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EXACT SUBCATEGORIES, SUBFUNCTORS OF EXT, AND SOME APPLICATIONS

HAILONG DAO, SOUVIK DEY AND MONALISA DUTTA

Abstract. Let $(\mathcal{A},\mathcal{E})$ be an exact category. We establish basic results that allow one to identify sub(bi)functors of $\operatorname{Ext}_{\mathcal{E}}(-,-)$ using additivity of numerical functions and restriction to subcategories. We also study a small number of these new functors over commutative local rings in detail and find a range of applications from detecting regularity to understanding Ulrich modules.

§1. Introduction

The Yoneda characterization of Ext is familiar to most students of homological algebra. Let A,B be two objects in an abelian category $\mathcal{A}$. Then $\operatorname{Ext}_{\mathcal{A}}(A,B)$ is the set of all equivalence classes of sequences of the form $0 \to B \to C \to A \to 0$, where two sequences α,β are equivalent if we have the following commutative diagram:

$$\begin{array}{llllllllll} \alpha: & 0 & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A & \longrightarrow & 0 \\ & & & \| & & \downarrow{\scriptstyle f} & & \| & & \\ \beta: & 0 & \longrightarrow & B & \longrightarrow & C' & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

Now, $\operatorname{Ext}_{\mathcal{A}}(A,B)$ can be given an abelian group structure by the well-known Baer sum as described in, for instance, [38, Tag 010I]. This consideration can be carried out more generally in any exact category (see [14, §1.2]).

The purpose of this note is to study the following rather natural questions: what if we place additional constraints on the short exact sequences? When do we get a subfunctor of Ext^1? Can one apply such functors to study ring and module theory, similar to the ways homological algebra has been very successfully applied in the last decades?

Let us elucidate our goals with a concrete example. Let $(R,\mathfrak{m},k)$ be a Noetherian local ring. Let A,B be finitely generated R-modules. We consider exact sequences $0 \to B \to C \to A \to 0$ of R-modules, with the added condition that $\mu(C) = \mu(A)+\mu(B)$, where $\mu(-)$ denotes the minimal number of generators. As we shall see later, the equivalence classes of such sequences do form a subfunctor of Ext^1_R, which we denote by $\operatorname{Ext}^1_R(-,-)^{\mu}$. More surprisingly, the vanishing of a single module $\operatorname{Ext}^1_R(-,-)^{\mu}$ can be used to characterize the regularity of R, a feature that is lacking with the classical Ext^1.

Analogous versions of classical homological functors have been studied by various authors, notably starting with Hochschild [23] who studied relative Ext and Tor groups of modules over a ring with respect to subrings of the original ring. This work is further developed by Butler–Horrocks as well as Auslander–Solberg, where the point of view is switched to allowable exact sequences that give rise to sub(bi)functors of Ext^1. The whole circle of

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ideas is now thriving on its own under the name *relative homological algebra*, with exact structures playing a fundamental role (see [4], [5], [8], [14], [16]–[18], [28], [33], [40]] for an incomplete list of literature and [6], [34] for some excellent surveys). In commutative algebra, as far as we know, this line of inquiry has not been exploited thoroughly; however, traces of it can be found in [31], [35], and [15, §§1 and 2].

Although the existing literature provides excellent starting points and inspiring ideas for this present work, it is not always easy to extract the precise results needed for our intended applications. For instance, while the connections between subfunctors of Ext^1 and certain sub-exact structures on a fixed category are well known [14, §1.2], checking the conditions of substructures in each case can be time-consuming.

We are able to find criteria that can be applied in broad settings to identify exact subcategories and subfunctors. Here is a sample result applicable to our motivating example above, which follows from Theorem 4.8 and Proposition 3.8.

THEOREM 1.1. *Let $(\mathcal{A},\mathcal{E})$ be an exact category. Let $\phi:\mathcal{A}\to\mathbb{Z}$ be a function such that ϕ is constant on isomorphism classes of objects in $\mathcal{A}$, ϕ is additive on finite biproducts, and ϕ is sub-additive on kernel–cokernel pairs in $\mathcal{E}$ (i.e., if $M \rightarrowtail N \twoheadrightarrow L$ is in $\mathcal{E}$, then $\phi(N)\le\phi(M)+\phi(L)$). Set $\mathcal{E}^{\phi}:=\{$kernel–cokernel pairs in $\mathcal{E}$ on which ϕ is additive$\}$. Then $\mathcal{E}^{\phi}$ gives rise, via the Yoneda construction, to a subfunctor of* $\mathrm{Ext}_{\mathcal{E}}(-,-)$.

Our above theorem is partly motivated by, and can be used to recover and extend, recent interesting work of Puthenpurakal in [31] (see Theorem 4.17).

Another situation we would like to have convenient criteria for subfunctors is when one *restricts* to a certain subcategory. A concrete example we have in mind concerns Ulrich modules and their recent generalizations. These modules form a subcategory of Cohen–Macaulay modules over a commutative ring and have been receiving increasing attention over the years due to very interesting and useful algebraic and geometric properties that their existence or abundance imply. We are able to show that even the generalized notion of *I-Ulrich modules*, recently introduced in [11], induces subfunctors of $\mathrm{Ext}^1_R(-,-)$. See Proposition 4.1 and Corollary 4.7 in this regard.

While this work is mainly concerned with foundational results, we also study the properties of some chosen new subfunctors, just to see if they are worth our efforts to show their existence! The early returns seem promising: these functors can be used to detect a wide range of ring and module-theoretic properties. Below we shall describe the organization of the paper and describe the most interesting findings in more detail.

Section 2 is devoted to preliminary results on subfunctors of additive functors. While these results are perhaps not new, we were unable to locate convenient references, hence their inclusion.

Section 3 establishes various foundational results on exact subcategories, which form the cornerstone of the theory. As mentioned above, our applications require some extra care in preparation, and we try to give complete proofs whenever possible.

Section 4 concerns our first main application. We study sub-additive numerical functions ϕ on an exact category and show that under mild conditions they induce exact subcategories and hence subfunctors of $\mathrm{Ext}^1(-,-)$, which we denote by $\mathrm{Ext}^1(-,-)^{\phi}$. See Theorem 4.8. A key consequence, Theorem 4.17, is inspired by, as well as extends, [31, Ths. 3.11 and 3.13]. We also give similar results about subfunctors of Ext^1 induced by half-exact functors, in the spirit of [1] (see Corollaries 4.13 and 4.14).

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In §5, we focus on two special types of subfunctors, which arise from simple applications of previous sections. Already these cases appear to be interesting and useful. The first one is $\mathrm{Ext}^1_R(-,-)^\mu$, where μ is the minimal number of generators function mentioned above. We compute the values of this subfunctor on all (pair of) finitely generated modules over a discrete valuation ring, i.e., DVR (Corollary 5.1.14), as well as for certain pairs of modules over a Cohen–Macaulay ring of minimal multiplicity (Proposition 5.1.22).

Using this subfunctor, we prove the following characterization of the regularity of local rings, which is the combination of Theorems 5.1.1 and 5.1.12, Proposition 5.1.14, and Corollary 5.2.7. Note that the regularity can be detected by the vanishing of a single $\mathrm{Ext}^1_R(-,-)^\mu$-module. In the following statement, we mention that for a finitely generated module M over the local ring R, $\Omega^i_R M$ denotes the ith syzygy in a minimal free resolution of M.

THEOREM 1.2. *Let $(R,\mathfrak{m},k)$ be a local ring of depth $t>0$. Then, the following are equivalent:*

(1) *R is regular.*
(2) $\mathrm{Ext}^1_R(k,R/(x_1,\dots,x_{t-1})R)^\mu=0$ *for some R-regular sequence $x_1,\dots,x_{t-1}$.*
(3) $\mathrm{Ext}^1_R(k,M)^\mu=0$ *for some finitely generated R-module M of projective dimension $t-1$.*
(4) *R is Cohen–Macaulay, and $\mathrm{Ext}^1_R(\Omega^{t-1}_R k,N)^\mu=0$ for some finitely generated nonzero R-module N of finite injective dimension.*

Moreover, if $t=1$, then the above are also equivalent to each of the following.

(5) $\mathrm{Ext}^1_R(M,N)^\mu=\mathfrak{m}\mathrm{Ext}^1_R(M,N)$ *for all finitely generated R-modules M and N.*
(6) *R is Cohen–Macaulay, and there exist Ulrich modules M,N such that N is faithful, $M\neq 0$, and for every R-module X that fits into a short exact sequence $0\to N\to X\to M\to 0$, one has X is also an Ulrich module.*

We note here that the usual Ext-modules (without the μ) in (2)–(4) of the above theorem are always nonzero. Moreover, the statement of part (6) of Theorem 1.2 apparently has nothing to do with subfunctor of Ext^1, but we do not know a proof of (1) $\iff$ (6) (which is contained in Corollary 5.2.7) without resorting to $\mathrm{Ext}^1_{\mathrm{Ul}(R)}(-,-):\mathrm{Ul}(R)^{op}\times\mathrm{Ul}(R)\to \mathrm{mod}\, R$.

It is worth mentioning that one can also use $\mathrm{Ext}^1_R(-,-)^\mu$ to detect the property of R being a hypersurface of minimal multiplicity (Corollary 5.1.23) or the weak $\mathfrak{m}$-fullness of a submodule (Proposition 5.1.11).

The second type arises from I-Ulrich modules, where I is any $\mathfrak{m}$-primary ideal in a Noetherian local ring $(R,\mathfrak{m})$. See Definition 4.5 and Corollary 4.7 for the definition of I-Ulrich modules and the fact that they form an exact category. In Theorem 5.2.14, using a subfunctor of Ext^1 corresponding to Ulrich modules over one-dimensional local Cohen–Macaulay rings, we give some characterizations of modules belonging to $\mathrm{add}_R(B(\mathfrak{m}))$ and also characterize when $B(\mathfrak{m})$ is a Gorenstein ring in terms of annihilator of Ext^1_R of Ulrich modules. Here, $B(-)$ denotes the blow-up. We give some applications of Theorem 5.2.14, one of which relates annihilation of $\mathrm{Ext}^1_R(\mathrm{Ul}(R),\mathfrak{m})$ with that of $\mathrm{Ext}^1_R(\mathrm{Ul}_\omega(R),B(\omega))$ (see Corollary 5.2.17).

Finally, we should mention that one of the main applications of our results has appeared in a separate work, where we study the splitting of short exact sequences of Ulrich modules and connections to other properties of singularities (see [9]).

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§2. Preliminaries on subfunctors of additive functors

Unless otherwise stated, all rings in this paper are assumed to be commutative, Noetherian, and with unity. For a ring R, $Q(R)$ will denote its total ring of fractions. For an R-module M, $\lambda_R(M)$ will denote its length. For a finitely generated R-module M, and $i \geq 1$, by $\Omega^i_R M$ we mean $\mathrm{Im} f_i$, where we have an exact sequence $F_i \xrightarrow{f_i} F_{i-1} \xrightarrow{f_{i-1}} \cdots \to F_0 \to M \to 0$, with each F_j being a finitely generated projective R-module. When R is moreover local, we choose this so that $\mathrm{Im}(f_j) \subseteq \mathfrak{m}F_{j-1}$ for each j (see [3, Prop. 1.3.1]).

For definitions, and basic properties of additive categories, additive functors, R-linear categories, and R-linear functors, we refer the reader to [38, Tags 09SE, 010M, and 09MI]. We now recall the definition of subfunctors as in [29].

Definition 2.1. Let $\mathcal{A},\mathcal{B}$ be two categories. Let $F : \mathcal{A} \to \mathcal{B}$ be a covariant (resp. contravariant) functor. A covariant (resp. contravariant) functor $G : \mathcal{A} \to \mathcal{B}$ is called a subfunctor of F if for every $M \in \mathcal{A}$, there exists a monomorphism $j_M : G(M) \to F(M)$, and moreover, for every $M, N \in \mathcal{A}$, and $f \in \mathrm{Mor}_{\mathcal{A}}(M,N)$, the following diagrams are commutative, where the left one stands for the covariant case and the right one for the contravariant case:

$$\begin{array}{ccc} F(M) & \xrightarrow{F(f)} & F(N) \\ {\scriptstyle j_M}\uparrow & & \uparrow{\scriptstyle j_N} \\ G(M) & \xrightarrow{G(f)} & G(N) \end{array} \qquad\qquad \begin{array}{ccc} F(N) & \xrightarrow{F(f)} & F(M) \\ {\scriptstyle j_N}\uparrow & & \uparrow{\scriptstyle j_M} \\ G(N) & \xrightarrow{G(f)} & G(M) \end{array}$$

2.2. For our purposes, we will always take $\mathcal{B}$ to be either the category of abelian groups **Ab**, Mod R or mod R (hence monomorphisms are just injective morphisms) for some commutative ring R, and j_M will usually be just the inclusion map.

The following lemma is probably well known, but we could not find an appropriate reference; hence, we include a proof. This will be used throughout the remainder of the article, possibly without further reference.

Lemma 2.3.

(1) *Subfunctor of an additive functor, between two additive categories, is additive.*
(2) *Subfunctor of an R-linear functor, between two R-linear categories, is R-linear.*

Proof. We will only prove the covariant case of both, since the contravariant case is similar.

(1) Let $\mathcal{A},\mathcal{B}$ be two additive categories, and let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor. Also, let $G : \mathcal{A} \to \mathcal{B}$ be a subfunctor of F. Then we have to show that the map $G : \mathrm{Mor}_{\mathcal{A}}(X,Y) \to \mathrm{Mor}_{\mathcal{B}}(G(X),G(Y))$ is a homomorphism of abelian groups for all $X,Y \in \mathcal{A}$. Fix two objects $X,Y \in \mathcal{A}$. Then we need to prove that $G(f+g) = G(f)+G(g)$ for all $f,g \in \mathrm{Mor}_{\mathcal{A}}(X,Y)$. Since G is a subfunctor of F, we have the following commutative diagrams:

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f+g)} & F(Y) \\ {\scriptstyle j_X}\uparrow & & \uparrow{\scriptstyle j_Y} \\ G(X) & \xrightarrow{G(f+g)} & G(Y) \end{array} \quad \begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle j_X}\uparrow & & \uparrow{\scriptstyle j_Y} \\ G(X) & \xrightarrow{G(f)} & G(Y) \end{array} \quad \begin{array}{ccc} F(X) & \xrightarrow{F(g)} & F(Y) \\ {\scriptstyle j_X}\uparrow & & \uparrow{\scriptstyle j_Y} \\ G(X) & \xrightarrow{G(g)} & G(Y) \end{array}$$

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Then we get

$$\begin{aligned}
j_Y \circ G(f+g) &= F(f+g) \circ j_X \\
&= (F(f)+F(g)) \circ j_X \text{ [Since } F \text{ is additive]} \\
&= F(f) \circ j_X + F(g) \circ j_X \\
&= j_Y \circ G(f) + j_Y \circ G(g) \\
&= j_Y \circ (G(f)+G(g)),
\end{aligned}$$

where the third and fifth equalities follow from the fact that $\mathcal{B}$ is an additive category. Now, since j_Y is a monomorphism, so $j_Y \circ G(f+g) = j_Y \circ (G(f)+G(g))$ implies that $G(f+g) = G(f)+G(g)$. Hence, G is additive.

(2) Let $\mathcal{A}, \mathcal{B}$ be two R-linear categories, and let $F : \mathcal{A} \to \mathcal{B}$ be an R-linear functor. Also, let $G : \mathcal{A} \to \mathcal{B}$ be a subfunctor of F. Then we have to show that the map $G : \operatorname{Mor}_{\mathcal{A}}(X,Y) \to \operatorname{Mor}_{\mathcal{B}}(G(X), G(Y))$ is an R-linear map for all $X, Y \in \mathcal{A}$. Fix two objects $X, Y \in \mathcal{A}$. Then, by part (1), we already have the additivity of G, so we only need to prove that $G(rf) = rG(f)$ for all $f \in \operatorname{Mor}_{\mathcal{A}}(X,Y)$ and for all $r \in R$. Since G is a subfunctor of F, we have the following commutative diagrams:

$$\begin{array}{ccc}
F(X) & \xrightarrow{F(rf)} & F(Y) \\
{\scriptstyle j_X}\uparrow & & {\scriptstyle j_Y}\uparrow \\
G(X) & \xrightarrow{G(rf)} & G(Y)
\end{array}
\qquad
\begin{array}{ccc}
F(X) & \xrightarrow{F(f)} & F(Y) \\
{\scriptstyle j_X}\uparrow & & {\scriptstyle j_Y}\uparrow \\
G(X) & \xrightarrow{G(f)} & G(Y)
\end{array}$$

Then we get

$$\begin{aligned}
j_Y \circ G(rf) &= F(rf) \circ j_X \\
&= (rF(f)) \circ j_X \text{ [Since } F \text{ is } R\text{-linear]} \\
&= r(F(f) \circ j_X) \\
&= r(j_Y \circ G(f)) \\
&= j_Y \circ (rG(f)),
\end{aligned}$$

where the third and fifth equalities follow from the fact that $\mathcal{B}$ is an R-linear category. Now, since j_Y is a monomorphism, so $j_Y \circ G(rf) = j_Y \circ (rG(f))$ implies that $G(rf) = rG(f)$. Hence, G is R-linear. ∎

We finish this section with a submodule inclusion result relating subfunctors of R-linear functors which will be applied in §5.

LEMMA 2.4. *Let $\mathcal{A}$ be an additive R-linear category, and let $G : \mathcal{A} \to \operatorname{mod} R$ be a subfunctor of an R-linear functor $F : \mathcal{A} \to \operatorname{Mod} R$, and for every object $A \in \mathcal{A}$, let $j_A : G(A) \to F(A)$ be the monomorphism as in the definition of a subfunctor. Let I be an ideal of R. Let $\{A_i\}_{i=1}^n$ be objects in $\mathcal{A}$ such that $j_{A_i}(G(A_i)) \subseteq IF(A_i)$ (resp. $j_{A_i}(G(A_i)) \supseteq IF(A_i)$) for all $i = 1, \ldots, n$. Let $X := \oplus_{i=1}^n A_i$. Then, it holds that $j_X(G(X)) \subseteq IF(X)$ (resp. $j_X(G(X)) \supseteq IF(X)$).*

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Proof. We will only do the covariant case, the contravariant case being similar. For every i, we have $\pi_i : X \to A_i$, which gives rise to the following commutative diagram:

$$\begin{tikzcd} G(X) \arrow[r, "G(\pi_i)"] \arrow[d, "j_X"] & G(A_i) \arrow[d, "j_{A_i}"] \\ F(X) \arrow[r, "F(\pi_i)"'] & F(A_i) \end{tikzcd}$$

Consequently, we get the following commutative diagram:

$$\begin{tikzcd} G(X) \arrow[r, "(G(\pi_i))_{i=1}^n"] \arrow[d, "j_X"] & \oplus_{i=1}^n G(A_i) \arrow[d, "\oplus_{i=1}^n j_{A_i}"] \\ F(X) \arrow[r, "(F(\pi_i))_{i=1}^n"'] & \oplus_{i=1}^n F(A_i) \end{tikzcd}$$

where the horizontal arrows are isomorphisms, since F and consequently G are additive functors (see Lemma 2.3). Call the top horizontal map θ, and the bottom one α, so that $\alpha \circ j_X = (\oplus_{i=1}^n j_{A_i}) \circ \theta$; hence, $j_X \circ \theta^{-1} = \alpha^{-1} \circ (\oplus_{i=1}^n j_{A_i})$. So, now, we get

$$\begin{aligned} j_X(G(X)) = (j_X \circ \theta^{-1})(\theta(G(X))) = (j_X \circ \theta^{-1})(\oplus_{i=1}^n G(A_i)) &= (\alpha^{-1} \circ (\oplus_{i=1}^n j_{A_i}))(\oplus_{i=1}^n G(A_i)) \\ &= \alpha^{-1}(\oplus_{i=1}^n j_{A_i}(G(A_i))). \end{aligned}$$

So, if $j_{A_i}(G(A_i)) \subseteq IF(A_i)$ (resp. $j_{A_i}(G(A_i)) \supseteq IF(A_i)$) for all $i = 1, \ldots, n$, then

$$j_X(G(X)) \subseteq \alpha^{-1}(\oplus_{i=1}^n IF(A_i)) = \alpha^{-1}(I(\oplus_{i=1}^n F(A_i))) = I\alpha^{-1}(\oplus_{i=1}^n F(A_i)) = IF(X)$$

(resp. $j_X(G(X)) \supseteq \alpha^{-1}(\oplus_{i=1}^n IF(A_i)) = \alpha^{-1}(I(\oplus_{i=1}^n F(A_i))) = I\alpha^{-1}(\oplus_{i=1}^n F(A_i)) = IF(X)$),

where we have used $\alpha^{-1}(IM) = I\alpha^{-1}(M)$, since α is an R-linear map. ∎

§3. Some generalities about exact subcategories

In this section, we record some generalities about exact subcategories of an exact category that we will later use for subcategories of mod R when R is a commutative Noetherian ring. All our subcategories are strict (closed under isomorphism classes) and full, and we often abbreviate this as strictly full. We will follow the definition of an exact category described in [6, Def. 2.1]. We try to provide complete proofs whenever possible.

Given an exact category $(\mathcal{A}, \mathcal{E})$, we call a monomorphism $X \xrightarrow{i} Y$ to be an $\mathcal{E}$-inflation (also, called an admissible monic) if it is the part of a kernel–cokernel pair $X \xrightarrow{i} Y \to Z$, which lies in $\mathcal{E}$. Dually, we call an epimorphism $Y \xrightarrow{p} Z$ to be an $\mathcal{E}$-deflation (also, called an admissible epic) if it is the part of a kernel–cokernel pair $X \to Y \xrightarrow{p} Z$, which lies in $\mathcal{E}$. We will often denote an admissible monic (resp. an admissible epic) by $\rightarrowtail$ (resp. $\twoheadrightarrow$).

We begin by stating a lemma on morphisms and kernel–cokernel pairs, which we will use frequently while proving that certain structures are closed under isomorphism classes of kernel–cokernel pairs. This should be standard and well known, but we could not find an appropriate reference; hence, we include a proof.

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LEMMA 3.1. *Let $\mathcal{A}$ be an additive category. Let $M,N,L \in \mathcal{A}$ be such that $M \xrightarrow{i} N \xrightarrow{d} L$ is a kernel–cokernel pair in $\mathcal{A}$. Also, let $M',N',L' \in \mathcal{A}$. Consider morphisms $M' \xrightarrow{i'} N'$, $N' \xrightarrow{d'} L'$. If we have the following diagram with commutative squares:*

$$\begin{tikzcd} M \arrow[r, "i"] \arrow[d, "\phi_1"] & N \arrow[r, "d"] \arrow[d, "\phi_2"] & L \arrow[d, "\phi_3"] \\ M' \arrow[r, "i'"] & N' \arrow[r, "d'"] & L' \end{tikzcd}$$

where the vertical arrows ϕ_1,ϕ_2,ϕ_3 are isomorphisms, then $M' \xrightarrow{i'} N' \xrightarrow{d'} L'$ is a kernel–cokernel pair in $\mathcal{A}$.

Proof. We will only prove that $\ker(d') = i'$, since the proof of $\operatorname{Coker}(i') = d'$ can be given by a dual argument. From the above commutative diagram, we have

$$\phi_2 \circ i = i' \circ \phi_1, \quad \phi_3 \circ d = d' \circ \phi_2. \tag{3.1}$$

Since $M \xrightarrow{i} N \xrightarrow{d} L$ is a kernel–cokernel pair in $\mathcal{A}$, we get $d \circ i = 0$. Then, by Equation (3.1), we have $d' \circ i' = (\phi_3 \circ d \circ \phi_2^{-1}) \circ (\phi_2 \circ i \circ \phi_1^{-1}) = \phi_3 \circ d \circ i \circ \phi_1^{-1} = \phi_3 \circ 0 \circ \phi_1^{-1} = 0$. Now, let $K \in \mathcal{A}$ and consider a morphism $f' : K \to N'$ such that $d' \circ f' = 0$. Define $f := \phi_2^{-1} \circ f' : K \to N$. Then, by Equation (3.1), we get that $d \circ f = d \circ (\phi_2^{-1} \circ f') = (d \circ \phi_2^{-1}) \circ f' = (\phi_3^{-1} \circ d') \circ f' = \phi_3^{-1} \circ (d' \circ f') = \phi_3^{-1} \circ 0 = 0$. Now, since $\ker(d) = i$, by the universal property of a kernel of a map, there exists a morphism $u : K \to M$ such that $i \circ u = f$. Now, define $u' := \phi_1 \circ u : K \to M'$. Then, by Equation (3.1), we get that $i' \circ u' = i' \circ (\phi_1 \circ u) = (i' \circ \phi_1) \circ u = (\phi_2 \circ i) \circ u = \phi_2 \circ f = \phi_2 \circ (\phi_2^{-1} \circ f') = f'$. This implies that $\ker(d') = i'$. ∎

We now record a lemma on the intersection of exact subcategories. Note that this is slightly different (and in view of [33, Cor. 2], more general) than [4, Lem. 5.2].

