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On Flag Curvature of Homogeneous Randers Spaces

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Abstract. In this paper we give an explicit formula for the flag curvature of homogeneous Randers spaces of Douglas type and apply this formula to obtain some interesting results. We first deduce an explicit formula for the flag curvature of an arbitrary left invariant Randers metric on a two-step nilpotent Lie group. Then we obtain a classification of negatively curved homogeneous Randers spaces of Douglas type. This results, in particular, in many examples of homogeneous non-Riemannian Finsler spaces with negative flag curvature. Finally, we prove a rigidity result that a homogeneous Randers space of Berwald type whose flag curvature is everywhere nonzero must be Riemannian.

Introduction

The purpose of this paper is to study flag curvature of homogeneous Randers spaces. Flag curvature is the natural generalization of section curvature in Riemannian geometry to Finsler geometry and is one of the most important quantities in the field. However, the nonlinearity of the Finsler metrics makes the computation of flag curvature of an explicit Finsler space extremely complicated. Although there is a formula for this quantity in a standard coordinate system, the computation involved is formidable and very difficult to handle, even with the help of computer programmes. Therefore, it is of special merit if we can present an explicit and simple formula for flag curvature.

Recent research shows that in the homogeneous case, problems will generally become much simpler, and we can get beautiful results. For example, in [DE08] the second author obtained a very simple formula for S-curvature of homogeneous Randers spaces and gave some interesting applications of the formula. Based on this formula and the previous work of Berger, Wallach, Aloff–Wallach and Bérard-Bergery on homogeneous Riemannian manifolds of positive sectional curvature, we classified all the homogeneous Randers spaces with isotropic S-curvature and positive flag curvature in [HD11]. It seems hopeful that we can get a simple formula for flag curvature of an arbitrary homogeneous Randers spaces. However, even in this case the computation is rather complicated. Therefore we first consider the special case of Douglas type. In this case, Shen obtained an explicit formula of the flag curvature under a coordinate system ([CS05]), and, in the homogeneous case, we can get a satisfactory formula without using coordinate systems.

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Here is a simple description of each section of this paper. In Section 1 we study the Levi-Civita connection of homogeneous Riemannian manifolds and compute some quantities that will be used to deduce the formula of flag curvature. In Section 2, we present a formula of the flag curvature of homogeneous Randers spaces of Douglas type, without using the local coordinate systems. In Sections 3, 4, and 5, we apply this formula to study three problems. We first give an explicit formula of flag curvature of an arbitrary left invariant Finsler metric on a two-step nilpotent Lie group. Then we give a complete description of all the homogeneous Randers spaces of Douglas type with negative flag curvature. This results in, among other things, a lot of examples of non-Riemannian homogeneous Randers spaces with negative flag curvature. Finally, we prove a rigidity theorem on Berwald spaces asserting that a homogeneous Randers space of Berwald type whose flag curvature is everywhere non-zero must be Riemannian.

1 The Levi-Civita Connection of Homogeneous Spaces

Let $(G/H, \alpha)$ be a homogeneous Riemannian manifold. Then the Lie algebra of G has a decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, where $\mathfrak{h}$ is the Lie algebra of H and $\operatorname{Ad}(h)(\mathfrak{m}) \subset \mathfrak{m}, \forall h \in H$. We identify $\mathfrak{m}$ with the tangent space $T_o(G/H)$ of the origin $o = H$. We shall use the notation $\langle \cdot\,, \cdot \rangle$ to denote the restriction of the Riemannian metric to $\mathfrak{m}$. Note that it is an $\operatorname{Ad} H$-invariant inner product on $\mathfrak{m}$. Hence we have

$$\langle [x,u], v\rangle + \langle [x,v], u\rangle = 0, \quad \forall x \in \mathfrak{h}, \forall u, v \in \mathfrak{m},$$

equivalently,

$$\langle [x,u], u\rangle = 0, \quad \forall x \in \mathfrak{h}, \forall u \in \mathfrak{m}.$$

Given $v \in \mathfrak{g}$, the fundamental vector field $\widehat{v}$ generated by v is ([KN63])

$$\widehat{v}_{gH} = \frac{d}{dt} \exp(tv) gH|_{t=0}, \quad \forall g \in G.$$

Since the one-parameter transformation group $\exp tv$ on G/H consists of isometries, $\widehat{v}$ is a Killing vector field.

