let

$$f_w(z)=\frac{(1-|w|^2)^{\frac{p+\eta-1}{p}}}{1-\overline{w}z}.$$

Then $\sup_{w\in\mathbb{D}}\|f_w\|_{\mathcal{AL}_{p,\eta}}\lesssim 1$.

By Lemma 2.5, we have

$$|f_w(z)|\lesssim\frac{\|f_w\|_{\mathcal{AL}_{p,\eta}}}{(1-|z|^2)^{\frac{1-\eta}{p}}}.$$

Since $\mathsf{M}_g:\mathcal{AL}_{p,\eta}\mapsto\mathcal{AL}_{2,1-\frac{2}{p}(1-\eta)}$ is bounded, we have

$$|f_w(z)g(z)|\lesssim\frac{\|M_gf_w\|_{\mathcal{AL}_{2,1-\frac{2}{p}(1-\eta)}}}{(1-|z|^2)^{\frac{1-\eta}{p}}}\lesssim\|M_g\|\frac{1}{(1-|z|^2)^{\frac{1-\eta}{p}}}.$$

Setting $z=w$, we obtain $|g(w)|\lesssim\|M_g\|$, which means $g\in\mathcal{H}^\infty$. Applying this to Theorem 1.4(i), we get $\mathsf{V}_g:\mathcal{AL}_{p,\eta}\mapsto\mathcal{AL}_{2,1-\frac{q}{p}(1-\eta)}$ is bounded. Notice that

$$\mathsf{U}_gf=\mathsf{M}_gf-f(0)g(0)-\mathsf{V}_g(f),$$

the boundedness of V_g gives U_g is bounded.

Now, it remains to prove $\mathsf{M}_g:\mathcal{AL}_{p,\eta}\mapsto\mathcal{AL}_{2,1-2/p(1-\eta)}(\mu)$ is compact if and only if $g\equiv 0$. Obviously, we need to prove the "only part". Since this operator M_g is bounded, we have $\|g\|_\infty<\infty$. For any nonzero sequence $\{w_j\}\subset\mathbb{D}$, set

$$f_j(z)=\frac{(1-|w_j|^2)^{\frac{p+\eta-1}{p}}}{1-\overline{w_j}z}.$$

Suppose $|w_j|\to 1$. Then $\|f_j\|_{\mathcal{AL}_{p,\eta}}\lesssim 1$ and $f_j\to 0$ uniformly on any compact set in $\mathbb{D}$. Hence, $\|\mathsf{M}_g(f_j)\|_{\mathcal{AL}_{2,1-\frac{2}{p}(1-\eta)}}\to 0$.

By the estimates

$$|g(z)f_j(z)|=|\mathsf{M}_g(f_j)(z)|\lesssim\|\mathsf{M}_gf_j\|_{\mathcal{AL}_{2,1-\frac{2}{p}(1-\eta)}}\frac{1}{(1-|z|^2)^{\frac{1-\eta}{p}}},$$

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and choosing $z = w_j$, we get

$$|g(w_j)| \lesssim \|\mathsf{M}_g(f_j)\|_{\mathcal{AL}_{2,1-\frac{2}{p}(1-\eta)}} \longrightarrow 0.$$

This means that $g(w_j) \to 0$, which, together with $g \in \mathcal{H}^\infty$, gives that $g \equiv 0$. ∎

Remark The Volterra-type operators, which connect with Hankel operator, Toeplitz operator, and multiplier, have been the interesting objects of analytic function spaces in complex analysis and operator theory. It is well known that $\mathsf{V}_g: \mathcal{H}^2 \to \mathcal{H}^2$ is bounded (resp. compact) if and only if $g \in \mathcal{BMOA}$ (resp. $g \in \mathcal{VMOA}$); see [9]. Further, the results of the boundedness of V_g on Hardy space $\mathcal{H}^p$ can be found from Siskakis [10] and Aleman and Cima [1]. During the last years, several authors have been investigated the operator V_g (see *e.g.*, [6,13,15,16] and references therein).

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Mathematics, Physics and Information Engineering, Zhejiang Normal University, Jinhua, Zhejiang, 321004, China
e-mail: wangjf@mail.ustc.edu.cn