LEMMA 3.2. *Let $(\mathcal{A},\mathcal{E})$ be an exact category. Also, let $(\mathcal{A}_\lambda,\mathcal{E}_\lambda)$ be an arbitrary family of exact subcategories of $(\mathcal{A},\mathcal{E})$. Then $(\cap_\lambda \mathcal{A}_\lambda, \cap_\lambda \mathcal{E}_\lambda)$ is exact subcategory of $(\mathcal{A},\mathcal{E})$.*

Proof. Clearly, $(\cap_\lambda \mathcal{A}_\lambda, \cap_\lambda \mathcal{E}_\lambda)$ is a strictly full additive subcategory of $(\mathcal{A},\mathcal{E})$. By [6, Def. 2.1] and [6, Rem. 2.4], it is enough to show that $\cap_\lambda \mathcal{E}_\lambda$ is closed under isomorphisms of kernel–cokernel pairs and $(\cap_\lambda \mathcal{A}_\lambda, \cap_\lambda \mathcal{E}_\lambda)$ satisfies the axioms [E0], [E0$^{\text{op}}$], [E1$^{\text{op}}$], [E2], and [E2$^{\text{op}}$] of [6, Def. 2.1]. First, we will show that $\cap_\lambda \mathcal{E}_\lambda$ is closed under isomorphisms of kernel–cokernel pairs. Let $M \rightarrowtail N \twoheadrightarrow L$ be a kernel–cokernel pair in $\cap_\lambda \mathcal{E}_\lambda$, so $M,N,L \in \mathcal{A}_\lambda$ and $M \rightarrowtail N \twoheadrightarrow L$ is a kernel–cokernel pair in $\mathcal{A}_\lambda$ for all λ. Also, let $M' \to N' \to L'$ be a kernel–cokernel pair in $\cap_\lambda \mathcal{A}_\lambda$ such that it is isomorphic to $M \rightarrowtail N \twoheadrightarrow L$. So, $M',N',L' \in \mathcal{A}_\lambda$ and $M' \to N'$, $N' \to L'$ are two morphisms in $\mathcal{A}_\lambda$ for all λ. Hence, by Lemma 3.1, we get that $M' \to N' \to L'$ is a kernel–cokernel pair in $\mathcal{A}_\lambda$ for all λ. Since $(\mathcal{A}_\lambda,\mathcal{E}_\lambda)$ is an exact category and $M \rightarrowtail N \twoheadrightarrow L$ is in $\mathcal{E}_\lambda$ for all λ, we have $M' \rightarrowtail N' \twoheadrightarrow L'$ is in $\mathcal{E}_\lambda$ for all λ. So, $M' \rightarrowtail N' \twoheadrightarrow L'$ is in $\cap_\lambda \mathcal{E}_\lambda$. Thus, $\cap_\lambda \mathcal{E}_\lambda$ is closed under isomorphisms. Next, we will show that $(\cap_\lambda \mathcal{A}_\lambda, \cap_\lambda \mathcal{E}_\lambda)$ satisfies the axiom [E0]. Let $A \in \cap_\lambda \mathcal{A}_\lambda$. Since $(\mathcal{A}_\lambda,\mathcal{E}_\lambda)$ is an exact category for all λ, by [6, Lem. 2.7], we get that $A \overset{1_A}{\rightarrowtail} A \oplus 0 \cong A \overset{0}{\twoheadrightarrow} 0$ is in $\mathcal{E}_\lambda$ for all λ. As $\mathcal{A}_\lambda$ is an additive subcategory of

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$\mathcal{A}$ for all λ, so $0_{\mathcal{A}} = 0_{\mathcal{A}_\lambda}$ for all λ. Hence, by definition, $A \overset{1_A}{\rightarrowtail} A \oplus 0 \cong A \overset{0}{\twoheadrightarrow} 0$ is in $\cap_\lambda \mathcal{E}_\lambda$, which implies that 1_A is an admissible monic in $\cap_\lambda \mathcal{E}_\lambda$. So, $(\cap_\lambda \mathcal{A}_\lambda, \cap_\lambda \mathcal{E}_\lambda)$ satisfies the axiom [E0]. Next, we will show that $(\cap_\lambda \mathcal{A}_\lambda, \cap_\lambda \mathcal{E}_\lambda)$ satisfies the axiom [E0$^{\text{op}}$]. Let $A \in \cap_\lambda \mathcal{A}_\lambda$. Since $(\mathcal{A}_\lambda, \mathcal{E}_\lambda)$ is an exact category for all λ, by [6, Lem. 2.7], we get that $0 \overset{0}{\rightarrowtail} 0 \oplus A \cong A \overset{1_A}{\twoheadrightarrow} A$ is in $\mathcal{E}_\lambda$ for all λ. As $\mathcal{A}_\lambda$ is an additive subcategory of $\mathcal{A}$ for all λ, so $0_{\mathcal{A}} = 0_{\mathcal{A}_\lambda}$ for all λ. Hence, by definition $0 \overset{0}{\rightarrowtail} A \oplus 0 \cong A \overset{1_A}{\twoheadrightarrow} A$ is in $\cap_\lambda \mathcal{E}_\lambda$, which implies that 1_A is an admissible epic in $\cap_\lambda \mathcal{E}_\lambda$. So, $(\cap_\lambda \mathcal{A}_\lambda, \cap_\lambda \mathcal{E}_\lambda)$ satisfies the axiom [E0$^{\text{op}}$]. Next, we will show that $(\cap_\lambda \mathcal{A}_\lambda, \cap_\lambda \mathcal{E}_\lambda)$ satisfies the axiom [E1$^{\text{op}}$]. Let $B' \overset{e}{\twoheadrightarrow} B$ and $B \overset{p}{\twoheadrightarrow} C$ be two admissible epics in $\cap_\lambda \mathcal{E}_\lambda$. Then we will show that $B' \overset{p \circ e}{\twoheadrightarrow} C$ is an admissible epic in $\cap_\lambda \mathcal{E}_\lambda$. Now, $B' \overset{e}{\twoheadrightarrow} B$ and $B \overset{p}{\twoheadrightarrow} C$ are admissible epics in $\mathcal{E}_\lambda$ for all λ. Since $(\mathcal{A}_\lambda, \mathcal{E}_\lambda)$ is an exact category for all λ, we get $B' \overset{p \circ e}{\twoheadrightarrow} C$ is an admissible epic in $\mathcal{E}_\lambda$ for all λ. Hence, there exist objects $D_\lambda \in \mathcal{A}_\lambda$ and kernel–cokernel pairs $\sigma_\lambda :\ D_\lambda \overset{i_\lambda}{\rightarrowtail} B' \overset{p \circ e}{\twoheadrightarrow} C$ in $\mathcal{E}_\lambda$ for all λ. So, $D_\lambda \overset{i_\lambda}{\rightarrowtail} B' \overset{p \circ e}{\twoheadrightarrow} C$ is in $\mathcal{E}$ for all λ. Hence, σ_λ is a kernel–cokernel pair in $\mathcal{A}$ for all λ, so by the universal property of kernels, we get that the kernel–cokernel pairs σ_λ's are all isomorphic to each other. Hence, all the D_λ's are isomorphic to each other. Now, fix a λ, say λ_0. Then $D_{\lambda_0} \cong D_\lambda$ for all λ. Since $\mathcal{A}_\lambda$ is a strict subcategory of $\mathcal{A}$ for all λ, we have $D_{\lambda_0} \in \cap_\lambda \mathcal{A}_\lambda$. Now, $\mathcal{A}_\lambda$ is a strictly full subcategory of $\mathcal{A}$ for all λ, so $D_{\lambda_0} \in \mathcal{A}_\lambda$ implies that σ_{λ_0} is a kernel–cokernel pair in $\mathcal{A}_\lambda$ for all λ. Since $\mathcal{E}_\lambda$ is closed under isomorphisms of kernel–cokernel pairs in $\mathcal{A}_\lambda$ and σ_λ's are all isomorphic to each other, we get $\sigma_{\lambda_0} \in \mathcal{E}_\lambda$ for all λ. So, $\sigma_{\lambda_0} \in \cap_\lambda \mathcal{E}_\lambda$. Hence, $D_{\lambda_0} \overset{i_{\lambda_0}}{\rightarrowtail} B' \overset{p \circ e}{\twoheadrightarrow} C$ is in $\cap_\lambda \mathcal{E}_\lambda$, so $B' \overset{p \circ e}{\twoheadrightarrow} C$ is an admissible epic in $\cap_\lambda \mathcal{E}_\lambda$. Thus, $(\cap_\lambda \mathcal{A}_\lambda, \cap_\lambda \mathcal{E}_\lambda)$ satisfies the axiom [E1$^{\text{op}}$]. Now, we will show that $(\cap_\lambda \mathcal{A}_\lambda, \cap_\lambda \mathcal{E}_\lambda)$ satisfies the axiom [E2]. Let $A \overset{i}{\rightarrowtail} B$ be an admissible monic in $\cap_\lambda \mathcal{E}_\lambda$, and let $A \xrightarrow{f} A'$ be an arbitrary morphism in $\cap_\lambda \mathcal{A}_\lambda$. Now, $A \overset{i}{\rightarrowtail} B$ is an admissible monic in $\mathcal{E}_\lambda$ for all λ. Since $(\mathcal{A}_\lambda, \mathcal{E}_\lambda)$ is an exact category for all λ, by [6, Prop. 2.12(iv)], we have the following pushout commutative diagram with rows being kernel–cokernel pairs in $\mathcal{E}_\lambda$:

$$\begin{tikzcd} & A \arrow[r, tail, "i"] \arrow[d, "f"'] \arrow[dr, phantom, "\text{PO}"] & B \arrow[r, two heads, "p"] \arrow[d, "f'_\lambda"] & C \arrow[d, equal] \\ \beta_\lambda : & A' \arrow[r, tail, "i'_\lambda"] & B'_\lambda \arrow[r, two heads, "p'_\lambda"] & C \end{tikzcd} \quad \text{(Note that } C \in \cap_\lambda \mathcal{A}_\lambda.)$$

Since $(\mathcal{A}_\lambda, \mathcal{E}_\lambda)$ is an exact subcategory of $(\mathcal{A}, \mathcal{E})$, by [6, Prop. 5.2], we get that the square

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "f"'] & B \arrow[d, "f'_\lambda"] \\ A' \arrow[r, tail, "i'_\lambda"] & B'_\lambda \end{tikzcd}$$

is a pushout square in $\mathcal{E}$ for all λ. Then, by the universal property of pushout, we get that the kernel–cokernel pairs β_λ's are all isomorphic to each other, so all the B'_λ's are isomorphic to each other. Now, fix a λ, say λ_0. Then $B'_{\lambda_0} \cong B'_\lambda$ for all λ. Since $\mathcal{A}_\lambda$ is a strict subcategory of $\mathcal{A}$ for all λ, we have $B'_{\lambda_0} \in \cap_\lambda \mathcal{A}_\lambda$. Now, $\mathcal{A}_\lambda$ is a strictly full subcategory of $\mathcal{A}$ for all λ, so $B'_{\lambda_0} \in \mathcal{A}_\lambda$ implies that β_{λ_0} is a kernel–cokernel pair in $\mathcal{A}_\lambda$ for all λ. Since $\mathcal{E}_\lambda$ is closed under isomorphisms of kernel–cokernel pairs in $\mathcal{A}_\lambda$ and β_λ's are all isomorphic to each other, we have $\beta_{\lambda_0} \in \mathcal{E}_\lambda$ for all λ. So, $\beta_{\lambda_0} \in \cap_\lambda \mathcal{E}_\lambda$. Hence, $A' \overset{i'_{\lambda_0}}{\rightarrowtail} B'_{\lambda_0} \overset{p'_{\lambda_0}}{\twoheadrightarrow} C$ is in $\cap_\lambda \mathcal{E}_\lambda$, so $A' \overset{i'_{\lambda_0}}{\rightarrowtail} B'_{\lambda_0}$ is an admissible monic in $\cap_\lambda \mathcal{E}_\lambda$.

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Thus, $(\cap_\lambda \mathcal{A}_\lambda, \cap_\lambda \mathcal{E}_\lambda)$ satisfies the axiom [E2]. A dual argument will show that $(\cap_\lambda \mathcal{A}_\lambda, \cap_\lambda \mathcal{E}_\lambda)$ satisfies the axiom [E2$^{\mathrm{op}}$]. Hence, $(\cap_\lambda \mathcal{A}_\lambda, \cap_\lambda \mathcal{E}_\lambda)$ is an exact subcategory of $(\mathcal{A}, \mathcal{E})$. ∎

Next, we record a useful lemma for proving the exactness of certain structures on additive subcategories of a given exact category.

LEMMA 3.3. *Let $(\mathcal{A}, \mathcal{E})$ be an exact category. Let $\mathcal{A}'$ be a strictly full additive subcategory of $\mathcal{A}$. Define $\mathcal{E}|_{\mathcal{A}'} := \{M \rightarrowtail N \twoheadrightarrow L \text{ is in } \mathcal{E} : M, N, L \in \mathcal{A}'\}$. If the pullback (resp. pushout) in $\mathcal{A}$ of every $\mathcal{E}$-deflation (resp. inflation) of sequences in $\mathcal{E}|_{\mathcal{A}'}$ along every morphism in $\mathcal{A}'$ is again in $\mathcal{A}'$, then $(\mathcal{A}', \mathcal{E}|_{\mathcal{A}'})$ is an exact subcategory of $(\mathcal{A}, \mathcal{E})$.*

The proof of Lemma 3.3 depends on Lemma 3.1 and the following proposition.

PROPOSITION 3.4. *Let $(\mathcal{A}, \mathcal{E})$ be an exact category. Suppose that we have the following diagram:*

$$\begin{tikzcd} A' \arrow[r, "i'"] \arrow[d, two heads, "e'"] \arrow[dr, phantom, "PB"] & B' \arrow[d, two heads, "e"] & \\ A \arrow[r, tail, "i"] & B \arrow[r, two heads, "p"] & C \end{tikzcd}$$

where the square commutative diagram is a pullback diagram and e, p are admissible epics and i is a kernel of p. Then i' is an admissible monic with a cokernel given by $B' \xrightarrow{p \circ e} C$.

Proof. The proof follows from the construction in the proof of [6, Prop. 2.15]. The only missing point in the proof of [6, Prop. 2.15], toward showing i' is a kernel of $p \circ e$, is the following: it was not shown that $(p \circ e) \circ i' = 0$. This can be easily checked as follows: $(p \circ e) \circ i' = p \circ (e \circ i') = p \circ (i \circ e') = (p \circ i) \circ e' = 0 \circ e' = 0$, since i is a kernel of p. ∎

Proof of Lemma 3.3. Clearly, each kernel–cokernel pair in $\mathcal{E}|_{\mathcal{A}'}$ is a kernel–cokernel pair in $\mathcal{A}$, hence also a kernel–cokernel pair in $\mathcal{A}'$. Thus, $\mathcal{E}|_{\mathcal{A}'}$ consists of kernel–cokernel pairs in $\mathcal{A}'$. Now, we will show that $(\mathcal{A}', \mathcal{E}|_{\mathcal{A}'})$ satisfies the axiom [E2]. Let $A, B, A' \in \mathcal{A}'$, let $A \overset{i}{\rightarrowtail} B$ be an admissible monic in $\mathcal{E}|_{\mathcal{A}'}$, and let $A \xrightarrow{f} A'$ be an arbitrary morphism in $\mathcal{A}'$. Now, $A \overset{i}{\rightarrowtail} B$ is an admissible monic in $\mathcal{E}$. Since $(\mathcal{A}, \mathcal{E})$ is an exact category, by [6, Prop. 2.12(iv)], we have the following pushout commutative diagram with rows being kernel–cokernel pairs in $\mathcal{E}$:

$$\begin{tikzcd} & A \arrow[r, tail, "i"] \arrow[d, "f"] \arrow[dr, phantom, "\mathrm{PO}"] & B \arrow[r, two heads, "p"] \arrow[d, "f'"] & C \arrow[d, equal] \\ \beta: & A' \arrow[r, tail, "i'"] & B' \arrow[r, two heads, "p'"] & C \end{tikzcd}$$

Now, by the assumption, we have $B' \in \mathcal{A}'$. Since $A \overset{i}{\rightarrowtail} B$ is an admissible monic in $\mathcal{E}|_{\mathcal{A}'}$, we have $C \in \mathcal{A}'$. Thus, $\beta \in \mathcal{E}|_{\mathcal{A}'}$. Since $\mathcal{A}'$ is a strictly full subcategory of $\mathcal{A}$, the above diagram is a pushout diagram in $\mathcal{A}'$ as well. Thus, $(\mathcal{A}', \mathcal{E}|_{\mathcal{A}'})$ satisfies the axiom [E2]. Similarly, $(\mathcal{A}', \mathcal{E}|_{\mathcal{A}'})$ satisfies the axiom [E2$^{\mathrm{op}}$]. Now, by [6, Def. 2.1] and [6, Rem. 2.4], it is enough to show that $\mathcal{E}|_{\mathcal{A}'}$ is closed under isomorphisms and $(\mathcal{A}', \mathcal{E}|_{\mathcal{A}'})$ satisfies the axioms [E0], [E0$^{\mathrm{op}}$], and [E1$^{\mathrm{op}}$]. First, we will show that $\mathcal{E}|_{\mathcal{A}'}$ is closed under isomorphisms. Let $M \rightarrowtail N \twoheadrightarrow L$ be a kernel–cokernel pair in $\mathcal{E}|_{\mathcal{A}'}$, so $M, N, L \in \mathcal{A}'$ and $M \rightarrowtail N \twoheadrightarrow L$ is a kernel–cokernel pair in $\mathcal{A}$. Also, let $M' \to N' \to L'$ be a

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kernel–cokernel pair in $\mathcal{A}'$ such that it is isomorphic to $M \rightarrowtail N \twoheadrightarrow L$. Hence, by Lemma 3.1, we get that $M' \to N' \to L'$ is a kernel–cokernel pair in $\mathcal{A}$. Since $(\mathcal{A},\mathcal{E})$ is an exact category, we have $M' \rightarrowtail N' \twoheadrightarrow L'$ is in $\mathcal{E}$. Hence, by definition, $M' \rightarrowtail N' \twoheadrightarrow L'$ is in $\mathcal{E}|_{\mathcal{A}'}$. Thus, $\mathcal{E}|_{\mathcal{A}'}$ is closed under isomorphisms. Next, we will show that $(\mathcal{A}',\mathcal{E}|_{\mathcal{A}'})$ satisfies the axiom [E0]. Let $A \in \mathcal{A}'$. Since $(\mathcal{A},\mathcal{E})$ is an exact category, by [6, Lem. 2.7], we get that $A \xrightarrow{1_A} A \oplus 0 \cong A \xrightarrow{0} 0$ is in $\mathcal{E}$. As $\mathcal{A}'$ is an additive subcategory of $\mathcal{A}$, so $0_{\mathcal{A}} = 0_{\mathcal{A}'}$. Hence, by definition, $A \xrightarrow{1_A} A \oplus 0 \cong A \xrightarrow{0} 0$ is in $\mathcal{E}|_{\mathcal{A}'}$, which implies that 1_A is an admissible monic in $\mathcal{E}|_{\mathcal{A}'}$. So, $(\mathcal{A}',\mathcal{E}|_{\mathcal{A}'})$ satisfies the axiom [E0]. Next, we will show that $(\mathcal{A}',\mathcal{E}|_{\mathcal{A}'})$ satisfies the axiom [E0$^{\mathrm{op}}$]. Let $A \in \mathcal{A}'$. Since $(\mathcal{A},\mathcal{E})$ is an exact category, by [6, Lem. 2.7], we get that $0 \xrightarrow{0} 0 \oplus A \cong A \xrightarrow{1_A} A$ is in $\mathcal{E}$. As $\mathcal{A}'$ is an additive subcategory of $\mathcal{A}$, so $0_{\mathcal{A}} = 0_{\mathcal{A}'}$. Hence, by definition, $0 \xrightarrow{0} A \oplus 0 \cong A \xrightarrow{1_A} A$ is in $\mathcal{E}|_{\mathcal{A}'}$, which implies that 1_A is an admissible epic in $\mathcal{E}|_{\mathcal{A}'}$. So, $(\mathcal{A}',\mathcal{E}|_{\mathcal{A}'})$ satisfies the axiom [E0$^{\mathrm{op}}$]. Next, we will show that $(\mathcal{A}',\mathcal{E}|_{\mathcal{A}'})$ satisfies the axiom [E1$^{\mathrm{op}}$]. Let $B' \xrightarrow{e} B$ and $B \xrightarrow{p} C$ be two admissible epics in $\mathcal{E}$ such that they are in $\mathcal{E}|_{\mathcal{A}'}$, which means that there exist two kernel–cokernel pairs $A \xrightarrow{i} B \xrightarrow{p} C$ and $D \rightarrowtail B' \xrightarrow{e} B$ in $\mathcal{E}$ such that $A,B,C,D,B' \in \mathcal{A}'$. Then we will show that $B' \xrightarrow{p\circ e} C$ is an admissible epic in $\mathcal{E}|_{\mathcal{A}'}$. From Proposition 3.4, we get that $A' \xrightarrow{i'} B' \xrightarrow{p\circ e} C$ is a kernel–cokernel pair in $\mathcal{E}$. By the hypothesis of Lemma 3.3 and the diagram of Proposition 3.4, we get that $A' \in \mathcal{A}'$. Hence, $B' \xrightarrow{p\circ e} C$ is an admissible epic in $\mathcal{E}|_{\mathcal{A}'}$. Thus, $(\mathcal{A}',\mathcal{E}|_{\mathcal{A}'})$ satisfies the axiom [E1$^{\mathrm{op}}$]. So, $(\mathcal{A}',\mathcal{E}|_{\mathcal{A}'})$ is an exact subcategory of $(\mathcal{A},\mathcal{E})$. ∎

From now on, given a subcategory $\mathcal{A}'$ of an exact category $(\mathcal{A},\mathcal{E})$, the notation $\mathcal{E}|_{\mathcal{A}'}$ will stand for as defined in Lemma 3.3. Using Lemma 3.3, we now record two quick consequences, which give a sufficient condition on a subcategory $\mathcal{A}'$ such that $(\mathcal{A}',\mathcal{E}|_{\mathcal{A}'})$ is an exact subcategory of $(\mathcal{A},\mathcal{E})$. The first of which we state below now is well known (see, e.g., [6, Lem. 10.20]). However, due to the absence of a proof in [6, Lem. 10.20], we give a proof using our Lemma 3.3.

Proposition 3.5. *(cf. [6, Lem. 10.20]) Let $(\mathcal{A},\mathcal{E})$ be an exact category. Let $\mathcal{A}'$ be a strictly full additive subcategory of $\mathcal{A}$. Assume that for every $X \to Y \to Z$ in $\mathcal{E}$, if $X,Z \in \mathcal{A}'$, then $Y \in \mathcal{A}'$ (i.e., $\mathcal{A}'$ is closed under extensions). Then $(\mathcal{A}',\mathcal{E}|_{\mathcal{A}'})$ is an exact subcategory of $(\mathcal{A},\mathcal{E})$.*

Proof. By Lemma 3.3, it is enough to show that the pullback (resp. pushout) in $\mathcal{A}$ of every $\mathcal{E}$-deflation (resp. inflation) of sequences in $\mathcal{E}|_{\mathcal{A}'}$ along every morphism in $\mathcal{A}'$ is again in $\mathcal{A}'$. Let $A \xrightarrow{i'} B$ be an inflation in $\mathcal{E}|_{\mathcal{A}'}$, and let $f: A \to A'$ be a morphism in $\mathcal{A}'$. Then, by [6, Prop. 2.12(iv)], we get the following pushout commutative diagram:

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "f"] \arrow[dr, phantom, "\mathrm{PO}"] & B \arrow[r, two heads, "p"] \arrow[d, "f'"] & C \arrow[d, equal] \\ A' \arrow[r, tail, "i'"] & B' \arrow[r, "p'"] & C \end{tikzcd}$$

Since $A', C \in \mathcal{A}'$, by assumption and the bottom row of the above diagram, we get that $B' \in \mathcal{A}'$. The pullback case follows by a dual argument. Hence, $(\mathcal{A}',\mathcal{E}|_{\mathcal{A}'})$ is an exact subcategory of $(\mathcal{A},\mathcal{E})$. ∎

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THEOREM 3.6. *Let $(\mathcal{A},\mathcal{E})$ be an exact category. Let $\mathcal{A}'$ be a strictly full additive subcategory of $\mathcal{A}$. Assume that $\mathcal{A}'$ is closed under kernels and co-kernels of admissible epics and monics in $\mathcal{E}$, respectively. Then, $(\mathcal{A}',\mathcal{E}|_{\mathcal{A}'})$ is an exact subcategory of $(\mathcal{A},\mathcal{E})$.*

Proof. By Lemma 3.3, it is enough to show that the pullback (resp. pushout) in $\mathcal{A}$ of every $\mathcal{E}$-deflation (resp. inflation) of sequences in $\mathcal{E}|_{\mathcal{A}'}$ along every morphism in $\mathcal{A}'$ is again in $\mathcal{A}'$. Let $A \overset{i'}{\rightarrowtail} B$ be an inflation in $\mathcal{E}|_{\mathcal{A}'}$, and let $f : A \to A'$ be a morphism in $\mathcal{A}'$. Now, we have the following pushout commutative diagram in $(A,\mathcal{E})$:

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "f"] \arrow[dr, phantom, "\mathrm{PO}"] & B \arrow[d, "f'"] \\ A' \arrow[r, tail, "i'"] & B' \end{tikzcd}$$

Then, by [6, Prop. 2.12(ii)], we have the following kernel–cokernel pair in $\mathcal{E}$: $A \rightarrowtail B \oplus A' \twoheadrightarrow B'$. As $B, A' \in \mathcal{A}'$ and $\mathcal{A}'$ is additive, so $B \oplus A' \in \mathcal{A}'$. Since $A \rightarrowtail B \oplus A'$ is an admissible monic in $\mathcal{E}$, we get the cokernel $B' \in \mathcal{A}'$. The pullback case follows by a dual argument. Hence, $(\mathcal{A}',\mathcal{E}|_{\mathcal{A}'})$ is an exact subcategory of $(\mathcal{A},\mathcal{E})$. $\square$

Given any exact category $(\mathcal{A},\mathcal{E})$ and $C, A \in \mathcal{A}$, one can define the Yoneda Ext group $\mathrm{Ext}_{\mathcal{E}}(C,A)$, which has an abelian group structure by Baer sum (see the beginning of [14, §1.2] for a description of $\mathrm{Ext}_{\mathcal{E}}(-,-)$). When $(\mathcal{A},\mathcal{E})$ is moreover an R-linear category, then $\mathrm{Ext}_{\mathcal{E}}(C,A)$ can be given an R-linear structure via either of the following constructions, both of which yield equivalent triples in $\mathrm{Ext}_{\mathcal{E}}(C,A)$:

Given a kernel–cokernel pair $\sigma : A \rightarrowtail B \twoheadrightarrow C$ in $\mathrm{Ext}_{\mathcal{E}}(C,A)$, the multiplication $r \cdot \sigma$ is either given by the following pullback diagram:

$$\begin{tikzcd} r\sigma: & A \arrow[r, tail] \arrow[d, equal] & B' \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "\mathrm{PB}"] & C \arrow[d, "r\cdot \mathrm{id}_C"] \\ \sigma: & A \arrow[r, tail] & B \arrow[r, two heads] & C \end{tikzcd}$$

or by the following pushout diagram:

$$\begin{tikzcd} \sigma: & A \arrow[r, tail] \arrow[d, "r\cdot \mathrm{id}_A"'] \arrow[dr, phantom, "\mathrm{PO}"] & B \arrow[r, two heads] \arrow[d] & C \arrow[d, equal] \\ r\sigma: & A \arrow[r, tail] & B' \arrow[r, two heads] & C \end{tikzcd}$$

That both of these yield equivalent triplet in $\mathrm{Ext}_{\mathcal{E}}(-,-)$ follows from [6, Prop. 3.1]. Moreover, this makes $\mathrm{Ext}_{\mathcal{E}}(-,-)$ into an R-linear functor in each component.

Now, we show that if $(\mathcal{A}',\mathcal{E}')$ is a strictly full exact subcategory of an exact category $(\mathcal{A},\mathcal{E})$, then $\mathrm{Ext}_{\mathcal{E}'}(-,-) : \mathcal{A}'^{\mathrm{op}} \times \mathcal{A}' \to \mathbf{Ab}$ is naturally a subfunctor of $\mathrm{Ext}_{\mathcal{E}}(-,-)|_{\mathcal{A}'^{\mathrm{op}} \times \mathcal{A}'} : \mathcal{A}'^{\mathrm{op}} \times \mathcal{A}' \to \mathbf{Ab}$. For this, we first record a remark.

REMARK 3.7. Let $(\mathcal{A}',\mathcal{E}')$ be a strictly full exact subcategory of $(\mathcal{A},\mathcal{E})$. Then, for $\sigma \in \mathcal{E}'(\subseteq \mathcal{E})$, it is actually true that $[\sigma]_{\mathcal{E}'} = [\sigma]_{\mathcal{E}}$; hence, the map $\mathrm{Ext}_{\mathcal{E}'}(-,-) \xrightarrow{[\sigma]_{\mathcal{E}'} \mapsto [\sigma]_{\mathcal{E}}} \mathrm{Ext}_{\mathcal{E}}(-,-)$ is the natural inclusion map. Indeed, to see $[\sigma]_{\mathcal{E}'} = [\sigma]_{\mathcal{E}}$: let σ be a kernel–cokernel pair $A \rightarrowtail B \twoheadrightarrow C$ in $\mathcal{E}'$, so $A, B, C \in \mathcal{A}'$. Let $\beta \in [\sigma]_{\mathcal{E}}$ be the kernel–cokernel pair $A \rightarrowtail B' \twoheadrightarrow C$ in $\mathcal{E}$, so $B' \in \mathcal{A}$. Hence, there exists $f \in \mathrm{Mor}_{\mathcal{A}}(B, B')$

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such that we have the following commutative diagram in $\mathcal{A}$:

$$\begin{tikzcd}
\sigma: & A \arrow[r, tail] \arrow[d, equal] & B \arrow[r, two heads] \arrow[d, "f"] & C \arrow[d, equal] \\
\beta: & A \arrow[r, tail] & B' \arrow[r, two heads] & C
\end{tikzcd}$$

By [6, Cor. 3.2], we get that f is an isomorphism. Since $B \in \mathcal{A}'$, $B' \in \mathcal{A}$, and $\mathcal{A}'$ is a strict subcategory of $\mathcal{A}$, $B \cong B'$ implies that $B' \in \mathcal{A}'$. Since $\mathcal{A}'$ is full, $f \in \mathrm{Mor}_{\mathcal{A}'}(B,B')$. Hence, the above commutative diagram is in $\mathcal{A}'$. Since $(\mathcal{A}',\mathcal{E}')$ is an exact category, $\mathcal{E}'$ is closed under isomorphisms. Hence, $\beta \in \mathcal{E}'$ and $\beta \in [\sigma]_{\mathcal{E}'}$. Thus, $[\sigma]_{\mathcal{E}} \subseteq [\sigma]_{\mathcal{E}'}$. Now, $(\mathcal{A}',\mathcal{E}')$ is a subcategory of $(\mathcal{A},\mathcal{E})$, so $[\sigma]_{\mathcal{E}'} \subseteq [\sigma]_{\mathcal{E}}$ as well. Hence, $[\sigma]_{\mathcal{E}'} = [\sigma]_{\mathcal{E}}$.