Let $\widehat{X}, \widehat{Y}, \widehat{Z}$ be Killing vector fields on G/H and let U, V be arbitrary smooth vector fields on G/H. Then we have ([BE87, pp. 40, 182, 183])

$$\begin{aligned}
[\widehat{X}, \widehat{Y}] &= -\widehat{[X,Y]},\\
\widehat{X}\langle U, V\rangle &= \langle [\widehat{X}, U], V\rangle + \langle [\widehat{X}, V], U\rangle,\\
\langle \nabla_{\widehat{X}} \widehat{Y}, \widehat{Z}\rangle &= -\frac{1}{2}\Big(\langle \widehat{[X,Y]}, \widehat{Z}\rangle + \langle \widehat{[X,Z]}, \widehat{Y}\rangle + \langle \widehat{[Y,Z]}, \widehat{X}\rangle \Big),
\end{aligned}$$

where ∇ is denotes the Levi-Civita connection of the Riemannian metric α.

Let $u_1, u_2, \dots, u_n$ be an orthonormal basis of $\mathfrak{m}$ with respect to $\langle \cdot\,, \cdot \rangle$ and extend it to a basis $u_1, u_2, \dots, u_m$ of $\mathfrak{g}$. By [HE78], there exists a local coordinate system on a neighborhood V of o, which is defined by the mapping

$$\big(\exp(x^1 u_1) \exp(x^2 u_2) \cdots \exp(x^n u_n) \big) H \to (x^1, x^2, \dots, x^n).$$

Suppose $gH = (x^1, x^2, \cdots, x^n) \in U$. Then

$$\frac{\partial}{\partial x^i}\Big|_{gH} = \frac{d}{dt}\Big(\exp(x^1 u_1)\cdots\exp(x^{i-1}u_{i-1})\exp\big((t+x^i)u_i\big)\ \exp(x^{i+1}u_{i+1})\cdots\exp(x^n u_n)H\Big)\Big|_{t=0}$$

$$= \frac{d}{dt}\Big(\exp(x^1 u_1)\cdots\exp(x^{i-1}u_{i-1})\exp(tu_i)\exp(-x^{i-1}u_{i-1})\cdots\exp(-x^1u_1)\cdot gH\Big)\Big|_{t=0}$$

$$= \frac{d}{dt}\Big(\exp te^{x^1 ad u_1}\cdots e^{x^{i-1}ad u_{i-1}}(u_i)\cdot gH\Big)\Big|_{t=0}.$$

Denote

$$e^{x^1 ad u_1}\cdots e^{x^{i-1}ad u_{i-1}}(u_i) = f_i^a u_a. \tag{1.1}$$

We have

$$\frac{\partial}{\partial x^i}\Big|_{gH} = f_i^a \widehat{u_a}|_{gH}.$$

Remark In the sequel, the indices $a, b, c, \ldots,$ range from 1 to m, $i, j, k, \ldots,$ range from 1 to n and $\lambda, \mu, \ldots,$ range from $n+1$ to m.

Let Γ_{ij}^l be the Christoffel symbols under the coordinate system, *i.e.*,

$$\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} = \Gamma_{ij}^k\frac{\partial}{\partial x^k}.$$

Then

$$\Gamma_{ij}^l\frac{\partial}{\partial x^l} = \nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} = \frac{\partial f_j^a}{\partial x^i}\widehat{u_a} + f_i^b f_j^a \nabla_{\widehat{u_b}}\widehat{u_a}. \tag{1.2}$$

From (1.1), we see that f_i^a are functions of $x^1, \ldots, x^{i-1}$. Thus for $i \geq j$, we have

$$\frac{\partial f_j^a}{\partial x^i} = 0.$$

Therefore (1.2) becomes

$$\Gamma_{ij}^l\frac{\partial}{\partial x^l} = f_i^b f_j^a \nabla_{\widehat{u_b}}\widehat{u_a}, \ \ i \geq j.$$