PROPOSITION 3.8. *Let $(\mathcal{A},\mathcal{E})$ be an exact category. Let $(\mathcal{A}',\mathcal{E}')$ be a strictly full exact subcategory of $(\mathcal{A},\mathcal{E})$. Then $\mathrm{Ext}_{\mathcal{E}'}(-,-) : \mathcal{A}'^{\mathrm{op}} \times \mathcal{A}' \to \mathbf{Ab}$ is a subfunctor of $\mathrm{Ext}_{\mathcal{E}}(-,-)|_{\mathcal{A}'^{\mathrm{op}} \times \mathcal{A}'} : \mathcal{A}'^{\mathrm{op}} \times \mathcal{A}' \to \mathbf{Ab}$, where for every $C, A \in \mathcal{A}'$, the natural inclusion map $\mathrm{Ext}_{\mathcal{E}'}(C,A) \to \mathrm{Ext}_{\mathcal{E}}(C,A)$ is given by $[\sigma]_{\mathcal{E}'} \mapsto [\sigma]_{\mathcal{E}}$. If $\mathcal{A}$ is moreover R-linear, then the natural inclusion map is also R-linear; hence, $\mathrm{Ext}_{\mathcal{E}'}(-,-) : \mathcal{A}'^{\mathrm{op}} \times \mathcal{A}' \to \mathrm{mod}\ R$ is a subfunctor of $\mathrm{Ext}_{\mathcal{E}}(-,-)|_{\mathcal{A}'^{\mathrm{op}} \times \mathcal{A}'} : \mathcal{A}'^{\mathrm{op}} \times \mathcal{A}' \to \mathrm{mod}\ R$.*

Proof. Let $C, A \in \mathcal{A}'$. Define $\phi_{C,A} : \mathrm{Ext}_{\mathcal{E}'}(C,A) \to \mathrm{Ext}_{\mathcal{E}}(C,A)$ by $\phi_{C,A}([\sigma]_{\mathcal{E}'}) = [\sigma]_{\mathcal{E}}$. From now on, we will call this map ϕ. The well-definedness and injectivity of ϕ follow from Remark 3.7. Next, we will show that ϕ is a group homomorphism. Let $[\sigma]_{\mathcal{E}'}, [\beta]_{\mathcal{E}'} \in \mathrm{Ext}_{\mathcal{E}'}(C,A)$, where $\sigma : A \rightarrowtail B \twoheadrightarrow C$ and $\beta : A \rightarrowtail B' \twoheadrightarrow C$ for some $B, B' \in \mathcal{A}'$. Then $[\sigma]_{\mathcal{E}'} + [\beta]_{\mathcal{E}'} = [\alpha]_{\mathcal{E}'}$ is given by the following commutative diagram in $(\mathcal{A}',\mathcal{E}')$:

$$\begin{tikzcd}
\sigma \oplus \beta: & A \oplus A \arrow[r, tail] \arrow[d, "\Sigma"] & B \oplus B' \arrow[r, two heads] \arrow[d] & C \oplus C \arrow[d, equal] \\
\gamma: & A \arrow[r, tail] & E \arrow[r, two heads] & C \oplus C \\
\alpha: & A \arrow[r, tail] \arrow[u, equal] & B + B' \arrow[r, two heads] \arrow[u] & C \arrow[u, "\Delta"']
\end{tikzcd}$$

We know that γ is the pushout of $\sigma \oplus \beta$ in $\mathcal{E}'$ by the sum map $A \oplus A \xrightarrow{\Sigma} A$. Since $(\mathcal{A}',\mathcal{E}')$ is an exact subcategory of $(\mathcal{A},\mathcal{E})$, by [6, Prop. 5.2], we get that γ is also the pushout of $\sigma \oplus \beta$ in $\mathcal{E}$ by the sum map $A \oplus A \xrightarrow{\Sigma} A$. Next, α is the pullback of γ in $\mathcal{E}'$ by the diagonal map $C \xrightarrow{\Delta} C \oplus C$, so by [6, Prop. 5.2] we get that α is also the pullback of γ in $\mathcal{E}$ by the diagonal map $C \xrightarrow{\Delta} C \oplus C$. This implies that α is a representative of $[\sigma]_{\mathcal{E}} + [\beta]_{\mathcal{E}}$ in $\mathrm{Ext}_{\mathcal{E}}(C,A)$, that is, $[\alpha]_{\mathcal{E}} = [\sigma]_{\mathcal{E}} + [\beta]_{\mathcal{E}}$. Since $\phi([\sigma]_{\mathcal{E}'} + [\beta]_{\mathcal{E}'}) = \phi([\alpha]_{\mathcal{E}'}) = [\alpha]_{\mathcal{E}}$ and $\phi([\sigma]_{\mathcal{E}'}) + \phi([\beta]_{\mathcal{E}'}) = [\sigma]_{\mathcal{E}} + [\beta]_{\mathcal{E}}$, we get ϕ is a group homomorphism.

Let $A, B, C \in \mathcal{A}'$, and let $f \in \mathrm{Mor}_{\mathcal{A}'}(A,B)$. First, we will show that $\mathrm{Ext}_{\mathcal{E}'}(C,-)$ is a subfunctor of $\mathrm{Ext}_{\mathcal{E}}(C,-)$, so we need to prove that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Ext}_{\mathcal{E}}(C,A) \arrow[r, "f^*"] & \mathrm{Ext}_{\mathcal{E}}(C,B) \\
\mathrm{Ext}_{\mathcal{E}'}(C,A) \arrow[r, "\tilde{f}"] \arrow[u, "\phi_{C,A}"] & \mathrm{Ext}_{\mathcal{E}'}(C,B) \arrow[u, "\phi_{C,B}"]
\end{tikzcd}$$

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where for every $[\sigma]_{\mathcal{E}} \in \mathrm{Ext}_{\mathcal{E}}(C,A)$, a representative of $f^*([\sigma]_{\mathcal{E}})$ is the pushout of σ by f in $\mathcal{E}$ and for every $[\beta]_{\mathcal{E}'} \in \mathrm{Ext}_{\mathcal{E}'}(C,A)$, a representative of $\tilde{f}([\beta]_{\mathcal{E}'})$ is the pushout of β by f in $\mathcal{E}'$. Let $[\sigma]_{\mathcal{E}'} \in \mathrm{Ext}_{\mathcal{E}'}(C,A)$. Now, $\phi_{C,A}([\sigma]_{\mathcal{E}'}) = [\sigma]_{\mathcal{E}}$ and $f^*([\sigma]_{\mathcal{E}}) = [\gamma]_{\mathcal{E}}$, where γ is the pushout of σ by f in $\mathcal{E}$. Next, we have $\tilde{f}([\sigma]_{\mathcal{E}'}) = [\alpha]_{\mathcal{E}'}$, where α is the pushout of σ by f in $\mathcal{E}'$. Hence, by [6, Prop. 5.2], we get that α is also the pushout of σ by f in $\mathcal{E}$. Hence, $[\alpha]_{\mathcal{E}} = [\gamma]_{\mathcal{E}}$, so $\phi_{C,B}(\tilde{f}([\sigma]_{\mathcal{E}'})) = \phi_{C,B}([\alpha]_{\mathcal{E}'}) = [\alpha]_{\mathcal{E}} = [\gamma]_{\mathcal{E}} = f^*([\sigma]_{\mathcal{E}}) = f^*(\phi_{C,A}([\sigma]_{\mathcal{E}'}))$, and therefore the above diagram commutes. Hence, $\mathrm{Ext}_{\mathcal{E}'}(C,-)$ is a subfunctor of $\mathrm{Ext}_{\mathcal{E}}(C,-)$ for any $C \in \mathcal{A}'$. Similarly, $\mathrm{Ext}_{\mathcal{E}'}(-,C)$ is a subfunctor of $\mathrm{Ext}_{\mathcal{E}}(-,C)$ for any $C \in \mathcal{A}'$. Thus, $\mathrm{Ext}_{\mathcal{E}'}(-,-) : \mathcal{A}'^{\mathrm{op}} \times \mathcal{A}' \to \mathbf{Ab}$ is a subfunctor of $\mathrm{Ext}_{\mathcal{E}}(-,-)|_{\mathcal{A}'^{\mathrm{op}} \times \mathcal{A}'} : \mathcal{A}'^{\mathrm{op}} \times \mathcal{A}' \to \mathbf{Ab}$. Now, let $\mathcal{A}$ be R-linear; hence, so is $\mathcal{A}'$ because it is a full subcategory of $\mathcal{A}$. Let $C, A \in \mathcal{A}'$, and let $[\sigma]_{\mathcal{E}'} \in \mathrm{Ext}_{\mathcal{E}'}(C,A)$. Let $[\gamma]_{\mathcal{E}'} = r \cdot [\sigma]_{\mathcal{E}'}$, so γ is obtained from σ by pullback in $\mathcal{E}'$ along the map $C \xrightarrow{r\cdot \mathrm{id}_C} C$. Now, $(\mathcal{A}',\mathcal{E}')$ is an exact subcategory of $(\mathcal{A},\mathcal{E})$; hence, this is also the pullback in $\mathcal{E}$ along the map $C \xrightarrow{r\cdot \mathrm{id}_C} C$ (see [6, Prop. 5.2]). So, $[\gamma]_{\mathcal{E}} = r \cdot [\sigma]_{\mathcal{E}} = r\phi([\sigma]_{\mathcal{E}'})$. Also, $[\gamma]_{\mathcal{E}} = \phi([\gamma]_{\mathcal{E}'}) = \phi(r \cdot [\sigma]_{\mathcal{E}'})$. This shows that the natural inclusion map ϕ is R-linear. ∎

In the following corollary, we denote $\mathrm{Hom}_{\mathcal{A}}(-,-)$ just by $\mathrm{Hom}(-,-)$ (we completely ignore the subcategory, since all our subcategories are full). In view of Definition 4.10, and the discussion following 4.11, the following corollary is crucial for recovering [35, Props. 1.37 and 1.38].

Corollary 3.9. *Let $(\mathcal{A},\mathcal{E})$ be an exact category. Let $(\mathcal{A}',\mathcal{E}')$ be a strictly full exact subcategory of $(\mathcal{A},\mathcal{E})$. Let $\sigma:\ M \overset{i}{\rightarrowtail} N \overset{p}{\twoheadrightarrow} L$ be a kernel–cokernel pair in $\mathcal{E}'$. Then, for every $A \in \mathcal{A}'$, we have the following commutative diagrams of long exact sequences:*

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(A,M) \arrow[r] & \mathrm{Hom}(A,N) \arrow[r] & \mathrm{Hom}(A,L) \arrow[r] & \mathrm{Ext}_{\mathcal{E}}(A,M) \arrow[r] & \mathrm{Ext}_{\mathcal{E}}(A,N) \arrow[r] & \mathrm{Ext}_{\mathcal{E}}(A,L) \\
0 \arrow[r] & \mathrm{Hom}(A,M) \arrow[r] \arrow[u, equal] & \mathrm{Hom}(A,N) \arrow[r] \arrow[u, equal] & \mathrm{Hom}(A,L) \arrow[r] \arrow[u, equal] & \mathrm{Ext}_{\mathcal{E}'}(A,M) \arrow[r] \arrow[u, hook] & \mathrm{Ext}_{\mathcal{E}'}(A,N) \arrow[r] \arrow[u, hook] & \mathrm{Ext}_{\mathcal{E}'}(A,L) \arrow[u, hook]
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(L,A) \arrow[r] & \mathrm{Hom}(N,A) \arrow[r] & \mathrm{Hom}(M,A) \arrow[r] & \mathrm{Ext}_{\mathcal{E}}(L,A) \arrow[r] & \mathrm{Ext}_{\mathcal{E}}(N,A) \arrow[r] & \mathrm{Ext}_{\mathcal{E}}(M,A) \\
0 \arrow[r] & \mathrm{Hom}(L,A) \arrow[r] \arrow[u, equal] & \mathrm{Hom}(N,A) \arrow[r] \arrow[u, equal] & \mathrm{Hom}(M,A) \arrow[r] \arrow[u, equal] & \mathrm{Ext}_{\mathcal{E}'}(L,A) \arrow[r] \arrow[u, hook] & \mathrm{Ext}_{\mathcal{E}'}(N,A) \arrow[r] \arrow[u, hook] & \mathrm{Ext}_{\mathcal{E}'}(M,A) \arrow[u, hook]
\end{tikzcd}$$

Proof. We will only prove the covariant version (the first diagram above), since the proof of the contravariant version is given by the dual argument. In the first diagram above, the commutativity of the first two squares is obvious. Also, the commutativity of the last two squares follows directly from the proof of Proposition 3.8 (i.e., $\mathrm{Ext}_{\mathcal{E}'}(-,-)$ is a subfunctor of $\mathrm{Ext}_{\mathcal{E}}(-,-)$). So, it is enough to prove that the following square is commutative:

$$\begin{tikzcd}
\mathrm{Hom}(A,L) \arrow[r, "f"] & \mathrm{Ext}_{\mathcal{E}}(A,M) \\
\mathrm{Hom}(A,L) \arrow[r, "f^*"] \arrow[u, equal] & \mathrm{Ext}_{\mathcal{E}'}(A,M) \arrow[u, hook, "\phi_{A,M}"']
\end{tikzcd}$$

Now, $f(g) = [\gamma]_{\mathcal{E}}$, where γ is the pullback of σ by g in $\mathcal{E}$ and $f^*(g) = [\alpha]_{\mathcal{E}'}$, where α is the pullback of σ by g in $\mathcal{E}'$. Since $(\mathcal{A}',\mathcal{E}')$ is an exact subcategory of $(\mathcal{A},\mathcal{E})$, by [6, Prop. 5.2],

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we get that $[\gamma]_{\mathcal{E}} = [\alpha]_{\mathcal{E}}$. Also, by definition, $\phi_{A,M}([\alpha]_{\mathcal{E}'}) = [\alpha]_{\mathcal{E}}$. Thus, $f(g) = \phi_{A,M}(f^*(g))$, so the above square commutes. ∎

Let $(\mathcal{A}, \mathcal{E})$ be an exact category. Let $\mathcal{A}'$ be a strictly full subcategory of $\mathcal{A}$. If $A_1 \overset{i}{\rightarrowtail} B \overset{p}{\twoheadrightarrow} A_2$ and $A_1 \overset{i}{\rightarrowtail} C \overset{p}{\twoheadrightarrow} A_2$ are two kernel–cokernel pairs in $\mathcal{E}$ belonging to the same class in $\operatorname{Ext}_{\mathcal{E}}(A_2, A_1)$, then $B \cong C$ (by short five lemma); hence, $B \in \mathcal{A}'$ if and only if $C \in \mathcal{A}'$. Hence, for every $M, N \in \mathcal{A}'$, the collection $\{[\sigma]_{\mathcal{E}} \in \operatorname{Ext}_{\mathcal{E}}(M,N) :$ the middle object of σ is in $\mathcal{A}'\}$ is well defined.

Proposition 3.10. *Let $(\mathcal{A}, \mathcal{E})$ be an exact category. Let $\mathcal{A}'$ be a strictly full additive subcategory of $\mathcal{A}$. For every $M, N \in \mathcal{A}'$, define $F(M,N) := \{[\sigma]_{\mathcal{E}} \in \operatorname{Ext}_{\mathcal{E}}(M,N) :$ the middle object of σ is in $\mathcal{A}'\}$. If $F(-,-)$ is a subfunctor of $\operatorname{Ext}_{\mathcal{E}}(-,-)|_{\mathcal{A}'^{\mathrm{op}} \times \mathcal{A}'} : \mathcal{A}'^{\mathrm{op}} \times \mathcal{A}' \to \mathbf{Ab}$ via the natural inclusion maps, then $(\mathcal{A}', \mathcal{E}|_{\mathcal{A}'})$ is an exact subcategory of $(\mathcal{A}, \mathcal{E})$.*

Proof. By Lemma 3.3, it is enough to show that the pullback (resp. pushout) in $\mathcal{A}$ of every $\mathcal{E}$-deflation (resp. inflation) of sequences in $\mathcal{E}|_{\mathcal{A}'}$ along every morphism in $\mathcal{A}'$ is again in $\mathcal{A}'$. We will only prove the pullback case, since the proof of the pushout case can be given by a similar argument. Let $B \twoheadrightarrow C$ be an admissible epic in $\mathcal{E}$ such that it is in $\mathcal{E}|_{\mathcal{A}'}$, which means that there exist a kernel–cokernel pair $\gamma : A \rightarrowtail B \twoheadrightarrow C$ in $\mathcal{E}$ such that $A, B, C \in \mathcal{A}'$. Also, let $B' \xrightarrow{f} C$ be a morphism in $\mathcal{A}'$. Then we have a map $f^* : \operatorname{Ext}_{\mathcal{E}}(C, A) \to \operatorname{Ext}_{\mathcal{E}}(B', A)$ defined as follows: for every $[\sigma]_{\mathcal{E}} \in \operatorname{Ext}_{\mathcal{E}}(C, A)$, a representative of $f^*([\sigma]_{\mathcal{E}})$ is the pullback of σ by f in $(\mathcal{A}, \mathcal{E})$. Now, by definition of $F(-,-)$, it is clear that $F(M,N) \subseteq \operatorname{Ext}_{\mathcal{E}}(M,N)$ for any $M, N \in \mathcal{A}'$. Since $F(-,-)$ is a subfunctor of $\operatorname{Ext}_{\mathcal{E}}(-,-)|_{\mathcal{A}'^{\mathrm{op}} \times \mathcal{A}'} : \mathcal{A}'^{\mathrm{op}} \times \mathcal{A}' \to \mathbf{Ab}$, we have the following commutative square:

$$\begin{tikzcd}
\operatorname{Ext}_{\mathcal{E}}(C, A) \arrow[r, "f^*"] & \operatorname{Ext}_{\mathcal{E}}(B', A) \\
F(C, A) \arrow[u, hook] \arrow[r, "\tilde{f}"] & F(B', A) \arrow[u, hook]
\end{tikzcd}$$

where the columns are natural inclusion maps. So, for every $[\beta]_{\mathcal{E}} \in F(C, A)$, a representative of $\tilde{f}([\beta]_{\mathcal{E}})$ is the pullback of β by f in $(\mathcal{A}, \mathcal{E})$. Now, consider the pullback of γ by f in $(\mathcal{A}, \mathcal{E})$ as follows:

$$\begin{tikzcd}
A \arrow[r, tail] & A' \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "\mathrm{PB}"] & B' \arrow[d, "f"] \\
\gamma : A \arrow[r, tail] & B \arrow[r, two heads] & C
\end{tikzcd}$$

Now, $[\gamma]_{\mathcal{E}} \in F(C, A)$, so $\tilde{f}([\gamma]_{\mathcal{E}}) \in F(B', A)$ implies that $A' \in \mathcal{A}'$. Hence, the pullback in $\mathcal{A}$ of every $\mathcal{E}$-deflation of sequences in $\mathcal{E}|_{\mathcal{A}'}$ along every morphism in $\mathcal{A}'$ is again in $\mathcal{A}'$. Hence, by Lemma 3.3, we get that $(\mathcal{A}', \mathcal{E}|_{\mathcal{A}'})$ is an exact subcategory of $(\mathcal{A}, \mathcal{E})$. ∎

§4. Subcategories and subfunctors of Ext^1 from numerical functions and applications to module categories

In this section, we present tools to identify exact subcategories of an exact category coming from certain numerical functions. Consequently, due to Proposition 3.8, we are also able to identify subfunctors of Ext^1 associated with certain numerical functions. Our first result in this direction is an application of Theorem 3.6.

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PROPOSITION 4.1. *Let $(\mathcal{A},\mathcal{E})$ be an exact category. Let $\phi:\mathcal{A}\to\mathbb{Z}_{\le 0}$ be a function such that ϕ is constant on isomorphism classes of objects in $\mathcal{A}$, ϕ is additive on finite biproducts, and ϕ is sub-additive on kernel–cokernel pairs in $\mathcal{E}$ (i.e., if $M \rightarrowtail N \twoheadrightarrow L$ is in $\mathcal{E}$, then $\phi(N)\le\phi(M)+\phi(L)$). Let $\mathcal{A}'$ be the strictly full subcategory of $\mathcal{A}$, whose objects are given by $\{M\in\mathcal{A}:\phi(M)=0\}$. Then $(\mathcal{A}',\mathcal{E}|_{\mathcal{A}'})$ is an exact subcategory of $(\mathcal{A},\mathcal{E})$.*

Proof. Note that $\mathcal{A}'$ is a strict subcategory of $\mathcal{A}$ since ϕ is constant on isomorphism classes of objects in $\mathcal{A}$. Since ϕ is additive on finite biproduct, $\phi(0_{\mathcal{A}})=\phi(0_{\mathcal{A}}\oplus 0_{\mathcal{A}})=\phi(0_{\mathcal{A}})+\phi(0_{\mathcal{A}})$, and hence $\phi(0_{\mathcal{A}})=0$. Hence, $0_{\mathcal{A}}\in\mathcal{A}'$. Since $\mathcal{A}$ is an additive category and ϕ is additive on finite biproducts, by definition of $\mathcal{A}'$, we get that for every $X,Y\in\mathcal{A}'$, the biproduct of X and Y in $\mathcal{A}$ also belongs to $\mathcal{A}'$. Hence, $\mathcal{A}'$ is an additive subcategory of $\mathcal{A}$. Let $B\in\mathcal{A}'$ and $A_1 \overset{i}{\rightarrowtail} B \overset{p}{\twoheadrightarrow} A_2$ be a kernel–cokernel pair in $\mathcal{E}$. We will show that $A_1,A_2\in\mathcal{A}'$. Since ϕ is subadditive on kernel–cokernel pairs in $\mathcal{E}$ and $B\in\mathcal{A}'$, we get $0=\phi(B)\le\phi(A_1)+\phi(A_2)\le 0$. Hence, $\phi(A_1)+\phi(A_2)=0$. Since we know ϕ always takes non-positive values, we get $\phi(A_1)=\phi(A_2)=0$; hence, $A_1,A_2\in\mathcal{A}'$. Thus, $\mathcal{A}'$ is closed under kernels and co-kernels of admissible epics and monics in $\mathcal{E}$, respectively. Then, by Theorem 3.6, we get that $(\mathcal{A}',\mathcal{E}|_{\mathcal{A}'})$ is an exact subcategory of $(\mathcal{A},\mathcal{E})$. ∎

We now proceed to give our main example of Proposition 4.1. First, we recall some terminologies and prove some preliminary lemmas. For any unexpected concepts and notations, we refer the reader to [3] and [27].

Let R be a commutative ring, and let $\mathcal{S}_R$ be the collection of all short exact sequences of R-modules, which gives the standard exact structure on mod R. When the ring in question is clear, we drop the suffix R and write only $\mathcal{S}$. Note that since mod R is extension closed in mod R, the collection of all short exact sequences in mod R gives an exact subcategory of mod R, and we take this as the standard exact structure of mod R. Note that if $(\mathcal{X},\mathcal{E})$ is an exact subcategory of mod R and $\mathcal{X}\subseteq$ mod R, then $(\mathcal{X},\mathcal{E})$ is also an exact subcategory of mod R.

Now, for a Noetherian local ring $(R,\mathfrak{m})$ of dimension d and for an integer $s\ge 0$, let $\mathrm{CM}^s(R)$ denote the full subcategory of mod R consisting of the zero-module, and all Cohen–Macaulay R-modules [3, Def. 2.1.1] of dimension s. Note that when $s=d$, $\mathrm{CM}^d(R)$ is just the category of all maximal Cohen–Macaulay modules, which we will also denote by $\mathrm{CM}(R)$.

4.2. We quickly note that for each $s\ge 0$, $\mathrm{CM}^s(R)$ is closed under finite direct sums, direct summands, and closed under extensions in mod R. Indeed, let $0\to L\to M\to N\to 0$ be a short exact sequence with $L,N\in\mathrm{CM}^s(R)$. If L or N is zero, then there is nothing to prove. So, assume L,N are nonzero. Now, $M\in$ mod R, and we also have the following calculation for $\dim M$: $\dim M=\dim\operatorname{Supp}(M)=\dim(\operatorname{Supp}(L)\cup\operatorname{Supp}(N))=\max\{\dim\operatorname{Supp}(L),\dim\operatorname{Supp}(N)\}=\max\{\dim L,\dim N\}=s$. Consequently, $s=\inf\{\operatorname{depth}_R L,\operatorname{depth}_R N\}\le\operatorname{depth}_R M\le\dim M=s$, which shows $M\in\mathrm{CM}^s(R)$. Thus, $\mathrm{CM}^s(R)$ is closed under extensions in mod R. To also show $\mathrm{CM}^s(R)$ is closed under direct summands, let M,N be nonzero modules with $M\oplus N\in\mathrm{CM}^s(R)$. Then, $M,N\in$ mod R, and $s=\operatorname{depth}_R(M\oplus N)=\inf\{\operatorname{depth}_R M,\operatorname{depth}_R N\}\le\operatorname{depth}_R M\le\dim M\le\dim(M\oplus N)=s$, and thus $M\in\mathrm{CM}^s(R)$.

Hence, if $\mathcal{S}$ is the standard exact structure on mod R, then $(\mathrm{CM}^s(R),\mathcal{S}|_{\mathrm{CM}^s(R)})$ is an exact subcategory of mod R by Proposition 3.5. Now, let I be an $\mathfrak{m}$-primary ideal and let $\phi_I:\mathrm{CM}^s(R)\to\mathbb{Z}$ be the function defined by $\phi_I(M):=\lambda_R(M/IM)-e_R(I,M)$, where

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$e_R(I,M)$ is the multiplicity of M with respect to I (see [3, Def. 4.6.1]). We first show that ϕ_I satisfy all the hypothesis of Proposition 4.1. To prove this, we will need to pass to the faithfully flat extension $S = R[X]_{\mathfrak{m}[X]}$ whose unique maximal ideal is $\mathfrak{m}S$. That we can harmlessly pass to this extension, is discussed in the following.

4.3. Let $(R,\mathfrak{m})$ be a local ring, and consider $S = R[X]_{\mathfrak{m}[X]}$, which is a faithfully flat extension of R and the unique maximal ideal of S is $\mathfrak{m}S$, whose residue field $S/\mathfrak{m}S$ is infinite (see [24, §8.4]). Let $M \in \operatorname{mod}(R)$. By [3, Th. 2.1.7] (and the sentences following it), we have $M \in \mathrm{CM}^s(R)$ if and only if $M \otimes_R S \in \mathrm{CM}^s(S)$. Let I be an $\mathfrak{m}$-primary ideal of R. As $\mathfrak{m}S$ is the maximal ideal of S, so $IS = I \otimes_R S$ is also primary to the maximal ideal of S. Now, $S \otimes_R \dfrac{M}{IM} \cong \dfrac{S \otimes_R M}{S \otimes_R (IM)} = \dfrac{S \otimes_R M}{(IS)(S \otimes_R M)}$ by Lemma 5.2.5(1). Hence, $\lambda_S\left(\dfrac{S \otimes_R M}{(IS)(S \otimes_R M)}\right) = \lambda_S\left(S \otimes_R \dfrac{M}{IM}\right) = \lambda_R\left(\dfrac{M}{IM}\right)$, where the last equality is by [38, Tag 02M1] remembering that $S/\mathfrak{m}S$ is the residue field of S. Also, $e_R(I,M) := \lim_{n\to\infty} \dfrac{(\dim M)!}{n^{\dim M}} \lambda_R\left(\dfrac{M}{I^{n+1}M}\right)$, and $\dim_S(M \otimes_R S) = \dim_R(M)$ [3, Th. A.11(b)], so a similar argument as the previous one shows $e_R(I,M) = e_S(IS, S \otimes_R M)$. Thus, $\phi_I(M) = \phi_{IS}(S \otimes_R M)$.

LEMMA 4.4. *The function* $\phi_I : \mathrm{CM}^s(R) \to \mathbb{Z}$ *satisfies* $\phi_I(M) \le 0$, *is constant on isomorphism classes of modules in* $\mathrm{CM}^s(R)$, *additive on finite biproducts, and subadditive on short exact sequences of modules in* $\mathrm{CM}^s(R)$.

Proof. It is obvious that ϕ_I is constant on isomorphism classes of modules in $\mathrm{CM}^s(R)$, and additive on finite biproducts (direct sums). Now, $e_R(I,-)$ is additive on short exact sequences of modules in $\mathrm{CM}^s(R)$ by [3, Cor. 4.7.7]. Since $\lambda_R((-) \otimes_R R/I)$ is always subadditive on short exact sequences, this proves ϕ_I is also subadditive on short exact sequences of modules in $\mathrm{CM}^s(R)$. Now, we finally prove that $\phi_I(M) \le 0$ for all $M \in \mathrm{CM}^s(R)$. If $\dim M = 0$, then $e_R(I,M) = \lambda_R(M) \ge \lambda_R(M/IM)$, so $\phi_I(M) \le 0$. Now, assume that $s = \dim M > 0$. We may assume that R has infinite residue field due to 4.3. By [3, Cor. 4.6.10], we have $e_R(I,M) = e_R((\mathbf{x}),M)$ for some system of parameters $\mathbf{x} = x_1,\dots,x_s$ on M, which is a reduction of I with respect to M. Then, $\mathbf{x}$ is an M-regular sequence, since M is Cohen–Macaulay (see [3, Th. 2.1.2(d)]). Let $J = (\mathbf{x})$. Since $\mathbf{x}$ is an M-regular sequence, by [3, Th. 1.1.8], we have $J^nM/J^{n+1}M \cong (M/JM)^{\oplus \frac{(s+n-1)!}{n!(s-1)!}}$. Hence, $e_R(I,M) = e_R(J,M) = (s-1)! \lim_{n\to\infty} \dfrac{\lambda_R(J^nM/J^{n+1}M)}{n^{s-1}} = \lambda_R(M/JM) \ge \lambda_R(M/IM)$ (the last inequality follows by noticing that $J \subseteq I$). Hence, $\phi_I(M) \le 0$. ∎

DEFINITION 4.5. Let $(R,\mathfrak{m})$ be a Noetherian local ring, and let I be an $\mathfrak{m}$-primary ideal. Let $s \ge 0$ be an integer. We denote by $\mathrm{Ul}_I^s(R)$ the full subcategory of all modules $M \in \mathrm{CM}^s(R)$ such that $\phi_I(M) = 0$, that is, $\lambda_R(M/IM) = e_R(I,M)$. When $s = \dim R$, we will denote this subcategory simply by $\mathrm{Ul}_I(R)$. When $I = \mathfrak{m}$, $\mathrm{Ul}_{\mathfrak{m}}^s(R)$ will be denoted by $\mathrm{Ul}^s(R)$. We will also denote $\mathrm{Ul}_{\mathfrak{m}}^{\dim R}(R)$ simply by $\mathrm{Ul}(R)$. Note that when $s = \dim R = 1$ and R is Cohen–Macaulay, $\mathrm{Ul}_I(R)$ is exactly the collection of all I-Ulrich modules as defined in [11, Def. 4.1].

REMARK 4.6. In terms of [21, Def. 2.1], a nonzero R-module M is Ulrich if $M \in \mathrm{Ul}^s(R)$ for some $s \ge 0$ in our notation. When R is Cohen–Macaulay, then the modules in $\mathrm{Ul}(R)$ are simply the maximally generated modules as studied in [2].

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Corollary 4.7. *For each integer $s \geq 0$ and $\mathfrak{m}$-primary ideal I, $(\mathrm{Ul}_I^s(R), \mathcal{S}|_{\mathrm{Ul}_I^s(R)})$ is an exact subcategory of* mod R *(hence of* mod R*). Hence,* $\mathrm{Ext}^1_{\mathrm{Ul}_I^s(R)}(-,-) : \mathrm{Ul}_I^s(R)^{op} \times \mathrm{Ul}_I^s(R) \to \mathrm{mod}\ R$ *is a subfunctor of* $\mathrm{Ext}^1_R(-,-) : \mathrm{Ul}_I^s(R)^{op} \times \mathrm{Ul}_I^s(R) \to \mathrm{mod}\ R$.

Proof. Since it has been noticed that $(\mathrm{CM}^s(R), \mathcal{S}|_{\mathrm{CM}^s(R)})$ is an exact subcategory of mod R, by Lemma 4.4 and Proposition 4.1, it follows that $(\mathrm{Ul}_I^s(R), \mathcal{S}|_{\mathrm{Ul}_I^s(R)})$ is an exact subcategory of $(\mathrm{CM}^s(R), \mathcal{S}|_{\mathrm{CM}^s(R)})$, and hence of mod R. The subfunctor part now follows from Proposition 3.8. ∎

Next, we record a proposition for constructing a special kind of exact substructure of an exact structure $\mathcal{E}$ on a category $\mathcal{A}$, without shrinking the category, that also comes from certain kinds of numerical functions. This is used in the next section to recover and improve some results on module categories.

Theorem 4.8. *Let $(\mathcal{A}, \mathcal{E})$ be an exact category. Let $\phi : \mathcal{A} \to \mathbb{Z}$ be a function such that ϕ is constant on isomorphism classes of objects in $\mathcal{A}$, ϕ is additive on finite biproducts, and ϕ is subadditive on kernel–cokernel pairs in $\mathcal{E}$ (i.e., if $M \rightarrowtail N \twoheadrightarrow L$ is in $\mathcal{E}$, then $\phi(N) \leq \phi(M) + \phi(L)$). Set $\mathcal{E}^\phi := \{$kernel–cokernel pairs in $\mathcal{E}$ on which ϕ is additive$\}$. Then, $(\mathcal{A}, \mathcal{E}^\phi)$ is an exact subcategory of $(\mathcal{A}, \mathcal{E})$.*

Proof. By [6, Def. 2.1] and [6, Rem. 2.4], it is enough to show that $\mathcal{E}^\phi$ is closed under isomorphisms and $(\mathcal{A}, \mathcal{E}^\phi)$ satisfies the axioms [E0], [E0$^{\mathrm{op}}$], [E1$^{\mathrm{op}}$], [E2], and [E2$^{\mathrm{op}}$]. First, we will show that $\mathcal{E}^\phi$ is closed under isomorphisms. Let $M \rightarrowtail N \twoheadrightarrow L$ be a kernel–cokernel pair in $\mathcal{E}^\phi$. Also, let $M' \to N' \to L'$ be a kernel–cokernel pair in $\mathcal{A}$ such that it is isomorphic to $M \rightarrowtail N \twoheadrightarrow L$, which implies that $M' \rightarrowtail N' \twoheadrightarrow L'$ is in $\mathcal{E}$ and $M \cong M'$, $N \cong N'$, $L \cong L'$. Since $\phi(N) = \phi(M) + \phi(L)$ and ϕ is constant on isomorphism classes of objects in $\mathcal{A}$, we have $\phi(N') = \phi(N) = \phi(M) + \phi(L) = \phi(M') + \phi(L')$. So, $M' \rightarrowtail N' \twoheadrightarrow L'$ is in $\mathcal{E}^\phi$. Hence, $\mathcal{E}^\phi$ is closed under isomorphisms. Next, we will show that $(\mathcal{A}, \mathcal{E}^\phi)$ satisfies the axiom [E0]. Let $A \in \mathcal{A}$. Since $(\mathcal{A}, \mathcal{E})$ is an exact category, by [6, Lem. 2.7], we get that $A \overset{1_A}{\rightarrowtail} A \oplus 0 \cong A \overset{0}{\twoheadrightarrow} 0$ is in $\mathcal{E}$. Since ϕ is additive on finite biproducts, ϕ is additive on the kernel–cokernel pair $A \overset{1_A}{\rightarrowtail} A \oplus 0 \cong A \overset{0}{\twoheadrightarrow} 0$. Hence, by definition, $A \overset{1_A}{\rightarrowtail} A \oplus 0 \cong A \overset{0}{\twoheadrightarrow} 0$ is in $\mathcal{E}^\phi$, which implies that 1_A is an admissible monic in $\mathcal{E}^\phi$. So, $(\mathcal{A}, \mathcal{E}^\phi)$ satisfies the axiom [E0]. Next, we will show that $(\mathcal{A}, \mathcal{E}^\phi)$ satisfies the axiom [E0$^{\mathrm{op}}$]. Let $A \in \mathcal{A}$. Since $(\mathcal{A}, \mathcal{E})$ is an exact category, by [6, Lem. 2.7], we get that $0 \overset{0}{\rightarrowtail} A \oplus 0 \cong A \overset{1_A}{\twoheadrightarrow} A$ is in $\mathcal{E}$. Since ϕ is additive on finite biproducts, ϕ is additive on the kernel–cokernel pair $0 \overset{0}{\rightarrowtail} A \oplus 0 \cong A \overset{1_A}{\twoheadrightarrow} A$. Hence, by definition, $0 \overset{0}{\rightarrowtail} A \oplus 0 \cong A \overset{1_A}{\twoheadrightarrow} A$ is in $\mathcal{E}^\phi$, which implies that 1_A is an admissible epic in $\mathcal{E}^\phi$. So, $(\mathcal{A}, \mathcal{E}^\phi)$ satisfies the axiom [E0$^{\mathrm{op}}$]. Next, we will show that $(\mathcal{A}, \mathcal{E}^\phi)$ satisfies the axiom [E1$^{\mathrm{op}}$]. Let $B' \overset{e}{\twoheadrightarrow} B$ and $B \overset{p}{\twoheadrightarrow} C$ be two admissible epics in $\mathcal{E}^\phi$. Then we will show that $B' \overset{p \circ e}{\twoheadrightarrow} C$ is an admissible epic in $\mathcal{E}^\phi$. Let $A \overset{i}{\rightarrowtail} B \overset{p}{\twoheadrightarrow} C$ be the complete kernel–cokernel pair of p, and consider a pullback square as in Proposition 3.4. From Proposition 3.4, we get that $A' \overset{i'}{\rightarrowtail} B' \overset{p \circ e}{\twoheadrightarrow} C$ is a kernel–cokernel pair in $\mathcal{E}$, so $\phi(B') \leq \phi(C) + \phi(A')$.

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Now, consider the pullback square of Proposition 3.4

$$\begin{tikzcd} A' \arrow[r, two heads, "e'"] \arrow[d, tail, "i'"] \arrow[dr, phantom, "\mathrm{PB}"] & A \arrow[d, tail, "i"] \\ B' \arrow[r, two heads, "e"] & B \end{tikzcd}$$

By the dual (pullback version) of [6, Prop. 2.12(i) $\Rightarrow$ (iv)] we get a commutative diagram in $\mathcal{E}$ as follows:

$$\begin{tikzcd} D \arrow[r, tail] \arrow[d, equal] & A' \arrow[r, two heads, "e'"] \arrow[d, "i'"] \arrow[dr, phantom, "\mathrm{PB}"] & A \arrow[d, "i"] \\ D \arrow[r, tail] & B' \arrow[r, two heads, "e"] & B \end{tikzcd}$$

The top row gives $\phi(A') \le \phi(A) + \phi(D)$. Thus, $\phi(C) + \phi(A') \le \phi(A) + \phi(C) + \phi(D)$. Since $B \xrightarrow{p} C$ is an admissible epic in $\mathcal{E}^\phi$ and $A \xrightarrow{i} B \xrightarrow{p} C$ is in $\mathcal{E}$ and $\phi(-)$ is constant on isomorphism classes of objects, we have $\phi(A) + \phi(C) = \phi(B)$. Thus, $\phi(C) + \phi(A') \le \phi(A) + \phi(C) + \phi(D)$ implies $\phi(C) + \phi(A') \le \phi(B) + \phi(D)$. Since $B' \xrightarrow{e} B$ is an admissible epic in $\mathcal{E}^\phi$, the bottom row of the above diagram gives $\phi(B') = \phi(B) + \phi(D)$. Hence, $\phi(C) + \phi(A') \le \phi(B')$. Thus, $A' \xrightarrow{i'} B' \xrightarrow{p\circ e} C$ in $\mathcal{E}$ is actually in $\mathcal{E}^\phi$. This shows $(\mathcal{A}, \mathcal{E}^\phi)$ satisfies the axiom [E1$^{\mathrm{op}}$].

Now, we will show that $(\mathcal{A}, \mathcal{E}^\phi)$ satisfies the axiom [E2]. Let $A \xrightarrow{i} B$ be an admissible monic in $\mathcal{E}^\phi$, and let $A \xrightarrow{f} A'$ be an arbitrary morphism in $\mathcal{A}$. Then we have the following pushout commutative square in $\mathcal{E}$:

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "f"] \arrow[dr, phantom, "\mathrm{PO}"] & B \arrow[d, "f'"] \\ A' \arrow[r, tail, "i'"] & B' \end{tikzcd}$$

Then, by [6, Prop. 2.12(ii)], we have the following kernel–cokernel pair in $\mathcal{E}$: $A \rightarrowtail B \oplus A' \twoheadrightarrow B'$, so $\phi(B) + \phi(A') = \phi(B \oplus A') \le \phi(A) + \phi(B')$. Also, by [6, Prop. 2.12(iv)], we have the following commutative diagram with rows being kernel–cokernel pairs in $\mathcal{E}$:

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "f"] \arrow[dr, phantom, "\mathrm{PO}"] & B \arrow[r, two heads, "p"] \arrow[d, "f'"] & C \arrow[d, equal] \\ A' \arrow[r, tail, "i'"] & B' \arrow[r, two heads, "p'"] & C \end{tikzcd}$$

Since $A \xrightarrow{i} B$ is an admissible monic in $\mathcal{E}^\phi$, we have $\phi(B) = \phi(A) + \phi(C)$. This implies $\phi(A) + \phi(C) + \phi(A') \le \phi(A) + \phi(B')$, so $\phi(C) + \phi(A') \le \phi(B')$. Also, $A' \xrightarrow{i'} B' \xrightarrow{p'} C$ is a kernel–cokernel pair in $\mathcal{E}$, so $\phi(B') \le \phi(C) + \phi(A')$. Thus, $\phi(C) + \phi(A') = \phi(B')$. Hence, $A' \xrightarrow{i'} B' \xrightarrow{p'} C$ is a kernel–cokernel pair in $\mathcal{E}^\phi$, so $A' \xrightarrow{i'} B'$ is an admissible monic in $\mathcal{E}^\phi$. Thus, $(\mathcal{A}, \mathcal{E}^\phi)$ satisfies the axiom [E2]. A dual argument will show that $(\mathcal{A}, \mathcal{E}^\phi)$ satisfies the axiom [E2$^{\mathrm{op}}$]. Hence, $(\mathcal{A}, \mathcal{E}^\phi)$ is an exact subcategory of $(\mathcal{A}, \mathcal{E})$. ∎

Moving forward, we record some applications of Theorem 4.8.

4.9. Let $\mathcal{X}$ be a subcategory of $\mathrm{Mod}(R)$ such that $(\mathcal{X},\mathcal{S}|_{\mathcal{X}})$ is an exact subcategory of mod R. In this case, for $M,N \in \mathcal{X}$, by $\mathrm{Ext}^1_{\mathcal{X}}(M,N)$, we will mean $\mathrm{Ext}_{\mathcal{S}|_{\mathcal{X}}}(M,N)$, that is, $\mathrm{Ext}^1_{\mathcal{X}}(M,N) = \{[\sigma] : \text{the middle object of } \sigma \text{ is in } \mathcal{X}\}$. (Note that it does not matter whether the equivalence class is taken in $\mathcal{S}|_{\mathcal{X}}$ or $\mathcal{S}$ by Remark 3.7.) Note that $\mathrm{Ext}^1_{\mathcal{X}}(-,-) : \mathcal{X}^{op} \times \mathcal{X} \to$ mod R is a subfunctor of $\mathrm{Ext}^1_R(-,-) : \mathcal{X}^{op} \times \mathcal{X} \to$ mod R by Proposition 3.8. For example, we can apply this discussion to $\mathcal{X} = \mathrm{Ul}^s_I(R)$ due to Corollary 4.7.

Note that if $\mathcal{X}$ is closed under taking extensions in mod R, then $(\mathcal{X},\mathcal{S}|_{\mathcal{X}})$ is exact subcategory of mod R by Proposition 3.5. In this case, $\mathrm{Ext}^1_{\mathcal{X}}(M,N) = \mathrm{Ext}^1_R(M,N)$ for all $M,N \in \mathcal{X}$.

Definition 4.10. Let $\mathcal{X}$ be a subcategory of mod R such that $(\mathcal{X},\mathcal{S}|_{\mathcal{X}})$ is an exact subcategory of mod R, and let $\phi : \mathcal{X} \to \mathbb{Z}$ be a function satisfying the hypothesis of Theorem 4.8, that is, ϕ is constant on isomorphism classes of modules in $\mathcal{X}$, ϕ is additive on finite direct sums, and subadditive on short exact sequences of modules in $\mathcal{X}$. Then, in the notation of Theorem 4.8, $(\mathcal{X},\mathcal{S}|_{\mathcal{X}^\phi})$ is an exact subcategory of $(\mathcal{X},\mathcal{S}|_{\mathcal{X}})$, hence an exact subcategory of mod R. For $M,N \in \mathcal{X}$, define $\mathrm{Ext}^1_{\mathcal{X}}(M,N)^\phi := \mathrm{Ext}_{\mathcal{S}|_{\mathcal{X}^\phi}}(M,N) = \{[\alpha] \in \mathrm{Ext}^1_{\mathcal{X}}(M,N) \;:\; \phi \text{ is additive on } \alpha\}$. Again, we notice that if $\mathcal{X} \subseteq$ mod R is closed under taking extensions, then $\mathrm{Ext}^1_{\mathcal{X}}(M,N)^\phi = \{[\alpha] \in \mathrm{Ext}^1_R(M,N) \;:\; \phi \text{ is additive on } \alpha\}$, and in this case, we denote it just by $\mathrm{Ext}^1_R(M,N)^\phi$.

4.11. Note that $\mathrm{Ext}^1_{\mathcal{X}}(-,-)^\phi : \mathcal{X}^{op} \times \mathcal{X} \to$ mod R is a subfunctor of $\mathrm{Ext}^1_{\mathcal{X}}(-,-) : \mathcal{X}^{op} \times \mathcal{X} \to$ mod R, which in turn is a subfunctor of $\mathrm{Ext}^1_R(-,-) : \mathcal{X}^{op} \times \mathcal{X} \to$ mod R (Proposition 3.8) and we have corresponding commutative diagram of long exact sequences by Corollary 3.9.

With the extension-closed subcategory $\mathcal{X} =$ mod R, where $(R,\mathfrak{m},k)$ is a Noetherian local ring, and the subadditive function $\phi(-) = \mu(-) : \text{mod } R \to \mathbb{Z}$ being the number of generators function in Definition 4.10, we see that $\mathrm{Ext}^1_R(M,N)^\mu = \langle M,N \rangle$ in the notation of [35, Def. 1.35]. So, by our discussion, Theorem 4.8, Proposition 3.8, and Corollary 3.9 recover [35, Cor. 1.36 and Props. 1.37 and 1.38].

Before we apply our discussion to more certain special cases, we record one preliminary lemma.

Lemma 4.12. *Let $(\mathcal{X},\mathcal{E})$ be an exact subcategory of* mod R. *Let $G : \mathcal{X} \to \mathrm{fl}(R)$ be an additive half-exact functor, where $\mathrm{fl}(R)$ denotes the full subcategory of* mod R *consisting of finite length modules. Then, $\phi(-) := \lambda_R(G(-)) : \mathcal{X} \to \mathbb{Z}$ satisfies the hypothesis of Theorem 4.8. Moreover, given a short exact sequence σ with objects in $\mathcal{X}$, $G(\sigma)$ is a short exact sequence of modules if and only if ϕ is additive on σ.*

Proof. We will only prove the statement when G is a covariant functor, since the proof for a contravariant functor is similar. Since G is a functor and λ_R is constant on isomorphism classes of objects in mod R, ϕ is constant on isomorphism classes of objects in $\mathcal{X}$. Let $A,B \in \mathcal{X}$. Then $\phi(A \oplus B) = \lambda_R(G(A \oplus B)) = \lambda_R(G(A) \oplus G(B)) = \lambda_R(G(A)) + \lambda_R(G(B)) = \phi(A) + \phi(B)$, so ϕ is additive on finite biproducts. Next, let $\sigma : 0 \to M \xrightarrow{f} N \xrightarrow{g} L \to 0$ be a short exact sequence with objects in $\mathcal{X}$. Then $G(\sigma)$ is the exact sequence $G(\sigma) : 0 \to K \to G(M) \xrightarrow{G(f)} G(N) \xrightarrow{G(g)} G(L) \to P \to 0$, where $K = \mathrm{Ker}(G(f))$ and $P = \mathrm{Coker}(G(g))$. Then we have $\phi(M) + \phi(L) - \phi(N) = \lambda_R(G(M)) + \lambda_R(G(L)) - \lambda_R(G(N)) = \lambda_R(K) + \lambda_R(P) \geq 0$, so $\phi(M) + \phi(L) \geq \phi(N)$. Hence, ϕ is subadditive on a short exact sequence with objects in

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$\mathcal{X}$. Thus, ϕ satisfies the hypothesis of Proposition 4.8. Next, let $\sigma : 0 \to M \xrightarrow{f} N \xrightarrow{g} L \to 0$ be a short exact sequence with objects in $\mathcal{X}$. Then $G(\sigma)$ is short exact if and only if $K = \mathrm{Ker}(G(f)) = 0$ and $P = \mathrm{Coker}(G(g)) = 0$ if and only if $\lambda_R(K) = \lambda_R(P) = 0$ if and only if $\phi(M) + \phi(L) - \phi(N) = \lambda_R(G(M)) + \lambda_R(G(L)) - \lambda_R(G(N)) = \lambda_R(K) + \lambda_R(P) = 0$ if and only if $\phi(M) + \phi(L) = \phi(N)$ if and only if ϕ is additive on σ. ∎

COROLLARY 4.13. *Let $(\mathcal{X}, \mathcal{E})$ be an exact subcategory of* mod R. *Let $G : \mathcal{X} \to \mathrm{fl}(R)$ be an additive half-exact functor, where $\mathrm{fl}(R)$ denotes the full subcategory of* mod R *consisting of finite length modules. Set $\mathcal{E}^G := \{\sigma \in \mathcal{E} : G(\sigma)$ is a short exact sequence of modules$\}$. Then, $(\mathcal{X}, \mathcal{E}^G)$ is an exact subcategory of* mod R.

Proof. From Lemma 4.12, it follows that $\mathcal{E}^G = \mathcal{E}^\phi$, where $\phi(-) := \lambda_R(G(-)) : \mathcal{X} \to \mathbb{Z}$. Hence, the claim now follows from Theorem 4.8. ∎

Now again, let $\mathcal{X} = \mathrm{mod}\ R$, where R is Noetherian. Basic examples of additive half-exact functors $G : \mathcal{X} \to \mathrm{fl}(R)$ are $G(-) := \mathrm{Ext}^i_R(-, X), \mathrm{Ext}^i_R(X, -), \mathrm{Tor}^R_i(X, -)$, where X is a fixed R-module of finite length and $i \geq 0$ is an integer.

Given a collection $\mathcal{C} \subseteq \mathrm{mod}\ R$, in [1], the authors considered the following:

$$F_{\mathcal{C}}(C, A) := \{\sigma : 0 \to A \to B \to C \to 0 \mid \mathrm{Hom}_R(X, \sigma) \text{ is short exact for all } X \in \mathcal{C}\},$$
$$F^{\mathcal{C}}(C, A) := \{\sigma : 0 \to A \to B \to C \to 0 \mid \mathrm{Hom}_R(\sigma, X) \text{ is short exact for all } X \in \mathcal{C}\}$$

and showed in [1, Prop. 1.7] that these define subfunctors of $\mathrm{Ext}^1_R(-,-) : \mathrm{mod}\ R^{op} \times \mathrm{mod}\ R \to \mathbf{Ab}$. Using Corollary 4.13, we recover this result when $\mathcal{C} \subseteq \mathrm{fl}(R)$ as follows.

COROLLARY 4.14. *Let R be a commutative Noetherian ring. Let $\mathcal{C} \subseteq \mathrm{fl}(R)$. Then, $F_{\mathcal{C}}(-,-)$ and $F^{\mathcal{C}}(-,-)$ are subfunctors of* $\mathrm{Ext}^1_R(-,-) : \mathrm{mod}\ R^{op} \times \mathrm{mod}\ R \to \mathrm{mod}\ R$.

Proof. We only prove the case of $F_{\mathcal{C}}(-,-)$, since the proof of $F^{\mathcal{C}}(-,-)$ is similar.

Let $\mathcal{S}^{\mathcal{C}}$ be the collection of all short exact sequences σ of finitely generated R-modules such that $\mathrm{Hom}_R(X, \sigma)$ is short exact for all $X \in \mathcal{C}$. We first prove that $(\mathrm{mod}\ R, \mathcal{S}^{\mathcal{C}})$ is an exact subcategory of mod R. Since $\mathcal{S}^{\mathcal{C}} = \cap_{C \in \mathcal{C}} \mathcal{S}^C$, it is enough to prove that for each $C \in \mathcal{C}$, $\mathcal{S}^C$ gives an exact substructure on mod R (intersection of exact substructures is again an exact substructure by Lemma 3.2). Now, note that $\mathcal{S}^C = \{\sigma : G(\sigma) \text{ is short exact}\}$, where $G : \mathrm{mod}\ R \to \mathrm{fl}(R)$ is given by $G(-) := \mathrm{Hom}_R(C, -)$, so by Corollary 4.13 we get that $(\mathrm{mod}\ R, \mathcal{S}^C)$ is an exact subcategory of mod R. Since $\mathrm{Ext}_{\mathcal{S}^{\mathcal{C}}}(-,-) = F_{\mathcal{C}}(-,-)$, by Proposition 3.8 we are done. ∎

Next, we recover and improve upon [31, Ths. 3.11 and 3.13].

For a Noetherian local ring $(R, \mathfrak{m})$, let $\mathrm{MD}(R)$ denote the full subcategory of mod R consisting of all modules M such that either $M = 0$ or $\dim M = \dim R$. We recall that if I is an $\mathfrak{m}$-primary ideal, then $e_R(I, -)$ is additive on $\mathrm{MD}(R)$. We also recall that a finitely generated module M satisfies Serre's condition (S_n) if depth $M_{\mathfrak{p}} \geq \inf\{n, \dim R_{\mathfrak{p}}\}$ for all $\mathfrak{p} \in \mathrm{Spec}\, R$. The full subcategory of mod R consisting of all modules satisfying (S_n) is denoted by $S_n(R)$. Note that $S_n(R)$ is closed under extensions in mod R.

LEMMA 4.15. *Let R be an equidimensional local ring. Let $S_1(R)$ be the collection of all modules in* mod R *satisfying Serre's condition (S_1). Then, $S_1(R) \subseteq \mathrm{MD}(R)$.*

Proof. The hypothesis on R implies that $\dim(R/\mathfrak{p}) = \dim R$ for all $\mathfrak{p} \in \mathrm{Min}(R)$. So, if $0 \neq M$ satisfy (S_1), then there exists $\mathfrak{p} \in \mathrm{Ass}(M) \subseteq \mathrm{Min}(R)$. Hence, $\dim(M) \geq \dim(R/\mathfrak{p}) = \dim R$, and we are done. ∎

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Now, inspecting the proof of [30, Prop. 17], the only place, where R is Cohen–Macaulay and M is maximal Cohen–Macaulay are required, is to ensure that $e_R(I,-)$ is additive on $0 \to \Omega_R M \to F \to M \to 0$ and that $M, \Omega_R M \in \mathrm{MD}(R)$. But if we assume R is equidimensional and satisfy (S_1), and $M \in S_1(R) \subseteq \mathrm{MD}(R)$, then since $\Omega_R M \in S_1(R) \subseteq \mathrm{MD}(R)$ (Lemma 4.15) and $e_R(I,-)$ is additive on short exact sequence of modules of the same dimension [3, Cor. 4.7.7]; hence, we get the following lemma by following the same proof as in [30, Prop. 17].