Differentiating the above equation with respect to x_k, we get (see [DE08])

$$\frac{\partial \Gamma_{ij}^l}{\partial x^k}\frac{\partial}{\partial x^l} + \Gamma_{ij}^s\Gamma_{ks}^l\frac{\partial}{\partial x^l} = \frac{\partial f_i^b f_j^a}{\partial x^k}\nabla_{\widehat{u_b}}\widehat{u_a} + f_i^b f_j^a f_k^c \nabla_{\widehat{u_c}}\nabla_{\widehat{u_b}}\widehat{u_a}, \ \ i \geq j.$$

Differentiating (1.1) with respect to x_k and letting $(x^1, \ldots, x^n) \to 0$, we obtain

$$\frac{\partial f_i^a}{\partial x^k}(0) = f(k,i)C_{ki}^a,$$

where C_{ab}^c are the structure constants of $\mathfrak{g}$ defined by $[u_a, u_b] = C_{ab}^c u_c$ and $f(k,l)$ is defined by

$$f(k,i) := \begin{cases} 1, & k < i, \\ 0 & k \geq i. \end{cases}$$

Considering the value at the origin o, we get the following lemma.

Lemma 1.1 *We have*

$$\Gamma^l_{ij}(o) = f(i,j)C^l_{ij} + \langle \nabla_{\widehat{u}_i}\widehat{u}_j, \widehat{u}_l\rangle,$$
$$\frac{\partial \Gamma^l_{ij}}{\partial x^k}\Big|_o = -\Gamma^s_{ij}(\Gamma^l_{ks} + \langle \nabla_{\widehat{u}_k}\widehat{u}_l, \widehat{u}_s\rangle) + f(k,j)C^a_{kj}\langle \nabla_{\widehat{u}_i}\widehat{u}_a, \widehat{u}_l\rangle$$
$$+ f(k,i)C^s_{ki}\langle \nabla_{\widehat{u}_s}\widehat{u}_j, \widehat{u}_l\rangle + \widehat{u}_k\langle \nabla_{\widehat{u}_i}\widehat{u}_j, \widehat{u}_l\rangle, i \geq j.$$

We will also need the following lemma.

Lemma 1.2 *For $u_i, u_j, u_k, u_l \in \mathfrak{m}, u_\lambda \in \mathfrak{h}$, we have*

$$\langle \nabla_{\widehat{u}_i}\widehat{u}_j, \widehat{u}_l\rangle|_o = -\frac{1}{2}\left(C^l_{ij} + C^j_{il} + C^i_{jl}\right),$$
$$\langle \nabla_{\widehat{u}_i}\widehat{u}_\lambda, \widehat{u}_j\rangle|_o = \left\langle [u_j, u_\lambda]_{\mathfrak{m}}, u_i\right\rangle = C^i_{j\lambda},$$
$$\widehat{u}_k\langle \nabla_{\widehat{u}_i}\widehat{u}_j, \widehat{u}_l\rangle|_o = \frac{1}{2}\left(C^l_{ka}C^a_{ij} + C^j_{ka}C^a_{il} + C^i_{ka}C^a_{jl} + C^s_{ij}C^t_{kl}\delta_{st} + C^s_{il}C^t_{kj}\delta_{st} + C^s_{jl}C^t_{ki}\delta_{st}\right),$$

where $[v_i, v_j]_{\mathfrak{m}}$ denotes the projection of $[v_i, v_j]$ to $\mathfrak{m}$.

By the above two lemmas, at the origin o we have

$$\Gamma^j_{ni} - \Gamma^i_{nj} = \langle \nabla_{\widehat{u}_n}\widehat{u}_i, \widehat{u}_j\rangle - \langle \nabla_{\widehat{u}_n}\widehat{u}_j, \widehat{u}_i\rangle = C^n_{ji}.$$

2 Flag Curvature of Homogeneous Randers Spaces of Douglas Type

Let

$$F = \alpha + \beta = \sqrt{a_{ij}(x)y^iy^j} + b_i(x)y^i$$

be a Randers metric of Douglas type on a manifold M, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a closed 1-form. Denote by $\nabla\beta = b_{i;j}y^idx^j$ the covariant derivative of β with respect to α. Then the Riemann curvature is given by (see[CS05])

$$R^i_k = \bar{R}^i_k + \left(3\left(\frac{\Phi}{2F}\right)^2 - \frac{\Psi}{2F}\right)\left\{\delta^i_k - \frac{F_{y^k}}{F}y^i\right\} + \tau_k y^i, \tag{2.1}$$

where

$$\Phi = b_{i;j}y^iy^j, \quad \Psi = b_{i;j;k}y^iy^jy^k, \quad \tau_k = \frac{1}{F}(b_{i;j;k} - b_{i;k;j})y^iy^j,$$

and $\bar{R}^i_k$ is the coefficient of the Riemann curvature tensor of α.