Lemma 4.16. *Let $(R,\mathfrak{m})$ be an equidimensional local ring satisfying (S_1). Let $M \in S_1(R)$. Let $d = \dim R \geq 1$, and let I be an $\mathfrak{m}$-primary ideal. Then, the function $n \mapsto \lambda(\operatorname{Tor}_1^R(M, R/I^{n+1}))$ is given by a polynomial of degree $\leq d-1$ for $n \gg 0$. So, in particular, the limit $\lim_{n\to\infty} \dfrac{\lambda(\operatorname{Tor}_1^R(M,R/I^{n+1}))}{n^{d-1}}$ exists.*

If R and M are as in Lemma 4.16, then let us denote $e_I^T(M) := (d-1)!\lim_{n\to\infty} \dfrac{\lambda(\operatorname{Tor}_1^R(M,R/I^{n+1}))}{n^{d-1}}$. (What [31] denotes by $e_R^T(M)$ is exactly the same as $e_{\mathfrak{m}}^T(M)$ in our notation.)

The following theorem generalizes [31, Ths. 3.11 and 3.13] owing to the fact that a local Cohen–Macaulay ring R is equidimensional, satisfies (S_1) and $\mathrm{CM}(R) \subseteq S_1(R)$.

Theorem 4.17. *Let $(R,\mathfrak{m})$ be an equidimensional local ring satisfying (S_1). Let $\mathcal{X} = S_1(R)$ be the subcategory of* mod R *consisting of all modules satisfying (S_1). Let I be an $\mathfrak{m}$-primary ideal. Then, $\operatorname{Ext}_R^1(-,-)^{e_I^T} : S_1(R)^{op} \times S_1(R) \to \operatorname{mod} R$ is a subfunctor of $\operatorname{Ext}_R^1(-,-) : S_1(R)^{op} \times S_1(R) \to \operatorname{mod} R$.*

Proof. Since $S_1(R)$ is an extension closed subcategory of mod R, we have $(S_1(R), \mathcal{S}|_{S_1(R)})$ is an exact subcategory of mod R. Since for each n, the function $\operatorname{mod} R \to \mathbb{Z}$ given by $M \mapsto \lambda(\operatorname{Tor}_1^R(M,R/I^{n+1}))$ is subadditive on short exact sequences of mod R, we have $e_I^T : S_1(R) \to \mathbb{Z}$ is also subadditive on short exact sequences of $S_1(R)$. Thus, $(S_1(R), \mathcal{S}|_{S_1(R)}^{e_I^T})$ is an exact subcategory of $(S_1(R), \mathcal{S}|_{S_1(R)})$ by Theorem 4.8. Hence, we are done by Proposition 3.8. ∎

§5. Special subfunctors of $\mathbf{Ext^1}$ and applications

Throughout this section, R will denote a Noetherian local ring with unique maximal ideal $\mathfrak{m}$ and residue field k. We will denote by $\mu_R(-)$ the minimal number of generators function for finitely generated modules, that is, $\mu_R(M) = \lambda_R(M \otimes_R R/\mathfrak{m})$ for all $M \in \operatorname{mod} R$. We drop the subscript R when the ring in question is clear. We shall study properties and applications of a number of subfunctors of Ext^1, whose existence follow from results in previous sections.

5.1 Some computations and applications of the subfunctor $\mathbf{Ext}_R^1(-,-)^{\mu}$

We start this subsection with a characterization of regularity among Cohen–Macaulay rings of positive dimension d in terms of vanishing of certain $\operatorname{Ext}_R^1(-,-)^{\mu}$ (see Definition 4.10 and the discussion after 4.11 for notation).

Theorem 5.1.1. *Let R be a local Cohen–Macaulay ring of dimension $d \geq 1$. Then, R is regular if and only if $\operatorname{Ext}_R^1(\Omega_R^{d-1} k, N)^{\mu} = 0$ for some finitely generated nonzero R-module N of finite injective dimension.*

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When R is regular, we show $\operatorname{Ext}^1_R(\Omega_R^{d-1}k, R)^\mu = 0$, which shows one direction of Theorem 5.1.1. For this, we first need two preliminary lemmas.

Lemma 5.1.2. *Let Y be a submodule of an R-module X such that $\mathfrak{m}X \subseteq Y$. If X is cyclic and $Y \neq X$, then $Y = \mathfrak{m}X$.*

Proof. We prove that if $Y \neq \mathfrak{m}X$, then $Y = X$. If $Y \neq \mathfrak{m}X$, then pick $y \in Y \setminus \mathfrak{m}X \subseteq X \setminus \mathfrak{m}X$. Then, y is part of a minimal system of generators of X, but X is cyclic, so $X = Ry$. Also, $Ry \subseteq Y \subseteq X$. Hence, $Y = X$. ∎

Lemma 5.1.3. *Let M be a finitely generated R-module with the first Betti number 1. Then, $\operatorname{Ext}^1_R(M, \Omega_R M)^\mu = \mathfrak{m}\operatorname{Ext}^1_R(M, \Omega_R M)$.*

Proof. $\operatorname{Ext}^1_R(M, \Omega_R M)^\mu$ is a submodule of $\operatorname{Ext}^1_R(M, \Omega_R M)$ such that $\mathfrak{m}\operatorname{Ext}^1_R(M, \Omega_R M) \subseteq \operatorname{Ext}^1_R(M, \Omega_R M)^\mu$ by Lemma 5.2.1. By hypothesis, we have $\Omega_R M \cong R/I$ for some ideal $I \neq R$, and we have an exact sequence $0 \to R/I \to F \to M \to 0$, for some free R-module F. So, in particular, $\mu(F) = \mu(M) \neq \mu(R/I) + \mu(M)$. Hence, this exact sequence is not μ-additive. So, $\operatorname{Ext}^1_R(M, \Omega_R M)^\mu \neq \operatorname{Ext}^1_R(M, \Omega_R M)$. Also, applying $\operatorname{Hom}_R(-, R/I)$ to the exact sequence $0 \to R/I \to F \to M \to 0$, we get an exact sequence $0 \to \operatorname{Hom}_R(M, R/I) \to \operatorname{Hom}_R(F, R/I) \to \operatorname{Hom}_R(R/I, R/I) \cong R/I \to \operatorname{Ext}^1_R(M, R/I) \cong \operatorname{Ext}^1_R(M, \Omega_R M) \to 0$. Hence, $\operatorname{Ext}^1_R(M, \Omega_R M)$ is a cyclic module. Now, applying Lemma 5.1.2 to $Y = \operatorname{Ext}^1_R(M, \Omega_R M)^\mu$ and $X = \operatorname{Ext}^1_R(M, \Omega_R M)$, we get $\operatorname{Ext}^1_R(M, \Omega_R M)^\mu = \mathfrak{m}\operatorname{Ext}^1_R(M, \Omega_R M)$. ∎

Now, one direction of Theorem 5.1.1 is the following.

Corollary 5.1.4. *Let $(R, \mathfrak{m}, k)$ be a regular local ring of dimension $d \geq 1$. Then, $\operatorname{Ext}^1_R(\Omega_R^{d-1}k, R)^\mu = 0$.*

Proof. We know $\Omega_R \Omega_R^{d-1} k \cong R$ by looking at the Koszul complex of $R/\mathfrak{m}$, so the first Betti number of $\Omega_R^{d-1}k$ is 1. Hence, by Lemma 5.1.3, we get that $\operatorname{Ext}^1_R(\Omega_R^{d-1}k, R)^\mu = \mathfrak{m}\operatorname{Ext}^1_R(\Omega_R^{d-1}k, R) \cong \mathfrak{m}\operatorname{Ext}^d_R(k, R) \cong \mathfrak{m} \cdot k = 0$. ∎

The other direction of Theorem 5.1.1 requires more work. First, we begin with the following setup.

5.1.5. Let $(R, \mathfrak{m}, k)$ be a local Cohen–Macaulay ring of dimension $d \geq 1$ admitting a canonical module ω_R. Then, $\operatorname{Ext}^1_R(\Omega_R^{d-1}k, \omega_R) \cong k$. Hence, for any two non-split exact sequences $\alpha, \beta \in \operatorname{Ext}^1_R(\Omega_R^{d-1}k, \omega_R)$, we have $[\beta] = r[\alpha] \in \operatorname{Ext}^1_R(\Omega_R^{d-1}k, \omega_R)$ for some unit $r \in R$. So, we have the following pushout diagram:

$$\begin{tikzcd}
\alpha: 0 \arrow[r] & \omega_R \arrow[r] \arrow[d, "\cdot r"'] & X_\alpha \arrow[r] \arrow[d] & \Omega_R^{d-1}k \arrow[r] \arrow[d, equal] & 0 \\
\beta: 0 \arrow[r] & \omega_R \arrow[r] & X_\beta \arrow[r] & \Omega_R^{d-1}k \arrow[r] & 0
\end{tikzcd}$$

Since $r \in R$ is a unit, we have $\omega_R \xrightarrow{\cdot r} \omega_R$ is an isomorphism. Hence, by five lemma, $X_\alpha \cong X_\beta$. So, there exists a unique module (up to isomorphism), call it E^R, such that the middle term of every non-split exact sequence $0 \to \omega_R \to X \to \Omega^{d-1}k \to 0$ is isomorphic to E^R. Since $\Omega_R^{d-1}k$ has co-depth 1, by [27, Prop. 11.21], we get that $0 \to \omega_R \to E^R \to \Omega_R^{d-1}k \to 0$ is the minimal maximal Cohen–Macaulay (MCM) approximation of $\Omega_R^{d-1}k$.

Moving forward, we denote $\operatorname{Hom}_R(-, \omega_R)$ by $(-)^\dagger$.

We need to collect some properties of E^R to prove the other direction of Theorem 5.1.1.

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LEMMA 5.1.6. *Let $(R,\mathfrak{m},k)$ be a local Cohen–Macaulay ring of dimension 1 admitting a canonical module ω_R. Then, $E^R \cong \mathfrak{m}^\dagger$.*

Proof. Dualizing $0 \to \mathfrak{m} \to R \to k \to 0$ by ω_R, we get $0 \to \omega_R \to \mathfrak{m}^\dagger \to \operatorname{Ext}^1_R(k,\omega_R) \cong k \to 0$. So, we have the exact sequence $0 \to \omega_R \to \mathfrak{m}^\dagger \to k \to 0$, which is clearly non-split, since $\mathfrak{m}^\dagger$ has positive depth. But $k \oplus \omega_R$ has depth 0. Thus, $E^R \cong \mathfrak{m}^\dagger$. $\square$

We record the following lemma, which will be used to deduce further properties of the module E^R.

LEMMA 5.1.7. *Let $(R,\mathfrak{m},k)$ be a local ring, and let I be an ideal of R, which is not principal. Then, for every $x \in I \setminus \mathfrak{m}I$, we have $(x\mathfrak{m} :_{\mathfrak{m}} I) = (x\mathfrak{m} :_R I) = ((x) :_R I)$.*

Proof. Since trivially $(x\mathfrak{m} :_{\mathfrak{m}} I) \subseteq (x\mathfrak{m} :_R I) \subseteq ((x) :_R I)$ always holds, it is enough to prove the inclusion $((x) :_R I) \subseteq (x\mathfrak{m} :_{\mathfrak{m}} I)$. So, let $y \in ((x) :_R I)$, which means $yI \subseteq (x)$. If y were a unit, then we would have $I \subseteq y^{-1}(x) = (x) \subseteq I$, implying $I = (x)$, contradicting our assumption that I is not principal. Thus, we must have $y \in \mathfrak{m}$. Now, pick an arbitrary element $r \in I$. Then $yr \in yI \subseteq (x)$, so $yr = xs$ for some $s \in R$. If $s \notin \mathfrak{m}$, then $x = s^{-1}yr \in \mathfrak{m}I$, which is a contradiction. Hence, $s \in \mathfrak{m}$. So, $yr = xs \in x\mathfrak{m}$. Since $r \in I$ was arbitrary, we get $yI \subseteq x\mathfrak{m}$. Hence, $y \in (x\mathfrak{m} :_{\mathfrak{m}} I)$. $\square$

In the following, $\mathrm{r}(-)$ will denote the type of a module, that is, $\mathrm{r}(M) = \dim_k \operatorname{Ext}_R^{\operatorname{depth}_R M}(k,M)$.

LEMMA 5.1.8. *Let $(R,\mathfrak{m},k)$ be a non-regular complete local Cohen–Macaulay ring of dimension $d \geq 1$, with canonical module ω_R. Then, we have $\mu(E^R) = \mathrm{r}(R) + \mu(\Omega_R^{d-1}k)$.*

Proof. We prove this by induction on d. First, let $d = 1$. By Lemma 5.1.6, we have $E^R \cong \mathfrak{m}^\dagger$. Pick $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ to be R-regular. Then $\mu_R(E^R) = \mu_{R/xR}(E^R/xE^R)$. Now, $E^R/xE^R \cong \mathfrak{m}^\dagger/x\mathfrak{m}^\dagger \cong \operatorname{Hom}_{R/xR}(\mathfrak{m}/x\mathfrak{m}, \omega_R/x\omega_R) \cong (\mathfrak{m}/x\mathfrak{m})^\dagger$ (where the last two isomorphisms follow from [3, Prop. 3.3.3(a) and Th. 3.3.5(a)]). Now, $\operatorname{Hom}_{R/xR}((\mathfrak{m}/x\mathfrak{m})^\dagger, k) \cong \operatorname{Hom}_{R/xR}(k^\dagger, \mathfrak{m}/x\mathfrak{m}) \cong \operatorname{Hom}_{R/xR}(k, \mathfrak{m}/x\mathfrak{m}) \cong (x\mathfrak{m} :_{\mathfrak{m}} \mathfrak{m})/x\mathfrak{m} = ((x) :_R \mathfrak{m})/x\mathfrak{m}$ (where the first isomorphism holds because R/xR is Artinian, so one can invoke [10, Lem. 3.14 and Rem. 3.15], and the last equality follows from Lemma 5.1.7 since R is not regular). Hence, $\mu_R(E^R) = \mu_{R/xR}(E^R/xE^R) = \dim_k \operatorname{Hom}_{R/xR}((\mathfrak{m}/x\mathfrak{m})^\dagger, k) = \dim_k((x) :_R \mathfrak{m})/x\mathfrak{m} = \dim_k((xR :_R \mathfrak{m})/xR) + \dim_k(xR/x\mathfrak{m}) = \dim_k \operatorname{Soc}(R/xR) + \dim_k(xR/x\mathfrak{m}) = \mathrm{r}(R) + \dim_k(xR/x\mathfrak{m})$ (since $xR/x\mathfrak{m}$ is annihilated by $\mathfrak{m}$, hence it is a k-vector space). Since $xR/x\mathfrak{m}$ is generated by x and it is a k-vector space, we have $xR/x\mathfrak{m} \cong k$. Hence, $\dim_k(xR/x\mathfrak{m}) = 1$, so $\mu(E^R) = \mathrm{r}(R) + 1 = \mathrm{r}(R) + \mu(k)$. This concludes the $d = 1$ case.

Now, let $\dim R = d > 1$ and suppose the claim has been proved for all non-regular local Cohen–Macaulay rings of dimension $1, \ldots, d-1$ admitting a canonical module. Since $\Omega_R^{d-1}k$ has co-depth 1, by [27, Prop. 11.21] (see 5.1.5), we get that $0 \to \omega_R \to E^R \to \Omega_R^{d-1}k \to 0$ is the minimal MCM approximation of $\Omega_R^{d-1}k$. Now, pick an R-regular element $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ (which is also $\Omega_R^{d-1}k$-regular, since $d - 1 > 0$). By [39, Cor. 2.5], we have $0 \to \omega_{R/xR} \to E^R/xE^R \to \Omega_R^{d-1}k/x(\Omega_R^{d-1}k) \to 0$ is the minimal MCM approximation of $\Omega_R^{d-1}k/x(\Omega_R^{d-1}k) \cong \Omega_{R/xR}^{d-1}k \oplus \Omega_{R/xR}^{d-2}k$ over R/xR (the isomorphism follows from [37, Cor. 5.3]). Since $0 \to 0 \to \Omega_{R/xR}^{d-1}k \to \Omega_{R/xR}^{d-1}k \to 0$ is the minimal MCM approximation of $\Omega_{R/xR}^{d-1}k$ over R/xR and $0 \to \omega_{R/xR} \to E^{R/xR} \to \Omega_{R/xR}^{d-2}k \to 0$ is the minimal MCM approximation of $\Omega_{R/xR}^{d-2}k$ over R/xR (by 5.1.5 and [27, Prop. 11.21]), their direct sum $0 \to$

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$\omega_{R/xR} \to E^{R/xR} \oplus \Omega^{d-1}_{R/xR} k \to \Omega^{d-2}_{R/xR} k \oplus \Omega^{d-1}_{R/xR} k \to 0$ is the minimal MCM approximation of $\Omega^{d-2}_{R/xR} k \oplus \Omega^{d-1}_{R/xR} k \cong \Omega^{d-1}_R k/x(\Omega^{d-1}_R k)$ over R/xR (direct sum preserves minimal MCM approximation by [39, Th. 1.4]). By uniqueness of minimal MCM approximation, we have $E^R/xE^R \cong E^{R/xR} \oplus \Omega^{d-1}_{R/xR} k$. So, $\mu_R(E^R) = \mu_{R/xR}(E^R/xE^R) = \mu_{R/xR}(E^{R/xR}) + \mu_{R/xR}(\Omega^{d-1}_{R/xR} k)$. Since R is not regular, R/xR is not regular. Hence, by induction hypothesis, we have $\mu_{R/xR}(E^{R/xR}) = \mathrm{r}(R/xR) + \mu_{R/xR}(\Omega^{d-2}_{R/xR} k) = \mathrm{r}(R) + \mu_{R/xR}(\Omega^{d-2}_{R/xR} k)$. So, $\mu_R(E^R) = \mathrm{r}(R) + \mu_{R/xR}(\Omega^{d-2}_{R/xR} k) + \mu_{R/xR}(\Omega^{d-1}_{R/xR} k) = \mathrm{r}(R) + \mu_{R/xR}(\Omega^{d-2}_{R/xR} k \oplus \Omega^{d-1}_{R/xR} k) = \mathrm{r}(R) + \mu_{R/xR}\left(\Omega^{d-1}_R k/x(\Omega^{d-1}_R k)\right) = \mathrm{r}(R) + \mu_R(\Omega^{d-1}_R k)$. This finishes the inductive step, and hence the proof. ∎

Since $\mu(\omega_R) = \mathrm{r}(R)$, Lemma 5.1.8 says that the sequence $0 \to \omega_R \to E^R \to \Omega^{d-1}_R k \to 0$ is μ-additive. As a consequence of this, we get the following.

PROPOSITION 5.1.9. *Let $(R, \mathfrak{m}, k)$ be a non-regular local Cohen–Macaulay ring of dimension $d \geq 1$, and let N be a finitely generated nonzero R-module with finite injective dimension. Then* $\operatorname{Ext}^1_R(\Omega^{d-1}_R k, N)^\mu = \operatorname{Ext}^1_R(\Omega^{d-1}_R k, N) \neq 0$.

Proof. That $\operatorname{Ext}^1_R(\Omega^{d-1}_R k, N) \cong \operatorname{Ext}^d_R(k, N) \neq 0$ follows from [3, Exer. 3.1.24]. So, it is enough to prove that $\operatorname{Ext}^1_R(\Omega^{d-1}_R k, N)^\mu = \operatorname{Ext}^1_R(\Omega^{d-1}_R k, N)$.

We first consider the case where R is complete, hence admitting a canonical module ω_R. Consider the μ-additive (by Lemma 5.1.8) exact sequence $\sigma : 0 \to \omega_R \to E^R \to \Omega^{d-1}_R k \to 0$. Since $\sigma \in \mathcal{S}|^{\mu}_{\text{mod } R}$, by applying $\operatorname{Hom}_R(-, N)$ to σ, we get the following part of a commutative diagram of exact sequences by Corollary 3.9:

$$\begin{array}{ccccc}
\operatorname{Hom}_R(\omega_R, N) & \longrightarrow & \operatorname{Ext}^1_R(\Omega^{d-1}_R k, N) & \longrightarrow & \operatorname{Ext}^1_R(E^R, N) \\
\Vert & & \uparrow & & \uparrow \\
\operatorname{Hom}_R(\omega_R, N) & \longrightarrow & \operatorname{Ext}^1_R(\Omega^{d-1}_R k, N)^\mu & \longrightarrow & \operatorname{Ext}^1_R(E^R, N)^\mu
\end{array}$$

Since E^R is maximal Cohen–Macaulay, we have $\operatorname{Ext}^1_R(E^R, N) = 0$ by [3, Exer. 3.1.24]. So, $\operatorname{Ext}^1_R(E^R, N)^\mu = 0$ as well. Hence, we get the following commutative diagram:

$$\begin{array}{ccccc}
\operatorname{Hom}_R(\omega_R, N) & \xrightarrow{f} & \operatorname{Ext}^1_R(\Omega^{d-1}_R k, N) & \longrightarrow & 0 \\
\Vert & & \uparrow h & & \\
\operatorname{Hom}_R(\omega_R, N) & \xrightarrow{g} & \operatorname{Ext}^1_R(\Omega^{d-1}_R k, N)^\mu & \longrightarrow & 0
\end{array}$$

Thus, $h \circ g = f$ is surjective, so h is surjective. Since h is the natural inclusion map, we have $\operatorname{Ext}^1_R(\Omega^{d-1}_R k, N)^\mu = \operatorname{Ext}^1_R(\Omega^{d-1}_R k, N)$.

Now, we consider the general case. Since $\operatorname{Ext}^1_R(\Omega^{d-1}_R k, N)^\mu \subseteq \operatorname{Ext}^1_R(\Omega^{d-1}_R k, N)$, it is enough to prove the other inclusion. So, let $\sigma : 0 \to N \to X \to \Omega^{d-1}_R k \to 0$ be a short exact sequence. We need to show that σ is μ-additive. Now, consider the completion $\widehat{\sigma} : 0 \to \widehat{N} \to \widehat{X} \to \widehat{\Omega^{d-1}_R k} \cong \Omega^{d-1}_{\widehat{R}} k \to 0$. Since $\widehat{R}$ is non-regular, Cohen–Macaulay of dimension d admits a canonical module, and $\widehat{N} \in \operatorname{mod} \widehat{R}$ has finite injective dimension over $\widehat{R}$, by the first part of the proof, we get $\operatorname{Ext}^1_{\widehat{R}}(\Omega^{d-1}_{\widehat{R}} k, \widehat{N}) = \operatorname{Ext}^1_{\widehat{R}}(\Omega^{d-1}_{\widehat{R}} k, \widehat{N})^\mu$. Thus, $[\widehat{\sigma}] \in \operatorname{Ext}^1_{\widehat{R}}(\Omega^{d-1}_{\widehat{R}} k, \widehat{N}) = \operatorname{Ext}^1_{\widehat{R}}(\Omega^{d-1}_{\widehat{R}} k, \widehat{N})^\mu$. Hence, $\widehat{\sigma}$ is μ-additive. So, $\mu_{\widehat{R}}(\widehat{N}) + \mu_{\widehat{R}}(\Omega^{d-1}_{\widehat{R}} k) = \mu_{\widehat{R}}(\widehat{X})$. Since the number of generators does not change under completion, we get $\mu_R(N) + \mu_R(\Omega^{d-1}_R k) = \mu_R(X)$. Thus, σ is μ-additive, which is what we wanted to prove. ∎

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Proof of Theorem 5.1.1. Follows by combining Corollary 5.1.4 and Proposition 5.1.9. ∎

For an arbitrary local ring of positive depth, we give a characterization of the ring being regular in terms of vanishing of $\operatorname{Ext}^1_R(k,M)^\mu$ for some $M \in \operatorname{mod} R$ of finite projective dimension. For this, we first recall the definitions of weakly $\mathfrak{m}$-full and Burch submodules of a module from [12, Defs. 3.1 and 4.1] and subsequently, we relate that property to the vanishing of certain $\operatorname{Ext}^1_R(k,-)^\mu$.

DEFINITION 5.1.10. Let $(R,\mathfrak{m},k)$ be a local ring, and let N be an R-submodule of a finitely generated R-module M. Then N is called a weakly $\mathfrak{m}$-full submodule of M if $(\mathfrak{m}N :_M \mathfrak{m}) = N$. Also, N is called a Burch submodule of M if $\mathfrak{m}(N :_M \mathfrak{m}) \neq \mathfrak{m}N$.

PROPOSITION 5.1.11. *Let $(R,\mathfrak{m},k)$ be a local ring, and let N be an R-submodule of a finitely generated R-module M such that $\operatorname{Ext}^1_R(k,N)^\mu = 0$. Then $(\mathfrak{m}N :_M \mathfrak{m}) = N + \operatorname{Soc}(M)$, that is, $N + \operatorname{Soc}(M)$ is a weakly $\mathfrak{m}$-full submodule of M. So, in particular, if we moreover have $\operatorname{depth}(M) > 0$, then N is a weakly $\mathfrak{m}$-full submodule of M.*

Proof. Clearly, $N \subseteq (\mathfrak{m}N :_M \mathfrak{m})$ and $\operatorname{Soc}(M) = (0 :_M \mathfrak{m}) \subseteq (\mathfrak{m}N :_M \mathfrak{m})$, so $N + \operatorname{Soc}(M) \subseteq (\mathfrak{m}N :_M \mathfrak{m})$. Now, choose an element $f \in (\mathfrak{m}N :_M \mathfrak{m})$, and we aim to show $f \in N + \operatorname{Soc}(M)$. If $f \in N$, then we are done. Otherwise, say $f \notin N$. As $f \in (\mathfrak{m}N :_M \mathfrak{m})$, so $\mathfrak{m}f \subseteq \mathfrak{m}N \subseteq N$, and hence $\mathfrak{m}(N + Rf) = \mathfrak{m}N$. Thus, $\dfrac{Rf+N}{N}$ is a nonzero (as $f \notin N$) k-vector space (as $\mathfrak{m}(N + Rf) \subseteq N$). Moreover, $\dfrac{Rf+N}{N}$ is cyclically generated by the image of f in M/N. Hence, $\dfrac{Rf+N}{N} \cong k$, so we have a short exact sequence $\sigma : 0 \to N \to Rf + N \to k \to 0$. This sequence is μ-additive, since $\mu(N+Rf) = \lambda\left(\frac{N+Rf}{\mathfrak{m}(N+Rf)}\right) = \lambda\left(\frac{N+Rf}{\mathfrak{m}N}\right) = \lambda\left(\frac{N+Rf}{N}\right) + \lambda\left(\frac{N}{\mathfrak{m}N}\right) = \lambda(k) + \mu(N) = \mu(k) + \mu(N)$. So, $\sigma \in \operatorname{Ext}^1_R(k,N)^\mu = 0$. Hence, σ splits. Thus, $N + Rf = N \oplus g(k)$, where $g : k \to N + Rf$ is the splitting map, so $k \cong g(k)$. Now, $f \in N + Rf = N \oplus g(k)$, so $f = x + y$ for some $x \in N$ and $y \in g(k) \subseteq M$. Since $k \cong g(k)$, we have $\mathfrak{m}g(k) = 0$. Now, $\mathfrak{m}y \in \mathfrak{m}g(k) = 0$, so $y \in (0 :_M \mathfrak{m}) = \operatorname{Soc}(M)$. Hence, $f = x + y \in N + \operatorname{Soc}(M)$. This finally shows that $(\mathfrak{m}N :_M \mathfrak{m}) \subseteq N + \operatorname{Soc}(M)$, so $(\mathfrak{m}N :_M \mathfrak{m}) = N + \operatorname{Soc}(M)$. Since $\mathfrak{m}(N + \operatorname{Soc}(M)) = \mathfrak{m}N$, we get $(\mathfrak{m}(N + \operatorname{Soc}(M)) :_M \mathfrak{m}) = N + \operatorname{Soc}(M)$, which implies that $N + \operatorname{Soc}(M)$ is a weakly $\mathfrak{m}$-full submodule M. Now note that if $\operatorname{depth}(M) > 0$, then $\operatorname{Soc}(M) = 0$. Hence, $N + \operatorname{Soc}(M) = N$ is a weakly $\mathfrak{m}$-full submodule of M. ∎

Now, we give another characterization of regular local rings in terms of vanishing of certain $\operatorname{Ext}^1_R(k,-)^\mu$.

THEOREM 5.1.12. *Let $(R,\mathfrak{m},k)$ be a local ring of depth $t > 0$. Then, the following are equivalent:*

(1) *R is regular.*
(2) *$\operatorname{Ext}^1_R(k, R/(x_1,\dots,x_{t-1})R)^\mu = 0$ for some R-regular sequence $x_1,\dots,x_{t-1}$.*
(3) *$\operatorname{Ext}^1_R(k,M)^\mu = 0$ for some finitely generated R-module M of projective dimension $t-1$.*

Proof. (1) $\implies$ (2) Since R is regular, we have $t = \dim R$. Since R is regular, we get that $\mathfrak{m}$ is generated by a regular sequence $x_1,\dots,x_t$. Now, $R/(x_1,\dots,x_{t-1})R$ has projective dimension $t-1$ over R and the $(t-1)$th Betti number of this module is 1 (by looking at the Koszul complex). Hence, $\operatorname{Ext}^1_R(k, R/(x_1,\dots,x_{t-1})R) \cong \operatorname{Tor}^R_{t-1}(k, R/(x_1,\dots,x_{t-1})R) \cong k$ by [3, Exer. 3.3.26]. Now, we have an exact sequence

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$0 \to R/(x_1,\dots,x_{t-1})R \xrightarrow{\cdot x_t} R/(x_1,\dots,x_{t-1})R \to R/(x_1,\dots,x_{t-1},x_t)R \cong k \to 0$, which is clearly not μ-additive. Hence, $\operatorname{Ext}^1_R(k,R/(x_1,\dots,x_{t-1})R) \neq \operatorname{Ext}^1_R(k,R/(x_1,\dots,x_{t-1})R)^\mu$. Since $\operatorname{Ext}^1_R(k,R/(x_1,\dots,x_{t-1})R) \cong k$ is cyclic, by Lemma 5.1.2, we get that $\operatorname{Ext}^1_R(k,R/(x_1,\dots,x_{t-1})R)^\mu = \mathfrak{m}\operatorname{Ext}^1_R(k,R/(x_1,\dots,x_{t-1})R) = 0$.

(2) $\implies$ (3) Obvious.

(3) $\implies$ (1) By the Auslander–Buchsbaum formula, depth $M = 1$, so $\operatorname{Soc}(M) = 0$. Hence, by prime avoidance, we can choose $x \in \mathfrak{m}$, which is both R and M-regular. Then $xM \cong M$, so $\operatorname{Ext}^1_R(k,xM)^\mu \cong \operatorname{Ext}^1_R(k,M)^\mu = 0$. By Proposition 5.1.11, we get that xM is a weakly $\mathfrak{m}$-full submodule of M. Since $\operatorname{depth}(M/xM) = 0$, xM is a Burch submodule of M by [12, Lem. 4.3]. Moreover, $\operatorname{pd}_{R/xR}(M/xM) = \operatorname{pd}_R M < \infty$, and $\operatorname{pd}_R R/xR < \infty$, so $\operatorname{pd}_R M/xM < \infty$. Hence, $\operatorname{Tor}^R_{\gg 0}(k,M/xM) = 0$. Thus, $\operatorname{pd}_R k < \infty$ by [12, Th. 1.2], so R is regular. ∎

Next, we try to relate $\operatorname{Ext}^1_R(M,N)^\mu$ to $\operatorname{Ext}^1_R(M,N)$ for all $M,N \in \operatorname{mod} R$, when $(R,\mathfrak{m})$ is a regular local ring of dimension 1 (i.e., a local PID). For this, we first record a preliminary lemma.

LEMMA 5.1.13. *Let $(R,\mathfrak{m})$ be a local ring, and let $x \in \mathfrak{m}$ be a nonzero divisor. Then,* $\operatorname{Ext}^1_R(R/xR,R/I)^\mu = \mathfrak{m}\operatorname{Ext}^1_R(R/xR,R/I) \cong \dfrac{\mathfrak{m}}{I+xR}$ *for every proper ideal I of R.*

Proof. Calculating $\operatorname{Ext}^1_R(R/xR,R/I)$ from the minimal free-resolution $0 \to R \xrightarrow{\cdot x} R \to R/xR \to 0$ of R/xR, we see that $\operatorname{Ext}^1_R(R/xR,R/I) \cong R/(xR+I)$ is a cyclic R-module. By Lemma 5.2.1, we have $\mathfrak{m}\operatorname{Ext}^1_R(R/xR,R/I) \subseteq \operatorname{Ext}^1_R(R/xR,R/I)^\mu$. We finally claim that $\operatorname{Ext}^1_R(R/xR,R/I)^\mu \neq \operatorname{Ext}^1_R(R/xR,R/I)$. Indeed, we have an exact sequence $\sigma : 0 \to xR/xI \to R/xI \to R/xR \to 0$. Now, we have a natural surjection $R \xrightarrow{r \mapsto rx+xI} xR/xI$, whose kernel is $\{r \in R : rx \in xI\}$. Since x is a nonzero divisor, $rx \in xI$ if and only if $r \in I$. Hence, the kernel is I. Hence, $R/I \cong xR/xI$. So, we get the exact sequence $\sigma : 0 \to R/I \to R/xI \to R/xR \to 0$. Moreover, σ is not μ-additive, since $\mu(R/xI) = 1 \neq 1+1 = \mu(R/I)+\mu(R/xR)$. Thus, $[\sigma] \in \operatorname{Ext}^1_R(R/xR,R/I) \setminus \operatorname{Ext}^1_R(R/xR,R/I)^\mu$. Hence, $\operatorname{Ext}^1_R(R/xR,R/I)^\mu = \mathfrak{m}\operatorname{Ext}^1_R(R/xR,R/I)$ by Lemma 5.1.2. ∎

Now, using this lemma, we can compare the structure of $\operatorname{Ext}^1_R(M,N)^\mu$ to $\operatorname{Ext}^1_R(M,N)$ for every pair of finitely generated modules M,N over a DVR.

PROPOSITION 5.1.14. *Let $(R,\mathfrak{m})$ be a regular local ring of dimension 1. Then,* $\operatorname{Ext}^1_R(M,N)^\mu = \mathfrak{m}\operatorname{Ext}^1_R(M,N)$ *for all finitely generated R-modules M and N. So, in particular,* $\operatorname{Ext}^1_R(k,N)^\mu = 0$ *for all finitely generated R-modules N.*

Proof. Let $\mathfrak{m} = xR$. Then, for every finitely generated R-module X, we have $X \cong R^{\oplus a} \oplus (\oplus_{i=1}^n R/x^{a_i}R)$ for some nonnegative integers (depending on X) a, a_i. Now, fix arbitrary $N \in \operatorname{mod} R$. Applying Lemma 2.4 to the subfunctor $\operatorname{Ext}^1_R(-,N)^\mu$ of $\operatorname{Ext}^1_R(-,N)$ and taking $I = \mathfrak{m}$, it is enough to prove that $\operatorname{Ext}^1_R(R,N)^\mu = \mathfrak{m}\operatorname{Ext}^1_R(R,N)$ and $\operatorname{Ext}^1_R(R/x^lR,N)^\mu = \mathfrak{m}\operatorname{Ext}^1_R(R/x^lR,N)$ for every integer $l \geq 1$. Now, $\operatorname{Ext}^1_R(R,N)^\mu = \mathfrak{m}\operatorname{Ext}^1_R(R,N)$ is obvious, as both sides are zero. Now, to prove $\operatorname{Ext}^1_R(R/x^lR,N)^\mu = \mathfrak{m}\operatorname{Ext}^1_R(R/x^lR,N)$ for every integer $l \geq 1$, first fix an $l \geq 1$, and look at the subfunctor $\operatorname{Ext}^1_R(R/x^lR,-)^\mu$ of $\operatorname{Ext}^1_R(R/x^lR,-)$. Again, by the structure of finitely generated R-modules and Lemma 2.4, it is enough to prove that $\operatorname{Ext}^1_R(R/x^lR,R)^\mu = \mathfrak{m}\operatorname{Ext}^1_R(R/x^lR,R)$ and $\operatorname{Ext}^1_R(R/x^lR,R/x^bR)^\mu = \mathfrak{m}\operatorname{Ext}^1_R(R/x^lR,R/x^bR)$ for every integer $b \geq 1$. Now, these equalities follow from Lemma 5.1.13, since x is a nonzero divisor. ∎

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When restricting to short exact sequences in Ext^1, on which certain subadditive function, other than $\mu(-)$, is additive, one obtains vanishing of the corresponding submodule of $\operatorname{Ext}^1_R(M,F)$ for any free R-module F. We make this precise in Proposition 5.1.16, whose proof uses the following lemma.

LEMMA 5.1.15. *Let $(R,\mathfrak{m})$ be a local Cohen–Macaulay ring of dimension 1. Let $x\in\mathfrak{m}$ be R-regular. Let $\sigma: 0\to F_1\to F_2\oplus\frac{R}{x^aR}\to\frac{R}{x^bR}\to 0$ be a short exact sequence, where a,b are nonnegative integers and F_1,F_2 are free R-modules. Let c be an integer such that $c\geq b$ and $\sigma\otimes\frac{R}{x^cR}$ is short exact. Then $a=b$ and $\operatorname{rank}(F_2)=\operatorname{rank}(F_1)$. So, in particular, σ is split exact.*

Proof. Since $x\in R$ is a nonzero divisor, so $\frac{R}{x^aR},\frac{R}{x^bR}$ are torsion modules, that is, have constant rank 0. Hence, calculating rank along σ, we get $\operatorname{rank}(F_2)=\operatorname{rank}(F_1)$. Next, we will show that $a\leq b$. Dualizing σ by R, we get the following part of a long exact sequence: $\operatorname{Ext}^1_R\left(\frac{R}{x^bR},R\right)\to\operatorname{Ext}^1_R\left(\frac{R}{x^aR},R\right)\to 0$. Now, since $x^a\in R$ is a nonzero divisor, by taking the resolution $0\to R\xrightarrow{\cdot x^a}R\to\frac{R}{x^aR}\to 0$ of $\frac{R}{x^aR}$, and dualizing by R, and calculating the cohomology, we get that $\operatorname{Ext}^1_R\left(\frac{R}{x^aR},R\right)\cong\frac{R}{x^aR}$. Similarly, $\operatorname{Ext}^1_R\left(\frac{R}{x^bR},R\right)\cong\frac{R}{x^bR}$. Hence, we have the exact sequence $\frac{R}{x^bR}\to\frac{R}{x^aR}\to 0$. Since x^b annihilates $\frac{R}{x^bR}$, we have x^b annihilates $\frac{R}{x^aR}$ as well, which implies $x^bR\subseteq x^aR$. Since $x\in\mathfrak{m}$ is R-regular, so $x^bR\subseteq x^aR$ implies that $a\leq b$. Since $\sigma\otimes\frac{R}{x^cR}$ is short exact, we have the following short exact sequence:

$$\sigma\otimes\frac{R}{x^cR}:0\to\left(\frac{R}{x^cR}\right)^{\oplus l_1}\to\left(\frac{R}{x^cR}\right)^{\oplus l_2}\oplus\frac{R}{x^aR}\to\frac{R}{x^bR}\to 0,$$

where we have used that $\frac{R}{x^cR}\otimes\frac{R}{x^aR}\cong\frac{R}{x^cR+x^aR}\cong\frac{R}{x^aR}$ and $\frac{R}{x^cR}\otimes\frac{R}{x^bR}\cong\frac{R}{x^cR+x^bR}\cong\frac{R}{x^bR}$, as $a\leq b\leq c$. Since R is local Cohen–Macaulay of dimension 1 and $x\in\mathfrak{m}$ is R-regular, we have x^aR,x^bR,x^cR are $\mathfrak{m}$-primary ideals. Hence, by calculating the length along the short exact sequence $\sigma\otimes\frac{R}{x^cR}$, we get that $\lambda\left(\frac{R}{x^bR}\right)-\lambda\left(\frac{R}{x^aR}\right)=(l_2-l_1)\lambda\left(\frac{R}{x^cR}\right)=0$. Since $x^bR\subseteq x^aR$, so $\lambda\left(\frac{R}{x^bR}\right)=\lambda\left(\frac{R}{x^aR}\right)$ now implies $x^aR=x^bR$ by calculating length along the short exact sequence $0\to\frac{x^aR}{x^bR}\to\frac{R}{x^bR}\to\frac{R}{x^aR}\to 0$. Since $x\in\mathfrak{m}$ is R-regular, we have $a=b$. Finally, since $F_1\cong F_2$ and $a=b$, we get σ is split exact. ∎

In the following, $\mathrm{H}^0_{\mathfrak{m}}(-)$ denotes the zeroth local cohomology module. For a finite length R-module M, $\ell\ell(M)$ will stand for the smallest integer $n\geq 0$ such that $\mathfrak{m}^nM=0$.

PROPOSITION 5.1.16. *Let $(R,\mathfrak{m})$ be a regular local ring of dimension 1. Let $L\in\operatorname{mod}(R)$. Let c be an integer such that $c\geq\ell\ell(\mathrm{H}^0_{\mathfrak{m}}(L))$. Consider the function $\phi_L(-):=\lambda(\frac{R}{\mathfrak{m}^c}\otimes -):\operatorname{mod}(R)\to\mathbb{N}\cup\{0\}$. Then, for any free R-module F, we have $\operatorname{Ext}^1_R(L,F)^{\phi_L}=0$.*

Proof. Note that R is not a field, since $\dim R=1$. Since R is a regular local ring, we have $L\cong G\oplus L'$, where G is a finite free R-module and L' is an R-module of finite length. We have $\mathrm{H}^0_{\mathfrak{m}}(L)\cong L'$. Since $\operatorname{Ext}^1_R(L,F)^{\phi_L}\cong\operatorname{Ext}^1_R(G,F)^{\phi_L}\oplus\operatorname{Ext}^1_R(L',F)^{\phi_L}\cong\operatorname{Ext}^1_R(L',F)^{\phi_L}$, we may replace L by L', and assume without loss of generality that L has finite length, and $c\geq\ell\ell(L)$. For simplicity, we denote $\phi_L=\phi$. Now, it is enough to show that $\operatorname{Ext}^1_R(L,R)^{\phi}=0$. Since $(R,\mathfrak{m})$ is a regular local ring of dimension 1, R is a PID. Hence, $\mathfrak{m}=xR$ for some $x\neq 0$. Since L is a finite length module, by the structure theorem of modules over a PID, we have $L\cong\oplus_{i=1}^n\frac{R}{x^{b_i}R}$, where $b_i>0$. So, $\ell\ell(L)=\max_{1\leq i\leq n}b_i$. Hence, it is enough to show that $\operatorname{Ext}^1_R\left(\frac{R}{x^{b_i}R},R\right)^{\phi}=0$ for all $i=1,\dots,n$. Fix an $i\in\{1,\dots,n\}$ and consider a short exact sequence $\sigma:0\to R\to X\to\frac{R}{x^{b_i}R}\to 0$ in $\operatorname{Ext}^1_R\left(\frac{R}{x^{b_i}R},R\right)^{\phi}$. By the structure theorem

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of modules over a PID, we also have $X \cong R^{\oplus s} \oplus \left(\oplus_{j=1}^{t} \frac{R}{x^{a_j}R}\right)$, where $a_j > 0$. As R is an integral domain and $\frac{R}{x^{b_i}R}$ and $\frac{R}{x^{a_j}R}$ are all torsion R-modules, so calculating rank along the short exact sequence σ, we obtain $s = 1$. Moreover, $1 + t = s + t = \mu(X) \leq 2$, since μ is subadditive. Thus, $t \leq 1$. If $t = 0$, then we obtain $\sigma : 0 \to R \to R \oplus \frac{R}{x^0 R} \to \frac{R}{x^{b_i}R} \to 0$. Since σ is ϕ-additive, by using Lemma 4.12 with the functor $\lambda_R\left(\frac{R}{\mathfrak{m}^c} \otimes_R -\right)$, we obtain that $\sigma \otimes \frac{R}{\mathfrak{m}^c} = \sigma \otimes \frac{R}{x^c R}$ is short exact. As $c \geq b_i > 0$, we obtain a contradiction from Lemma 5.1.15. Thus, $t = 1$. So, we get $X \cong R \oplus \frac{R}{x^a R}$ for some $a = a_j$. Thus, we have the short exact sequence $\sigma : 0 \to R \to R \oplus \frac{R}{x^a R} \to \frac{R}{x^{b_i}R} \to 0$.

Since $c \geq \ell\ell(L) = \max_{1 \leq i \leq n} b_i$, we have $c \geq b_i$ for all $i = 1, \ldots, n$. Hence, by applying Lemma 5.1.15 on σ, we get that σ is split exact. Since σ is an arbitrary element of $\operatorname{Ext}^1_R\left(\frac{R}{x^{b_i}R}, R\right)^{\phi}$, we get $\operatorname{Ext}^1_R\left(\frac{R}{x^{b_i}R}, R\right)^{\phi} = 0$ for all $i = 1, \ldots, n$. This implies that $\operatorname{Ext}^1_R(L, R)^{\phi} = 0$, so $\operatorname{Ext}^1_R(L, F)^{\phi} = 0$. ∎

Taking $L = k$ and $c = 1 = \ell\ell(k)$, we see that $\phi_L(-) = \mu(-)$ in Proposition 5.1.16. So, we also get another proof of Corollary 5.1.4 in dimension 1.

Next, we compare $\operatorname{Ext}^1_R(M, R)^{\mu}$ with $\operatorname{Ext}^1_R(M, R)$. In arbitrary dimension, we only consider this for local Cohen–Macaulay rings of minimal multiplicity. We first record some general preliminary lemmas.

Lemma 5.1.17. *Let $(R, \mathfrak{m}, k)$ be a local ring such that $\mathfrak{m}^2 = 0$ and $\mathfrak{m} \neq 0$. Let $e := \mu(\mathfrak{m})$. Then $\mu(\omega_R) = e$, $\mu(\Omega_R \omega_R) = e^2 - 1$, and $\Omega_R \omega_R \cong k^{\oplus(e^2-1)}$.*

Proof. Since $\mathfrak{m}^2 = 0$ and $\mathfrak{m} \neq 0$, we have $\mathfrak{m} \subseteq (0 :_R \mathfrak{m}) \subsetneq R$. Hence, $\mathfrak{m} = (0 :_R \mathfrak{m})$. Then we have $\mu(\omega_R) = \mathrm{r}(R) = \dim_k(0 :_R \mathfrak{m}) = \dim_k \mathfrak{m} = \dim_k\left(\frac{\mathfrak{m}}{\mathfrak{m}^2}\right) = \mu(\mathfrak{m}) = e$. Also, note that $\lambda(R) = \lambda\left(\frac{R}{\mathfrak{m}^2}\right) = \lambda\left(\frac{R}{\mathfrak{m}}\right) + \lambda\left(\frac{\mathfrak{m}}{\mathfrak{m}^2}\right) = 1 + \mu(\mathfrak{m})$. Next, consider the short exact sequence $0 \to \Omega_R \omega_R \to R^{\oplus \mu(\omega_R)} \to \omega_R \to 0$, so $\Omega_R \omega_R \subseteq \mathfrak{m} R^{\oplus \mu(\omega_R)}$. Hence, $\mathfrak{m} \Omega_R \omega_R = 0$. This implies that $\mu(\Omega_R \omega_R) = \lambda\left(\frac{\Omega_R \omega_R}{\mathfrak{m} \Omega_R \omega_R}\right) = \lambda(\Omega_R \omega_R) = \lambda(R^{\oplus \mu(\omega_R)}) - \lambda(\omega_R) = \mu(\omega_R)\lambda(R) - \lambda(R) = \lambda(R)(\mu(\omega_R) - 1) = \lambda(R)(\mu(\mathfrak{m}) - 1) = (1 + \mu(\mathfrak{m}))(\mu(\mathfrak{m}) - 1) = e^2 - 1$, where $\lambda(\omega_R) = \lambda(R)$ follows from Matlis duality. ∎

Lemma 5.1.18. *Let $(R, \mathfrak{m}, k)$ be a local ring, and let M be a finitely generated R-module. If x is M-regular, then $\frac{\Omega_R M}{x \Omega_R M} \cong \Omega_{R/xR}\left(\frac{M}{xM}\right)$.*

Proof. Since x is M-regular, we have $\operatorname{Tor}^R_1(M, \frac{R}{xR}) = 0$. Hence, tensoring the short exact sequence $0 \to \Omega_R M \to R^{\oplus \mu_R(M)} \to M \to 0$ with R/xR, we get the exact sequence $0 \to \frac{\Omega_R M}{x \Omega_R M} \to (\frac{R}{xR})^{\oplus \mu_R(M)} \to \frac{M}{xM} \to 0$. Since $\mu_R(M) = \mu_{\frac{R}{xR}}(\frac{M}{xM})$, we have $\frac{\Omega_R M}{x \Omega_R M} \cong \Omega_{R/xR}\left(\frac{M}{xM}\right)$. ∎

5.1.19. Since $\mu_R(-) = \lambda_R\left(\frac{(-)}{\mathfrak{m}(-)}\right)$, it follows by 4.3 that $\mu_R((-)) = \mu_S(S \otimes_R (-))$, where $S = R[X]_{\mathfrak{m}[X]}$. Since tensoring with S preserves exactness, so tensoring a minimal free resolution $(F_\bullet, \partial_\bullet)$ of an R-module M with S and remembering $S \otimes \partial$ now have entries in $\mathfrak{m}S$, the maximal ideal of S, we see that $S \otimes_R \Omega_R M \cong \Omega_S(S \otimes_S M)$. Also, if ω_R exists, then owing to the fact that $S/\mathfrak{m}S$ is a field, we see that ω_S also exists and $\omega_S \cong S \otimes_R \omega_R$ by [3, Th. 3.3.14(a)]. Hence, $S \otimes_R \operatorname{Hom}_R(-, \omega_R) \cong \operatorname{Hom}_S(S \otimes_R (-), \omega_S)$. Finally, we also have $\mathrm{r}(R) = \mathrm{r}(S)$ by [3, Prop. 1.2.16(b)].

Proposition 5.1.20. *Let $(R, \mathfrak{m}, k)$ be a local Cohen–Macaulay ring of minimal multiplicity admitting a canonical module ω_R, and let $\mathfrak{m} \neq 0$. Then $\mu((\Omega_R \omega_R)^\dagger) = \mathrm{r}(R)^2 - 1$.*

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Proof. Due to 5.1.19, we may pass to the faithfully flat extension $S := R[X]_{\mathfrak{m}[X]}$ and assume that the residue field is infinite.

We will prove the claim by induction on $\dim R = d$. First, let $d = 0$. Since R has minimal multiplicity, we have $\mathfrak{m}^2 = 0$. Hence, by Lemma 5.1.17, we get that $\Omega_R\omega_R \cong k^{\oplus(\mathrm{r}(R)^2-1)}$. This implies $(\Omega_R\omega_R)^\dagger \cong k^{\oplus(\mathrm{r}(R)^2-1)}$, so $\mu((\Omega_R\omega_R)^\dagger) = \mathrm{r}(R)^2 - 1$. Now, let $\dim R = d \geq 1$, and let the claim be true for rings with dimension $d-1$. Let $x \in \mathfrak{m}$ be such that $\mathfrak{m}^2 = (x, x_1, \ldots, x_{d-1})\mathfrak{m}$ (see [3, Exer. 4.6.14(c)]). So, $\frac{R}{xR}$ has minimal multiplicity. Now, we have

$$\frac{(\Omega_R\omega_R)^\dagger}{x(\Omega_R\omega_R)^\dagger} \cong \left(\frac{\Omega_R\omega_R}{x\Omega_R\omega_R}\right)^\dagger \cong \left(\Omega_{\frac{R}{xR}}\left(\frac{\omega_R}{x\omega_R}\right)\right)^\dagger \cong \left(\Omega_{\frac{R}{xR}}\omega_{\frac{R}{xR}}\right)^\dagger,$$

where the first isomorphism follows from [3, Prop. 3.3.3(a)], and the second isomorphism is by Lemma 5.1.18. So,

$$\begin{aligned}\mu_R((\Omega_R\omega_R)^\dagger) = \mu_{\frac{R}{xR}}\left(\frac{(\Omega_R\omega_R)^\dagger}{x(\Omega_R\omega_R)^\dagger}\right) &= \mu_{\frac{R}{xR}}\left(\left(\Omega_{\frac{R}{xR}}\omega_{\frac{R}{xR}}\right)^\dagger\right)\\ &= r\left(\frac{R}{xR}\right)^2 - 1 \text{ [By induction hypothesis]}\\ &= \mathrm{r}(R)^2 - 1.\end{aligned}$$

□

Proposition 5.1.21. *Let $(R, \mathfrak{m}, k)$ be a local Cohen–Macaulay ring of minimal multiplicity admitting a canonical module ω_R. Then the exact sequence $0 \to R \to \omega_R^{\oplus\mu(\omega_R)} \to (\Omega_R\omega_R)^\dagger \to 0$ is μ-additive.*

Proof. Consider the exact sequence $0 \to \Omega_R\omega_R \to R^{\oplus\mu(\omega_R)} \to \omega_R \to 0$. Since $\mathrm{Ext}^1_R(\omega_R, \omega_R) = 0$, we have the exact sequence $0 \to R \to \omega_R^{\oplus\mu(\omega_R)} \to (\Omega_R\omega_R)^\dagger \to 0$. From Proposition 5.1.20, we have $\mu((\Omega_R\omega_R)^\dagger) = \mathrm{r}(R)^2 - 1 = \mu(\omega_R)^2 - \mu(R) = \mu(\omega_R^{\oplus\mu(\omega_R)}) - \mu(R)$. Hence, $0 \to R \to \omega_R^{\oplus\mu(\omega_R)} \to (\Omega_R\omega_R)^\dagger \to 0$ is μ-additive. □

Proposition 5.1.22. *Let $(R, \mathfrak{m}, k)$ be a local Cohen–Macaulay ring of minimal multiplicity. Then $\mathrm{Ext}^1_R(M, F) = \mathrm{Ext}^1_R(M, F)^\mu$ for any maximal Cohen–Macaulay R-module M and any finitely generated free R-module F.*

Proof. By Lemma 2.4, it is enough to prove the claim for $F = R$.

We first consider the case when R has a canonical module ω_R. Consider the μ-additive (by Proposition 5.1.21) exact sequence $\sigma : 0 \to R \to \omega_R^{\oplus\mu(\omega_R)} \to (\Omega_R\omega_R)^\dagger \to 0$. Applying $\mathrm{Hom}_R(M, -)$ to σ, we get the following part of a commutative diagram of exact sequences by Corollary 3.9:

$$\begin{tikzcd}
\mathrm{Hom}_R(M, (\Omega_R\omega_R)^\dagger) \arrow[r] & \mathrm{Ext}^1_R(M, R) \arrow[r] & \mathrm{Ext}^1_R(M, \omega_R^{\oplus\mu(\omega_R)}) \\
\mathrm{Hom}_R(M, (\Omega_R\omega_R)^\dagger) \arrow[r] \arrow[u, equal] & \mathrm{Ext}^1_R(M, R)^\mu \arrow[r] \arrow[u, hook] & \mathrm{Ext}^1_R(M, \omega_R^{\oplus\mu(\omega_R)})^\mu \arrow[u, hook]
\end{tikzcd}$$

Since ω_R has finite injective dimension, we have $\mathrm{Ext}^1_R(M, \omega_R^{\oplus\mu(\omega_R)}) = 0$ by [3, Exer. 3.1.24]. So, $\mathrm{Ext}^1_R(M, \omega_R^{\oplus\mu(\omega_R)})^\mu = 0$ as well. Hence, we get the following commutative diagram:

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$$\begin{array}{ccccc}
\operatorname{Hom}_R(M,(\Omega_R\omega_R)^\dagger) & \xrightarrow{f} & \operatorname{Ext}^1_R(M,R) & \longrightarrow & 0 \\
\| & & \uparrow h & & \\
\operatorname{Hom}_R(M,(\Omega_R\omega_R)^\dagger) & \xrightarrow{g} & \operatorname{Ext}^1_R(M,R)^\mu & \longrightarrow & 0
\end{array}$$

Thus, $h\circ g = f$ is surjective, so h is surjective. Since h is the natural inclusion map, we have $\operatorname{Ext}^1_R(M,R) = \operatorname{Ext}^1_R(M,R)^\mu$.

Now, we consider the general case. Since $\operatorname{Ext}^1_R(M,R)^\mu \subseteq \operatorname{Ext}^1_R(M,R)$, it is enough to prove the other inclusion. So, let $\sigma: 0\to R\to X\to M\to 0$ be an exact sequence. We need to show σ is μ-additive. Now, consider the completion $\widehat{\sigma}: 0\to \widehat{R}\to\widehat{X}\to\widehat{M}\to 0$. Since $\widehat{R}$ is Cohen–Macaulay, having minimal multiplicity, admitting a canonical module, and $\widehat{M}$ is maximal Cohen–Macaulay over $\widehat{R}$, by the first part of the proof, we get $\operatorname{Ext}^1_{\widehat{R}}(\widehat{M},\widehat{R}) = \operatorname{Ext}^1_{\widehat{R}}(\widehat{M},\widehat{R})^\mu$. Thus, $[\widehat{\sigma}]\in \operatorname{Ext}^1_{\widehat{R}}(\widehat{M},\widehat{R}) = \operatorname{Ext}^1_{\widehat{R}}(\widehat{M},\widehat{R})^\mu$. Hence, $\widehat{\sigma}$ is μ-additive. Since the number of generators does not change under completion, we get σ is μ-additive, which is what we wanted to prove. □

Corollary 5.1.23. *Let $(R,\mathfrak{m},k)$ be a local Cohen–Macaulay ring of dimension d and minimal multiplicity. If $\operatorname{Ext}^1_R(\Omega^{i+d}_R k,R)^\mu = 0$ for some $i\geq 0$, then R is a hypersurface.*

Proof. By Proposition 5.1.22, we have $\operatorname{Ext}^{d+i+1}_R(k,R)\cong \operatorname{Ext}^1_R(\Omega^{d+i}_R k,R) = \operatorname{Ext}^1_R(\Omega^{d+i}_R k, R)^\mu = 0$. Hence, R has finite injective dimension by [32, II, Th. 2], so R is Gorenstein. Since R has minimal multiplicity, R is a hypersurface. □

One can also detect when a local ring has depth 0 by comparing $\operatorname{Ext}^1_R(M,R)$ to $\operatorname{Ext}^1_R(M,R)^\mu$ as shown in the following proposition. In the following, $\operatorname{Tr}(-)$ stands for Auslander transpose (see [27, Def. 12.3]).

Proposition 5.1.24. *Let I be an ideal of a local ring $(R,\mathfrak{m},k)$. Then, $\operatorname{Ext}^1_R(\operatorname{Tr}(R/I),R/\operatorname{ann}_R(I))^\mu = \mathfrak{m}\operatorname{Ext}^1_R(\operatorname{Tr}(R/I),R/\operatorname{ann}_R(I))$. Moreover, the following are equivalent:*

(1) $\operatorname{depth} R = 0$.
(2) $\operatorname{Ext}^1_R(M,F)^\mu = \operatorname{Ext}^1_R(M,F)$ *for all finitely generated R-modules M and F, where F is free.*
(3) $\operatorname{Ext}^1_R(\operatorname{Tr} k,R)^\mu = \operatorname{Ext}^1_R(\operatorname{Tr} k,R)$.

Proof. Since everywhere in this proposition the Auslander transpose is in the first component of Ext^1, our claim does not depend on the choice of Tr. First, we prove $\operatorname{Ext}^1_R(\operatorname{Tr}(R/I),R/\operatorname{ann}_R(I))^\mu = \mathfrak{m}\operatorname{Ext}^1_R(\operatorname{Tr}(R/I),R/\operatorname{ann}_R(I))$. We may assume $\operatorname{Tr}(R/I)$ is non-free. Dualizing the exact sequence $R^{\oplus\mu(I)}\to R\xrightarrow{\pi} R/I\to 0$ by R, we get $0\to \operatorname{Hom}_R(R/I,R)\xrightarrow{\pi^*} R\to R^{\oplus\mu(I)}\to \operatorname{Tr}(R/I)\to 0$. Under the natural identification $\operatorname{Hom}_R(R/I,R)\cong \operatorname{ann}_R(I)$, the map $\operatorname{Hom}_R(R/I,R)\xrightarrow{\pi^*} R$ can be identified with the inclusion map $\operatorname{ann}_R(I)\to R$, giving us $0\to\operatorname{ann}_R(I)\to R\to R^{\oplus\mu(I)}\to\operatorname{Tr}(R/I)\to 0$. Hence, we get an exact sequence $0\to R/\operatorname{ann}_R(I)\to R^{\oplus\mu(I)}\to \operatorname{Tr}(R/I)\to 0$. Since $\operatorname{Tr}(R/I)$ is non-free, this sequence gives us $\Omega_R\operatorname{Tr}(R/I)\cong R/\operatorname{ann}_R(I)$ and the first Betti number of $\operatorname{Tr}(R/I)$ is 1. Hence, $\operatorname{Ext}^1_R(\operatorname{Tr}(R/I),R/\operatorname{ann}_R(I))^\mu = \mathfrak{m}\operatorname{Ext}^1_R(\operatorname{Tr}(R/I),R/\operatorname{ann}_R(I))$ by Lemma 5.1.3.

Now, we prove the equivalence of the three conditions as follows:

(1) $\Longrightarrow$ (2): Let $\sigma: 0\to F\to X\to M\to 0$ be an exact sequence, where F is a finitely generated free R-module. Since $\operatorname{depth} R = 0$, we get that k embeds inside R, that is, k is torsionless. Hence, $\operatorname{Ext}^1_R(\operatorname{Tr} k,R) = 0$ by [27, Prop. 12.5]. Hence, $\operatorname{Ext}^1_R(\operatorname{Tr} k,F) = 0$. So, we

get an exact sequence $0 \to \mathrm{Hom}_R(\mathrm{Tr}\,k, F) \to \mathrm{Hom}_R(\mathrm{Tr}\,k, X) \to \mathrm{Hom}_R(\mathrm{Tr}\,k, M) \to 0$. Then, by [27, Exer. 13.36], we get that the sequence $0 \to F \otimes_R k \to X \otimes_R k \to M \otimes_R k \to 0$ is also exact, which means σ is μ-additive. Thus, $\mathrm{Ext}^1_R(M,F)^\mu = \mathrm{Ext}^1_R(M,F)$.

(2) $\implies$ (3) Obvious.

(3) $\implies$ (1): Assume $\mathrm{Ext}^1_R(\mathrm{Tr}\,k, R)^\mu = \mathrm{Ext}^1_R(\mathrm{Tr}\,k, R)$. Now, if possible let $\operatorname{depth} R > 0$. Then $\mathfrak{m}$ contains a nonzero divisor, so $\mathrm{ann}_R(\mathfrak{m}) = 0$. Then, by the first part of this proposition, we get $\mathrm{Ext}^1_R(\mathrm{Tr}\,k, R)^\mu = \mathfrak{m}\mathrm{Ext}^1_R(\mathrm{Tr}\,k, R)$. Hence, $\mathrm{Ext}^1_R(\mathrm{Tr}\,k, R) = \mathfrak{m}\mathrm{Ext}^1_R(\mathrm{Tr}\,k, R)$. Then, by Nakayama's lemma, $\mathrm{Ext}^1_R(\mathrm{Tr}\,k, R) = 0$. Hence, by [27, Prop. 12.5], we have an embedding $k \to k^{**}$. Consequently, $k^* \neq 0$, that is, $\operatorname{depth} R = 0$. ∎

For general local Cohen–Macaulay rings of dimension 1, we now show that if $\mathrm{Ext}^1_R(M,R)^\mu = 0$ for some I-Ulrich module [11, Def. 4.1] $M \subseteq Q(R)$ containing a nonzero divisor of R, then I is principal. For this, we first need the following lemma. For the remainder of this section, given R-submodules M, N of $Q(R)$, by $(M : N)$, we will mean $\{x \in Q(R) : xN \subseteq M\}$.

Lemma 5.1.25. *Let $(R,\mathfrak{m})$ be a local Cohen–Macaulay ring of dimension 1. Let I be an $\mathfrak{m}$-primary ideal of R admitting a principal reduction $a \in I$. Assume $(I : I) = R$. If $M \subseteq Q(R)$ is an I-Ulrich module, and contains a nonzero divisor of R, then the natural inclusion map $\mathrm{Hom}_R(M,(a)) \to \mathrm{Hom}_R(M,I)$, induced by the inclusion $(a) \to I$, is an isomorphism.*

Proof. The natural inclusion $0 \to (a) \xrightarrow{i} I$ induces the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(M,(a)) \arrow[r] & \mathrm{Hom}_R(M,I) \\
0 \arrow[r] & ((a):M) \arrow[r] \arrow[u, "\alpha\to\{x\mapsto\alpha x\}"] & (I:M) \arrow[u, "\alpha\to\{x\mapsto\alpha x\}"']
\end{tikzcd}$$

where the rows are natural inclusion maps, and the vertical arrows are isomorphisms due to [26, Prop. 2.4(1)]. So, it is enough to show that $(I : M) \subseteq ((a) : M)$. Indeed, if $x \in (I : M)$, then $xM \subseteq I$. Since M is I-Ulrich, M is a $B(I) = R\left[\dfrac{I}{a}\right]$-module (see [11, Rem. 4.4 and Th. 4.6]). Hence, $\dfrac{I}{a}M \subseteq M$, so $\dfrac{I}{a}xM \subseteq xM \subseteq I$. Thus, $\dfrac{1}{a}xM \subseteq (I : I) = R$, so $xM \subseteq (a)$. Therefore, $x \in ((a) : M)$. ∎

Proposition 5.1.26. *Let $(R,\mathfrak{m})$ be a local Cohen–Macaulay ring of dimension 1. Let I be an $\mathfrak{m}$-primary ideal of R admitting a principal reduction $a \in I$. Assume $(I : I) = R$. If there exists an I-Ulrich module $M \subseteq Q(R)$, containing a nonzero divisor of R such that $\mathrm{Ext}^1_R(M,R)^\mu = 0$, then $I \cong R$.*

Proof. Consider the short exact sequence $0 \to (a) \xrightarrow{i} I \to I/(a) \to 0$. Since $I^{n+1} = aI^n$ for all $n \gg 0$ and I contains a nonzero divisor, we have that a is a nonzero divisor. So, $(a) \cong R$. If $a \in \mathfrak{m}I$, then $I^{n+1} \subseteq \mathfrak{m}II^n = \mathfrak{m}I^{n+1}$, which implies that $I^{n+1} = 0$ by Nakayama's lemma. This contradicts the fact that I is $\mathfrak{m}$-primary. So, $a \notin \mathfrak{m}I$, and hence $\mu(I/(a)) = \mu(I) - 1$. Thus, the above short exact sequence is μ-additive. So, by Corollary 3.9, we get the following induced exact sequence:

$$0 \to \mathrm{Hom}_R(M,(a)) \to \mathrm{Hom}_R(M,I) \to \mathrm{Hom}_R(M,I/(a)) \to \mathrm{Ext}^1_R(M,(a))^\mu \cong \mathrm{Ext}^1_R(M,R)^\mu = 0,$$

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where the induced map $\mathrm{Hom}_R(M,(a)) \to \mathrm{Hom}_R(M,I)$ is an isomorphism by Lemma 5.1.25. Hence, we get $\mathrm{Hom}_R(M,I/(a)) = 0$. Now, $I/(a)$ has finite length. So, if $I/(a) \neq 0$, then $\mathrm{Ass}(I/(a)) = \{\mathfrak{m}\}$. Hence, $\mathrm{Ass}(\mathrm{Hom}_R(M,I/(a))) = \mathrm{Supp}(M) \cap \mathrm{Ass}(I/(a)) = \mathrm{Supp}(M) \cap \{\mathfrak{m}\} = \{\mathfrak{m}\}$, contradicting $\mathrm{Hom}_R(M,I/(a)) = 0$. Thus, we must have $I/(a) = 0$, that is, $I = (a) \cong R$. ∎

5.2 Some applications of the subfunctor $\mathrm{Ext}^1_{\mathrm{Ul}_I(R)}(-,-)$

Let I be an $\mathfrak{m}$-primary ideal. In this subsection, we give various applications of the subfunctor $\mathrm{Ext}^1_{\mathrm{Ul}^s_I(R)}(-,-) : \mathrm{Ul}^s_I(R)^{op} \times \mathrm{Ul}^s_I(R) \to \mathrm{mod}\ R$ (see 4.9 for notation), where we recall from Corollary 4.7 that $\mathrm{Ul}^s_I(R)$ along with its all short exact sequences form an exact subcategory of mod R. Hence, $\mathrm{Ext}^1_{\mathrm{Ul}^s_I(R)}(-,-) : \mathrm{Ul}^s_I(R)^{op} \times \mathrm{Ul}^s_I(R) \to \mathrm{mod}\ R$ is a subfunctor of $\mathrm{Ext}^1_R(-,-) : \mathrm{Ul}^s_I(R)^{op} \times \mathrm{Ul}^s_I(R) \to \mathrm{mod}\ R$ by Proposition 3.8.

We begin by observing a general connection between $\mathrm{Ext}^1_R(M,N)$ and $\mathrm{Ext}^1_R(M,N)^{\nu_I}$, where $M,N \in \mathrm{mod}\ R$ and $\nu_I(-) := \lambda((-) \otimes_R R/I) : \mathrm{mod}\ R \to \mathbb{Z}$ and (see Definition 4.10 for notation of $\mathrm{Ext}^1_R(-,-)^{\phi}$).

Lemma 5.2.1. *Let $(R,\mathfrak{m})$ be a Noetherian local ring, and let I be an $\mathfrak{m}$-primary ideal. Then, $I\ \mathrm{Ext}^1_R(M,N) \subseteq \mathrm{Ext}^1_R(M,N)^{\nu_I}$ for all $M,N \in \mathrm{mod}\ R$.*

Proof. Let $\sigma : 0 \to N \to X \to M \to 0$ be a short exact sequence in $I\ \mathrm{Ext}^1_R(M,N)$. Then $\sigma \otimes_R R/I$ splits by [36, Th. 1.1], so $M/IM \oplus N/IN \cong X/IX$. Hence, taking length, we get $\lambda(M/IM) + \lambda(N/IN) = \lambda(X/IX)$, so σ is ν_I-additive. ∎

This allows us to prove a general connection between $\mathrm{Ext}^1_{\mathrm{Ul}^s_I(R)}(M,N)$ and $\mathrm{Ext}^1_R(M,N)^{\nu_I}$, when $M,N \in \mathrm{Ul}^s_I(R)$ (recall the definition of $\mathrm{Ul}^s_I(R)$ from Definition 4.5).

Lemma 5.2.2. *Let $(R,\mathfrak{m})$ be a local ring, let $s \geq 0$ be an integer, and let I be an $\mathfrak{m}$-primary ideal. Then, $\mathrm{Ext}^1_R(M,N)^{\nu_I} = \mathrm{Ext}^1_{\mathrm{Ul}^s_I(R)}(M,N)$ for all $M,N \in \mathrm{Ul}^s_I(R)$. So, in particular, $I\mathrm{Ext}^1_R(M,N) \subseteq \mathrm{Ext}^1_{\mathrm{Ul}^s_I(R)}(M,N)$ for all $M,N \in \mathrm{Ul}^s_I(R)$.*

Proof. Let $M,N \in \mathrm{Ul}^s_I(R)$, and let $\sigma : 0 \to N \to X \to M \to 0$ be a short exact sequence of R-modules. Then $X \in \mathrm{CM}^s(R)$ (see 4.2), and $\lambda(M/IM) + \lambda(N/IN) = e_R(I,M) + e_R(I,N) = e_R(I,X)$. Now, $\sigma \in \mathrm{Ext}^1_R(M,N)^{\nu_I}$, if and only if $\lambda(X/IX) = \lambda(M/IM) + \lambda(N/IN)$, if and only if $\lambda(X/IX) = e_R(I,X)$, if and only if $X \in \mathrm{Ul}^s_I(R)$ if and only if $\sigma \in \mathrm{Ext}^1_{\mathrm{Ul}^s_I(R)}(M,N)$. This proves the desired equality $\mathrm{Ext}^1_R(M,N)^{\nu_I} = \mathrm{Ext}^1_{\mathrm{Ul}^s_I(R)}(M,N)$. The last inclusion in the statement now follows from this and Lemma 5.2.1. ∎

When R is Cohen–Macaulay of dimension 1 and $M,N \in \mathrm{Ul}_I(R)$, then one can improve the inclusion $I\mathrm{Ext}^1_R(M,N) \subseteq \mathrm{Ext}^1_{\mathrm{Ul}_I(R)}(M,N)$ of Lemma 5.2.2 quite a bit. For this, we first record a general lemma about trace ideals. In the following, we say that $\dfrac{a}{b} \in Q(R)$ is a nonzero divisor if a is a nonzero divisor in R. Note that this does not depend on the choice of representative, since $\frac{a}{b} = \frac{a'}{b'}$ in $Q(R)$ implies $ab' = a'b$, and since b, b' are nonzero divisors in R, so a is a nonzero divisor in R if and only if a' is a nonzero divisor in R.

Lemma 5.2.3. *Let I be an ideal of R containing a nonzero divisor. Then the following holds:*

(1) *There exist nonzero divisors $x_1,\dots,x_n \in R$ such that $I = (x_1,\dots,x_n)$.*
(2) *There exist nonzero divisors $y_1,\dots,y_n \in (R:I)$ such that $\mathrm{tr}_R(I) = \sum_{i=1}^n y_i I$.*

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Proof. (1) Let S be the collection of all nonzero divisors of R that are in I. Let $\langle S\rangle$ be the ideal of R generated by S. Then $I \subseteq (\cup_{\mathfrak{p}\in \operatorname{Ass}(R)}\mathfrak{p}) \cup \langle S\rangle$. By prime avoidance, either $I \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Ass}(R)$, or $I \subseteq \langle S\rangle$. But I contains a nonzero divisor, so $I \nsubseteq \mathfrak{p}$ for every $\mathfrak{p} \in \operatorname{Ass}(R)$. Hence, $I \subseteq \langle S\rangle$. Since S is a subset of I, we conclude $I = \langle S\rangle$. Since R is Noetherian, there exist finitely many elements $x_1,\dots,x_n \in S$ such that $I = (x_1,\dots,x_n)$.

(2) We know that $\operatorname{tr}_R(I) = (R:I)I$ in $Q(R)$ (see [26, Prop. 2.4(2)]). Pick a nonzero divisor $a \in I$, so $J = a(R:I) \subseteq R$ is an ideal of R. Then $\operatorname{tr}_R(I) = J(\frac{1}{a}I)$. Now, J contains a nonzero divisor, so by part (1), we have $J = (x_1,\dots,x_n)$ for some nonzero divisors $x_1,\dots,x_n$. Then $\operatorname{tr}_R(I) = \sum_{i=1}^n \frac{1}{a}x_iI$. Denoting $y_i := \frac{1}{a}x_i$, we see that each $y_i \in Q(R)$ is a nonzero divisor, and $y_i \in (R:I)$ as $x_i \in a(R:I)$. Hence the claim. ∎

In the proof of the next result, for an $\mathfrak{m}$-primary ideal I, $B(I)$ denotes blow-up of I in the sense of [11, Def. 4.3 and Rem. 4.4], namely $B(I) := \cup_{n>0}(I^n :_{Q(R)} I^n)$.

PROPOSITION 5.2.4. *Let $(R,\mathfrak{m})$ be a local Cohen–Macaulay ring of dimension 1, and let I be an $\mathfrak{m}$-primary ideal of R. Then $\operatorname{tr}_R(I)\operatorname{Ext}^1_R(M,N) \subseteq \operatorname{Ext}^1_{\operatorname{Ul}_I(R)}(M,N)$ for all $M,N \in \operatorname{Ul}_I(R)$.*

Proof. Let $M,N \in \operatorname{Ul}_I(R)$, and let $a \in (R:I)$ be a nonzero divisor. Then aI is an $\mathfrak{m}$-primary ideal of R and $B(I) = B(aI)$. So, $\operatorname{Ul}_I(R) = \operatorname{Ul}_{aI}(R)$ by [11, Prop. 4.24]. Hence, for every nonzero divisor $a \in (R:I)$, we have $aI\operatorname{Ext}^1_R(M,N) \subseteq \operatorname{Ext}^1_{\operatorname{Ul}_I(R)}(M,N)$ by Lemma 5.2.2. Now, by Lemma 5.2.3, there exist nonzero divisors $y_1,\dots,y_n \in (R:I)$ such that $\operatorname{tr}_R(I) = \sum_{i=1}^n y_iI$. Thus, $\operatorname{tr}_R(I)\operatorname{Ext}^1_R(M,N) = \sum_{i=1}^n y_iI\operatorname{Ext}^1_R(M,N) \subseteq \operatorname{Ext}^1_{\operatorname{Ul}_I(R)}(M,N)$. ∎

When $s = 1$, $\operatorname{depth} R > 0$, and $I = \mathfrak{m}$ (so $\nu_I(-) = \mu(-)$), the inclusion $I\operatorname{Ext}^1_R(M,N) \subseteq \operatorname{Ext}^1_{\operatorname{Ul}^s_I(R)}(M,N)$ of Lemma 5.2.2 is actually an equality as we prove next. For this, we first record an easy lemma about flat extensions.

LEMMA 5.2.5. *Let $R \to S$ be a flat extension of rings. Let M be an R-module, and let I an ideal of R. Then, the following holds:*

(1) *$S \otimes_R (IM) = (IS)(S \otimes_R M)$ when identified as submodules of $S \otimes_R M$.*
(2) *If $N \subseteq M$ is an R-submodule, S is a faithfully flat extension of R, and $S \otimes_R M = S \otimes_R N$, then $M = N$.*

Proof. (1) Consider the exact sequence $0 \to IM \to M \to M/IM \to 0$, which after tensoring with S gives $0 \to S \otimes_R (IM) \to S \otimes_R M \to S \otimes_R M/IM \to 0$. Now, $S \otimes_R M/IM \cong S \otimes_R (M \otimes_R R/I) \cong (S \otimes_R M) \otimes_R R/I \cong (S \otimes_R M)/I(S \otimes_R M)$. Now, by the natural S-module structure on $S \otimes_R M$, we see that $I(S \otimes_R M) = (IS)(S \otimes_R M)$. Thus, we get the exact sequence $0 \to S \otimes_R (IM) \to S \otimes_R M \to (S \otimes_R M)/(IS)(S \otimes_R M) \to 0$, and by naturality of the isomorphisms, we see that the map $S \otimes_R M \to (S \otimes_R M)/(IS)(S \otimes_R M)$ in the exact sequence has kernel $(IS)(S \otimes_R M)$. Hence, $S \otimes_R (IM) = (IS)(S \otimes_R M)$.

(2) Tensoring the exact sequence $0 \to N \to M \to M/N \to 0$ with S and using $S \otimes_R M = S \otimes_R N$, we get $(M/N) \otimes_R S = 0$. Since S is faithfully flat, we have $M/N = 0$. Hence, $M = N$. ∎

Now, we prove the desired equality between $\mathfrak{m}\operatorname{Ext}^1_R(M,N)$ and $\operatorname{Ext}^1_{\operatorname{Ul}^1(R)}(M,N)$, when R has positive depth.

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PROPOSITION 5.2.6. *Let $(R,\mathfrak{m},k)$ be a local ring of positive depth. Let $M,N \in \mathrm{Ul}^1(R)$. Then, the following holds:*

(1) *We always have $\mathfrak{m}\mathrm{Ext}^1_R(M,N) = \mathrm{Ext}^1_{\mathrm{Ul}^1(R)}(M,N)$.*
(2) *If R is moreover Cohen–Macaulay of dimension 1 (so $\mathrm{Ul}^1(R) = \mathrm{Ul}(R)$) and if $x \in \mathfrak{m}$ is a minimal reduction of $\mathfrak{m}$, then $x\mathrm{Ext}^1_R(M,N) = \mathrm{Ext}^1_{\mathrm{Ul}(R)}(M,N)$.*

Proof. (1) Due to Lemma 5.2.2, we only need to prove $\mathrm{Ext}^1_{\mathrm{Ul}^1(R)}(M,N) \subseteq \mathfrak{m}\mathrm{Ext}^1_R(M,N)$. We may assume $M,N \neq 0$.

First, we assume that the residue field is infinite. Let $\sigma : 0 \to N \to X \to M \to 0$ be a short exact sequence such that $X \in \mathrm{Ul}^1(R)$. Choose $x \in \mathfrak{m}$ to be $R \oplus M \oplus N \oplus X$-superficial (which exists by [24, Prop. 8.5.7], since we are assuming R has infinite residue field). Now, $\mathrm{depth}_R(R \oplus M \oplus N \oplus X) = \inf\{\mathrm{depth}\, R, \mathrm{depth}_R M, \mathrm{depth}_R N, \mathrm{depth}_R X\} > 0$, so x is $R \oplus M \oplus N \oplus X$-regular by [24, Lem. 8.5.4]. Hence, $x \in \mathfrak{m}$ is regular on R, M, and N, and superficial on M, N, and X. Thus, $M/xM, N/xN, X/xX$ are zero-dimensional Ulrich modules by [21, Prop. 2.2(4)], so these are k-vector spaces by [21, Prop. 2.2(1)]. Hence, $\sigma \otimes_R R/xR$, being a short exact sequence of k-vector spaces, is split exact. Since x is R, M, N-regular, by [36, Prop. 2.8], we have $\sigma \in x\mathrm{Ext}^1_R(M,N) \subseteq \mathfrak{m}\mathrm{Ext}^1_R(M,N)$.

Now, we prove the general case. This part will use extensively that $\mathrm{Ext}^1_{\mathrm{Ul}^1(R)}(M,N)$ is a submodule of $\mathrm{Ext}^1_R(M,N)$, where $M,N \in \mathrm{Ul}^1(R)$. Consider the faithfully flat extension of R as follows: $S := R[X]_{\mathfrak{m}[X]}$, with maximal ideal $\mathfrak{m}S$, and infinite residue field. Since we already know, $\mathfrak{m}\mathrm{Ext}^1_R(M,N) \subseteq \mathrm{Ext}^1_{\mathrm{Ul}^1(R)}(M,N)$ by Lemma 5.2.2 and since S is faithfully flat, it is enough to prove $S \otimes_R \mathfrak{m}\mathrm{Ext}^1_R(M,N) = S \otimes_R \mathrm{Ext}^1_{\mathrm{Ul}^1(R)}(M,N)$ (by Lemma 5.2.5(2)). Now, $\mathfrak{m}\mathrm{Ext}^1_R(M,N) \subseteq \mathrm{Ext}^1_{\mathrm{Ul}^1(R)}(M,N)$, with flatness of S, already implies $S \otimes_R \mathfrak{m}\mathrm{Ext}^1_R(M,N) \subseteq S \otimes_R \mathrm{Ext}^1_{\mathrm{Ul}^1(R)}(M,N)$. So, to prove equality, it is enough to prove that $S \otimes_R \mathrm{Ext}^1_{\mathrm{Ul}^1(R)}(M,N) \subseteq S \otimes_R \mathfrak{m}\mathrm{Ext}^1_R(M,N)$. Now, due to Lemma 5.2.5(1), it is enough to prove $S \otimes_R \mathrm{Ext}^1_{\mathrm{Ul}^1(R)}(M,N) \subseteq (\mathfrak{m}S)(S \otimes_R \mathrm{Ext}^1_R(M,N))$, and the latter object here is naturally identified with $(\mathfrak{m}S)\mathrm{Ext}^1_S(S \otimes_R M, S \otimes_R N)$. Since the S-module $S \otimes_R \mathrm{Ext}^1_{\mathrm{Ul}^1(R)}(M,N)$ is generated by $1 \otimes_R \sigma$ as σ runs over all elements of $\mathrm{Ext}^1_{\mathrm{Ul}^1(R)}(M,N)$, we need to prove that $1 \otimes_R \sigma \in (\mathfrak{m}S)\mathrm{Ext}^1_S(S \otimes_R M, S \otimes_R N)$ for all $\sigma \in \mathrm{Ext}^1_{\mathrm{Ul}^1(R)}(M,N)$. So, let $\sigma : 0 \to N \to X \to M \to 0$ be a short exact sequence such that $X \in \mathrm{Ul}^1(R)$. Then, $1 \otimes_R \sigma \in S \otimes_R \mathrm{Ext}^1_{\mathrm{Ul}^1(R)}(M,N) \subseteq S \otimes_R \mathrm{Ext}^1_R(M,N) \cong \mathrm{Ext}^1_S(S \otimes_R M, S \otimes_R N)$ is naturally identified with the exact sequence $S \otimes_R \sigma : 0 \to S \otimes_R N \to S \otimes_R X \to S \otimes_R M \to 0$ in $\mathrm{Ext}^1_S(S \otimes_R M, S \otimes_R N)$. Now, $S \otimes_R \sigma$ is a short exact sequence of modules in $\mathrm{Ul}^1(S)$ by [21, Prop. 2.2(3)]. Hence, $1 \otimes_R \sigma \in \mathrm{Ext}^1_{\mathrm{Ul}^1(S)}(S \otimes_R M, S \otimes_R N)$. Since S also has positive depth and infinite residue field, by the proof of the infinite residue field case, we get $1 \otimes_R \sigma \in \mathrm{Ext}^1_{\mathrm{Ul}^1(S)}(S \otimes_R M, S \otimes_R N) \subseteq (\mathfrak{m}S)\mathrm{Ext}^1_S(S \otimes_R M, S \otimes_R N)$, which is what we wanted to prove.

(2) Due to Lemma 5.2.2, we only need to prove $\mathrm{Ext}^1_{\mathrm{Ul}(R)}(M,N) \subseteq x\mathrm{Ext}^1_R(M,N)$. So, let $\sigma : 0 \to N \to X \to M \to 0$ be a short exact sequence such that $X \in \mathrm{Ul}(R)$. Since xR is a reduction of $\mathfrak{m}$, we have xR is $\mathfrak{m}$-primary. Hence, x is R-regular, so x is M, N, X-regular. Also, $\mathfrak{m}M = xM, \mathfrak{m}N = xN, \mathfrak{m}X = xX$. So, $\sigma \otimes_R R/xR : 0 \to N/xN \to X/xX \to M/xM \to 0$ is an exact sequence of k-vector spaces. Hence, $\sigma \otimes_R R/xR$ is split exact. Since x is R, M, N, X-regular, by [36, Prop. 2.8], we have $\sigma \in x\mathrm{Ext}^1_R(M,N)$. Thus, we get $\mathrm{Ext}^1_{\mathrm{Ul}(R)}(M,N) \subseteq x\mathrm{Ext}^1_R(M,N)$. ∎

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Following is an interesting consequence of Proposition 5.2.6. Here, for R-modules X, Y, we denote by $X * Y$ the collection of all R-modules Z that fit into a short exact sequence $0 \to X \to Z \to Y \to 0$.

COROLLARY 5.2.7. *Let $(R, \mathfrak{m})$ be a local one-dimensional Cohen–Macaulay ring. Then, the following are equivalent:*

(1) *R is regular.*
(2) *$\mathrm{Ul}(R)$ is closed under taking extensions.*
(3) *There exists $M, N \in \mathrm{Ul}(R)$ such that N is faithful, $M \neq 0$, and $N * M \subseteq \mathrm{Ul}(R)$.*

Proof. (1) $\Longrightarrow$ (2): If R is regular, then $\mathrm{Ul}(R) = \mathrm{CM}(R)$, which is closed under taking extensions.

(2) $\Longrightarrow$ (3): One can take $M = N = \mathfrak{m}^n$ for some $n \gg 0$.

(3) $\Longrightarrow$ (1): By hypothesis, we get $\mathrm{Ext}^1_R(M,N) = \mathrm{Ext}^1_{\mathrm{Ul}(R)}(M,N)$. Then from Proposition 5.2.6 (remembering $\mathrm{Ul}^1(R) = \mathrm{Ul}(R)$ in our case), we get $\mathrm{Ext}^1_R(M,N) = \mathfrak{m}\mathrm{Ext}^1_R(M,N)$. So, by Nakayama's lemma, we get $\mathrm{Ext}^1_R(M,N) = 0$. By [11, Th. 4.6] (see also [12, Lem. 5.2]), we get $M \cong \mathfrak{m}M$ and $N \cong \mathfrak{m}N$. So, $\mathrm{Ext}^1_R(\mathfrak{m}M, \mathfrak{m}N) = 0$. Hence, $\mathrm{pd}_R(\mathfrak{m}M) < \infty$ by [12, Prop. 2.6]. Thus, $\mathrm{Tor}^R_{\gg 0}(\mathfrak{m}, \mathfrak{m}M) = 0$. Since $0 \neq M$, we have $\mathfrak{m}M \cong M \neq 0$. Thus, $\mathrm{pd}_R(\mathfrak{m}) < \infty$ by [12, Cor. 3.17]. Hence, R is regular. ∎

We mention in passing that when R is local Cohen–Macaulay of minimal multiplicity, (2) $\Longrightarrow$ (1) of Corollary 5.2.7 also follows from [13, Prop. 3.2(5)].

Before proceeding further with applications to $\mathrm{Ul}_I(R)$, we outline an alternative proof that $\mathrm{Ul}_I(R)$ is an exact subcategory of mod R when R is a local one-dimensional Cohen–Macaulay ring, via birational extensions. For this, we first record some preliminary results.

The following is easy to see from the definition of an exact category.

LEMMA 5.2.8. *Let R, S be two commutative rings such that $R \to S$ is a ring extension. Let $(\mathcal{X}, \mathcal{E})$ be a strictly full exact subcategory of* Mod S. *Assume that $\mathcal{X}$ is closed under R-linear isomorphism of R-modules. Also, assume that every R-linear map between any two modules in $\mathcal{X}$ is S-linear, that is, $\mathrm{Hom}_R(M,N) = \mathrm{Hom}_S(M,N)$ for all $M, N \in \mathcal{X}$. Then, $(\mathcal{X}, \mathcal{E})$ is a strictly full exact subcategory of* Mod R.

For the consequences of Lemma 5.2.8, we need the following lemma, which is essentially [27, Prop. 4.14(i)]. Before proceeding, we note that for a birational extension $R \subseteq S \subseteq Q(R)$ and an S-module N, being torsion-free as an S-module and as an R-module are the same.

LEMMA 5.2.9. *Let R be a commutative Noetherian ring, and let $R \subseteq S \subseteq Q(R)$ be a ring extension. Then, $\mathrm{Hom}_R(M,N) = \mathrm{Hom}_S(M,N)$ for all $M, N \in \mathrm{Mod}(S)$ where N is torsion-free S-module.*

Proof. $\mathrm{Hom}_S(M,N) \subseteq \mathrm{Hom}_R(M,N)$ is clear. Hence, it is enough to show that $\mathrm{Hom}_R(M,N) \subseteq \mathrm{Hom}_S(M,N)$. Let $\dfrac{a}{b} \in S \subseteq Q(R)$, where $a, b \in R$, so $b \in R$ is a nonzero divisor. Let $m \in M$ and $f \in \mathrm{Hom}_R(M,N)$. Then $b\left(f\left(\dfrac{a}{b}m\right) - \dfrac{a}{b}f(m)\right) = bf\left(\dfrac{a}{b}m\right) - af(m) = f\left(b\dfrac{a}{b}m\right) - af(m) = f(am) - af(m) = 0$. Since N is torsion-free and b is a nonzero

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divisor on R, it is a nonzero divisor on N. So, $f\left(\frac{a}{b}m\right) = \frac{a}{b}f(m)$. As $\frac{a}{b} \in S, m \in M$, and $f \in \operatorname{Hom}_R(M,N)$ were arbitrary, we conclude $\operatorname{Hom}_R(M,N) \subseteq \operatorname{Hom}_S(M,N)$. ∎

Now, as a consequence, we can deduce the following.

Proposition 5.2.10. *Let $R \subseteq S$ be a birational extension ($S \subseteq Q(R)$) of commutative rings. Let $\mathcal{S}_R$ be the standard exact structure on* mod R. *Let $\mathcal{X}$ be the strictly full subcategory of* Mod$S$ *consisting of all torsion-free S-modules. Then it holds that $\mathcal{S}_R|_{\mathcal{X}} = \mathcal{S}_S|_{\mathcal{X}}$, and $(\mathcal{X}, \mathcal{S}_R|_{\mathcal{X}})$ is a strictly full exact subcategory of* mod R, *and $(\mathcal{X} \cap \text{mod } R, \mathcal{S}_R|_{\mathcal{X} \cap \text{mod } R})$ is an exact subcategory of* Mod R *(hence, also of* mod R*).*

Proof. By definition, $\mathcal{S}_R|_{\text{mod } R \cap \mathcal{X}} = \mathcal{S}_R|_{\text{mod } R} \cap \mathcal{S}_R|_{\mathcal{X}}$. Since $(\text{mod } R, \mathcal{S}_R|_{\text{mod } R})$ is an exact subcategory of Mod R, the second part of the statement would readily follow from the first part of the statement and Lemma 3.2. First, we show that $\mathcal{X}$ is strictly full in mod R. If M,N are two R-modules and $f: M \to N$ is an R-linear isomorphism and N is moreover a torsion-free S-module extending its R-module structure, then M has an S-module structure, extending its R-module structure, given by $s \cdot m := f^{-1}(sf(m))$ for all $s \in S, m \in M$. Note that, with this structure, M is moreover a torsion-free S-module. Indeed, let $s \in S$ be a nonzero divisor such that $s \cdot m = 0$. Then $sf(m) = f(s \cdot m) = 0$, so $f(m) = 0$ as N is S-torsion-free. Thus, $m = 0$ as f is an isomorphism. Moreover, $\operatorname{Hom}_R(M,N) = \operatorname{Hom}_S(M,N)$ for all torsion-free S-modules M,N by Lemma 5.2.9. Consequently, we notice that $\mathcal{S}_R|_{\mathcal{X}} = \mathcal{S}_S|_{\mathcal{X}}$. So, to show that $(\mathcal{X}, \mathcal{S}_R|_{\mathcal{X}})$ is an exact subcategory of Mod R, it is enough to show that $(\mathcal{X}, \mathcal{S}_S|_{\mathcal{X}})$ is an exact subcategory of Mod S (by Lemma 5.2.8). Now, it is well known that for any ring S, the subcategory of all S-torsion-free modules is closed under extensions in Mod S. Hence, $(\mathcal{X}, \mathcal{S}_S|_{\mathcal{X}})$ is an exact subcategory of Mod S by Proposition 3.5. ∎

Corollary 5.2.11. *Let R be a local Cohen–Macaulay ring of dimension* 1. *Let $R \subseteq S$ be a finite birational extension. Then, $(\mathrm{CM}(S), \mathcal{S}_R|_{\mathrm{CM}(S)})$ is a strictly full exact subcategory of* mod R.

Proof. We note that mod$S$ = Mod $S \cap$ mod R. Since S is also one-dimensional Cohen–Macaulay, we have $\mathrm{CM}(S)$ = collection of all finitely generated torsion-free S-modules = collection of all torsion-free S-modules $\cap$mod R. Hence, the conclusion follows from Proposition 5.2.10. ∎

When $(R,\mathfrak{m})$ is a local Cohen–Macaulay ring of dimension 1 and I an $\mathfrak{m}$-primary ideal, then considering the finite birational extension $R \subseteq B(I)$, where $B(I)$ denotes blow-up of I [11, Def. 4.3 and Rem. 4.4], it is shown in [11, Th. 4.6] that $\mathrm{CM}(B(I)) = \mathrm{Ul}_I(R)$. Thus, in this particular case, we get another proof of the fact that $\mathrm{Ul}_I(R)$ is an exact subcategory of mod R, which is very different from Corollary 4.7.

For further applications, we first record a lemma connecting $\operatorname{Ext}^1_{\mathrm{Ul}_I(R)}(-,-)$ to $\operatorname{Ext}^1_{B(I)}(-,-)$. Here, we will keep in mind that when $\dim R = 1$, we write $\mathrm{Ul}^1_I(R) = \mathrm{Ul}_I(R)$ in our notation (see Definition 4.5).

Lemma 5.2.12. *Let $(R,\mathfrak{m})$ be a local Cohen–Macaulay ring of dimension* 1. *Let I be an $\mathfrak{m}$-primary ideal of R. Then, for all I-Ulrich modules M,N, we have a natural map $\operatorname{Ext}^1_{\mathrm{Ul}_I(R)}(M,N) \xrightarrow{[\sigma]_R \to [\sigma]_{B(I)}} \operatorname{Ext}^1_{B(I)}(M,N)$, which is an isomorphism of R-modules.*

Proof. We recall that $\operatorname{Ext}^1_{\mathrm{Ul}_I(R)}(M,N)$ is the equivalence class of all short exact sequences $0 \to N \to X \to M \to 0$ of I-Ulrich modules and R-linear maps. Since $\mathrm{Ul}_I(R) =$

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$\mathrm{CM}(B(I))$ [11, Th. 4.6], any R-linear map between two Ulrich modules is $B(I)$-linear (see [27, Prop. 4.14(i)]). Hence, any short exact sequence $0 \to N \to X \to M \to 0$ of I-Ulrich modules and R-linear maps is a short exact sequence of maximal Cohen–Macaulay $B(I)$-modules and $B(I)$-linear maps, and any R-linear morphism between two short exact sequences of I-Ulrich R-modules is actually a $B(I)$-morphism. This proves the well-definedness of the map. Injectivity is similarly obvious. To prove surjectivity, we note that if Y is a $B(I)$-module, and $0 \to N \to Y \to M \to 0$ is a short exact sequence in $\mathrm{Mod}(B(I))$, then $M, N \in \mathrm{CM}(B(I))$ implies $Y \in \mathrm{CM}(B(I))$. Hence, $Y \in \mathrm{Ul}_I(R)$. Since $B(I)$-linear maps are R-linear, this gives a pre-image in $\mathrm{Ext}^1_{\mathrm{Ul}_I(R)}(M,N)$. Finally, to show the map is R-linear, we recall that the Baer-sum structure and multiplication by R on Ext^1 are given by certain pullback and pushout diagrams. So, it is enough to prove that given an exact sequence $\sigma : 0 \to N \to X \to M \to 0$ in $\mathrm{Ul}_I(R)$ and R-linear maps $f : N \to N', g : M' \to M$ with $M', N' \in \mathrm{Ul}_I(R)$, the pushout and pullback of σ by f and g in mod R, respectively, are actually pushout and pullback in $\mathrm{mod}(B(I))$. Let us look at the pushout case, as the pullback case is similar. If we have the following pushout diagram:

$$\begin{tikzcd}
\sigma : 0 \arrow[r] & N \arrow[r] \arrow[d, "f"'] & X \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N' \arrow[r] & Y \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

then in the bottom row, we again have $Y \in \mathrm{Ul}_I(R)$, since $\mathrm{Ul}_I(R)$ along with all its short exact sequences is an exact subcategory of mod R by Corollary 4.7. Hence, in the above diagram, all the modules are in $\mathrm{Ul}_I(R) = \mathrm{CM}(B(I))$, so all the R-linear maps are moreover $B(I)$-linear. Since $\mathrm{CM}(B(I))$ is an extension closed subcategory of $\mathrm{mod}(B(I))$, it is an exact subcategory of $\mathrm{mod}(B(I))$ by Proposition 3.5. Thus, this is a pushout diagram in $\mathrm{mod}(B(I))$ as well by [6, Prop. 2.12]. This is what we wanted to show. ∎

Corollary 5.2.13. *Let $(R,\mathfrak{m})$ be a local Cohen–Macaulay ring of dimension 1, and let I be an $\mathfrak{m}$-primary ideal of R. If $M, N \in \mathrm{Ul}_I(R)$ are such that $\mathrm{Ext}^1_{B(I)}(M,N) = 0$, then $\mathrm{tr}_R(I)\mathrm{Ext}^1_R(M,N) = 0$.*

Proof. This follows from Lemma 5.2.12 and Proposition 5.2.4. ∎

In the following, for a module X over a ring R, by $\mathrm{add}_R(X)$, we denote the collection of all R-modules Y such that there exist an R-module Z and an isomorphism of R-modules $Y \oplus Z \cong X^{\oplus n}$ for some integer $n \geq 0$.

Theorem 5.2.14. *Let $(R,\mathfrak{m})$ be a local Cohen–Macaulay ring of dimension 1. Let I be an $\mathfrak{m}$-primary ideal of R. Then the following hold:*

(1) *If $M \in \mathrm{add}_R(B(I))$, then $\mathrm{tr}_R(I)\mathrm{Ext}^1_R(M, \mathrm{Ul}_I(R)) = 0$.*
(2) *Let $M \in \mathrm{Ul}(R)$. Then $M \in \mathrm{add}_R(B(\mathfrak{m}))$ if and only if $\mathfrak{m}\mathrm{Ext}^1_R(M, \mathrm{Ul}(R)) = 0$.*
(3) *If $B(I)$ is a Gorenstein ring, then $\mathrm{tr}_R(I)\mathrm{Ext}^1_R(\mathrm{Ul}_I(R), B(I)) = 0$. The converse holds when $I = \mathfrak{m}$.*

Proof. (1) Denote $S := B(I)$. If $M \in \mathrm{add}_R(S)(\subseteq \mathrm{Ul}_I(R))$, then there exist $X \in \mathrm{mod}(R)$ and an isomorphism of R-modules $M \oplus X \cong B(I)^{\oplus n}$ for some $n \geq 0$. Then M, X are I-Ulrich, so $M, X \in \mathrm{CM}(S)$. Hence, the isomorphism is also S-linear by [27, Prop. 4.14(i)], so M is a projective S-module. Thus, $\mathrm{Ext}^1_S(M,N) = 0$ for all $N \in \mathrm{CM}(S) = \mathrm{Ul}_I(R)$. Now, the conclusion follows from Corollary 5.2.13.

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(2) If $M \in \operatorname{add}_R(B(\mathfrak{m}))$, then the conclusion follows by (1) as $\mathfrak{m} \subseteq \operatorname{tr}_R(\mathfrak{m})$. So, assume $M \in \operatorname{Ul}(R)$ and $\mathfrak{m}\operatorname{Ext}^1_R(M,N) = 0$ for all $N \in \operatorname{Ul}(R)$. By Proposition 5.2.6, we then have $\operatorname{Ext}^1_{\operatorname{Ul}(R)}(M,N) = 0$ for all $N \in \operatorname{Ul}(R)$. Now, denote $S = B(\mathfrak{m})$. Then, by Lemma 5.2.12, we have $\operatorname{Ext}^1_S(M,N) = 0$ for all $N \in \operatorname{Ul}(R) = \operatorname{CM}(S)$. We know that, for any Cohen–Macaulay ring S, and $M \in \operatorname{CM}(S)$, we have $\operatorname{Ext}^1_S(M, \operatorname{CM}(S)) = 0$ if and only if $M \in \operatorname{add}_S(S)$. Indeed, one direction is clear since $M \in \operatorname{add}_S(S)$ implies M is projective. For the converse, we notice that as M is finitely generated over S, so we have a short exact sequence $\sigma : 0 \to M' \to S^{\oplus n} \to M \to 0$ for some $n > 0$ and S-module M'. Moreover, S being Cohen–Macaulay implies $M' \in \operatorname{CM}(S)$, and now the vanishing of $\operatorname{Ext}^1_S(M,M')$ gives that the sequence σ splits. Hence, $M \in \operatorname{add}_S(S)$. Now, applying this observation to our scenario with $S = B(\mathfrak{m})$, we get $M \in \operatorname{add}_S(B(\mathfrak{m})) = \operatorname{add}_R(B(\mathfrak{m}))$, where this last equality follows from the fact that direct summand of $B(\mathfrak{m})$ is in $\operatorname{CM}(B(\mathfrak{m}))$ and consequently, R-linear maps are S-linear (see [27, Prop. 4.14(i)]).

(3) Nothing to prove if R is regular, so we assume R is singular. If $B(I)$ is a Gorenstein ring, then $\operatorname{Ext}^1_{B(I)}(M, B(I)) = 0$ for all $M \in \operatorname{CM}(B(I)) = \operatorname{Ul}_I(R)$. Since $B(I)$ is I-Ulrich, by Lemma 5.2.12, we get $\operatorname{Ext}^1_{\operatorname{Ul}_I(R)}(M, B(I)) = 0$ for all $M \in \operatorname{Ul}_I(R)$. Then, by Corollary 5.2.13, we get $\operatorname{tr}_R(I)\operatorname{Ext}^1_R(M, B(I)) = 0$ for all $M \in \operatorname{Ul}_I(R)$. For the converse part, assume $I = \mathfrak{m}$ and $\mathfrak{m}\operatorname{Ext}^1_R(M, B(\mathfrak{m})) = 0$ for all $M \in \operatorname{Ul}(R)$. Then, by Proposition 5.2.6, we have $\operatorname{Ext}^1_{\operatorname{Ul}(R)}(M, B(\mathfrak{m})) = 0$ for all $M \in \operatorname{Ul}(R)$. Hence, by Lemma 5.2.12, we have $\operatorname{Ext}^1_{B(\mathfrak{m})}(M, B(\mathfrak{m})) = 0$ for all $M \in \operatorname{Ul}(R) = \operatorname{CM}(B(\mathfrak{m}))$. Now, we note that for a finite-dimensional Cohen–Macaulay ring S, $\operatorname{Ext}^1_S(M,S) = 0$ for all $M \in \operatorname{CM}(S)$ if and only if S is Gorenstein. Indeed, if S is Gorenstein, then the vanishing is clear by [3, Def. 3.1.18 and Props. 3.1.9 and 3.1.24]. Conversely, let $d = \dim S$. Then, for every $\mathfrak{p} \in \operatorname{Spec}(S)$, we have $\Omega^d_S(S/\mathfrak{p}) \in \operatorname{CM}(S)$ (by localizing and applying depth lemma). So, now the vanishing of $\operatorname{Ext}^{d+1}_S(S/\mathfrak{p}, S) \cong \operatorname{Ext}^1_S(\Omega^d_S(S/\mathfrak{p}), S)$ for every $\mathfrak{p} \in \operatorname{Spec}(S)$ implies $\operatorname{injdim}_S S < \infty$. Thus, S is Gorenstein by [3, Def. 3.1.18 and Prop. 3.1.9]. Now, applying this observation to our scenario with $S = B(\mathfrak{m})$, we get that $B(\mathfrak{m})$ is Gorenstein. □

Since for one-dimensional local Cohen–Macaulay rings $(R,\mathfrak{m})$ of minimal multiplicity it holds that $B(\mathfrak{m}) = (\mathfrak{m} : \mathfrak{m}) \cong \mathfrak{m}$, the following corollary is an immediate consequence of Theorem 5.2.14(3) and [20, Th. 5.1].

Corollary 5.2.15. *Let $(R,\mathfrak{m})$ be a local Cohen–Macaulay ring of dimension 1 and minimal multiplicity. Then, R is almost Gorenstein if and only if $\mathfrak{m}\operatorname{Ext}^1_R(\operatorname{Ul}(R), \mathfrak{m}) = 0$.*

Finally, we give one more application of Theorem 5.2.14(3), for which we first record an easy observation.

Lemma 5.2.16. *Let $(R,\mathfrak{m})$ be a local Cohen–Macaulay ring of dimension 1. Let I be an $\mathfrak{m}$-primary ideal of R admitting a principal reduction $x \in I$. Then, $\mathfrak{m}$ is I-Ulrich if and only if $\mathfrak{m} \subseteq ((x) : I)$.*

Proof. If I is principal, then all MCM modules are I-Ulrich. So, it is enough to assume I is not principal. If $\mathfrak{m}$ is I-Ulrich, then $I\mathfrak{m} = x\mathfrak{m}$ (see [11, Prop. 4.5]). So, $\mathfrak{m} \subseteq ((x) : I)$. Conversely, let $\mathfrak{m} \subseteq ((x) : I)$. Then, $\mathfrak{m} \subseteq ((x) :_R I)$. We claim that $x \notin \mathfrak{m}I$. Since $x \in I$ is a principal reduction of I, we have $I^{n+1} = xI^n$ for some $n \geq 1$. Hence, if $x \in \mathfrak{m}I$, then $I^{n+1} \subseteq \mathfrak{m}I^{n+1}$. So, $I^{n+1} = \mathfrak{m}I^{n+1}$. Hence, by Nakayama's lemma, $I^{n+1} = 0$, contradicting I is $\mathfrak{m}$-primary. Thus, $x \in I \setminus \mathfrak{m}I$. Since $\mathfrak{m} \subseteq ((x) :_R I)$ and I is not principal, we have $\mathfrak{m} \subseteq (x\mathfrak{m} :_R I)$ by Lemma 5.1.7. So, $\mathfrak{m}I \subseteq x\mathfrak{m}$. Hence, $\mathfrak{m}I = x\mathfrak{m}$, so $\mathfrak{m}$ is I-Ulrich. □

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COROLLARY 5.2.17. *Let* $(R,\mathfrak{m})$ *be a local Cohen–Macaulay ring of dimension* 1*, with infinite residue field, and minimal multiplicity, admitting a canonical module. If* $\mathfrak{m}\mathrm{Ext}^1_R(\mathrm{Ul}(R),\mathfrak{m})=0$*, then* R *admits a canonical ideal* ω *and* $B(\omega)$ *is Gorenstein. Consequently,* $\mathfrak{m}\mathrm{Ext}^1_R(\mathrm{Ul}_\omega(R),B(\omega))=0$.

Proof. Let ω be a canonical module of R. We show that ω can be identified with an ideal of R, $\mathfrak{m}\subseteq \mathrm{tr}_R(\omega)$, and $B(\omega)$ is Gorenstein. Since R has minimal multiplicity and $\mathfrak{m}\mathrm{Ext}^1_R(\mathrm{Ul}(R),\mathfrak{m})=0$, R is almost Gorenstein by Corollary 5.2.15. Hence, $\mathfrak{m}\subseteq \mathrm{tr}_R(\omega)$ from [22, Def. 2.2 and Prop. 6.1], so R is generically Gorenstein (see [22, Lem. 2.1]). Hence, ω can be identified with an ideal of R by [3, Prop. 3.3.18]. Since R is almost Gorenstein, we have $\mathfrak{m}\subseteq((x):\omega)$ for some principal reduction x of ω (see [20, Setting 3.4 and Th. 3.11]). Thus, $\mathfrak{m}$ is ω-Ulrich by Lemma 5.2.16. So, $\mathfrak{m}\in\mathrm{CM}(B(\omega))$ by [11, Th. 4.6]. Hence, $\mathfrak{m}\subseteq\mathrm{tr}_R(\mathfrak{m})\subseteq(R:_{Q(R)}B(\omega))$ by [11, Th. 2.9]. Thus, the conductor $c_R(B(\omega)):=(R:_{Q(R)}B(\omega))$ of $B(\omega)$ is either $\mathfrak{m}$ or R. As R has minimal multiplicity, by using [3, Exercise 4.6.14(c)], we see that $c_R(B(\omega))$ satisfies the condition (2) of [20, Cor. 3.8]. Thus, $B(\omega)$ is Gorenstein by [20, Cor. 3.8]. Now, the claim follows from Theorem 5.2.14(3). ∎

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Hailong Dao
Department of Mathematics
University of Kansas
Lawrence
Kansas 66045-7523
USA
hdao@ku.edu

Souvik Dey
Department of Mathematics
University of Kansas
Lawrence
Kansas
66045-7523
USA
and

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Department of Algebra
Faculty of Mathematics and Physics
Charles University
Sokolovska'
83 186 75
Praha Czech Republic
souvik.dey@matfyz.cuni.cz

Monalisa Dutta
Department of Mathematics
University of Kansas
Lawrence
Kansas 66045-7523
USA
m819d903@ku.edu

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