Now we consider the homogeneous case. Let $(G/H, \alpha)$ and $\mathfrak{m}$ be as above. It is well known that a Randers metric $F = \alpha + \beta$ is G-invariant if and only if α and β are both invariant under G (see [BR04]). Moreover, through α, the 1-form β corresponds to a vector field W that is invariant under G and satisfies $\alpha(W) < 1$.

This implies that there is a one-to-one correspondence between the invariant Randers metrics on G/M with the underlying Riemannian metric and the set

$$V = \{u \in \mathfrak{m} | Ad(h)u = u,\ \langle u, u\rangle < 1, \quad \forall h \in H\}$$

(see [DH04]). Recall that the corresponding Randers metric is of Douglas type if and only if $\langle u, [\mathfrak{m}, \mathfrak{m}]_{\mathfrak{m}}\rangle = 0$ (see [DE08]).

Let $(G/H, F)$ be a homogeneous Randers space and $(U, (x^1, \cdots, x^n))$ be the local coordinate system as before. We suppose the vector field W generated by $u = cu_n (0 \le c < 1)$ corresponds to the invariant 1-form β. Then

$$[\mathfrak{h}, u_n] = 0, \quad C^a_{\lambda n} = 0,$$

and

$$\begin{aligned}\widetilde{u}|_{gH} &= \frac{d}{dt} g\exp(tu)H|_{t=0}\\ &= \frac{d}{dt}\big(\exp x^1 u_1 \exp x^2 u_2 \cdots \exp(x^n + ct)u_n\big) H|_{t=0}\\ &= c\frac{\partial}{\partial x^n}|_{gH}.\end{aligned}$$

A direct computation shows that

$$\begin{aligned} b_i &= \beta\Big(\frac{\partial}{\partial x^i}\Big) = \Big\langle \widetilde{U}, \frac{\partial}{\partial x^i}\Big\rangle = c\Big\langle \frac{\partial}{\partial x^n}, \frac{\partial}{\partial x^i}\Big\rangle = ca_{ni},\\ \frac{\partial b_i}{\partial x^j} &= c\frac{\partial a_{ni}}{\partial x^j} = c\big(\Gamma^k_{nj} a_{ki} + \Gamma^k_{ji} a_{kn}\big),\\ b_{i;j} &= \frac{\partial b_i}{\partial x^j} - b_l \Gamma^l_{ij} = c\Gamma^k_{nj} a_{ki},\\ b_{i;j;k}|_o &= c\Big(\frac{\partial \Gamma^i_{nj}}{\partial x^k} + \Gamma^s_{nj}\Gamma^i_{sk} - \Gamma^i_{ns}\Gamma^s_{jk}\Big),\\ b_{i;j;k}|_o - b_{i;k;j}|_o &= c\Big(\frac{\partial \Gamma^i_{nj}}{\partial x^k} - \frac{\partial \Gamma^i_{nk}}{\partial x^j} + \Gamma^s_{nj}\Gamma^i_{sk} - \Gamma^s_{nk}\Gamma^i_{sj}\Big) = c\bar{R}^i_{nkj}|_o. \end{aligned}$$

By the antisymmetry of $\bar{R}$ and the fact that $a_{ij}(o) = \delta_{ij}$, we obtain

$$\tau_k = \frac{c}{F}\bar{R}^i_{nkj} y^i y^j = -\frac{c}{F}\bar{R}^n_k.$$

Since β is a closed 1-form, we also have $b_{i;j} = b_{j;i}$. By [DE08], we obtain

$$\big\langle [u_i, u_j], u_n\big\rangle = 0, \quad C^n_{ij} = 0.$$

Since the inner product on $\mathfrak{m}$ is $Ad(H)$-invariant, we get $C^j_{\lambda i} + C^i_{\lambda j} = 0$.

Let $U\colon \mathfrak{m}\times\mathfrak{m}\to\mathfrak{m}$ be the bilinear symmetric map defined by

$$2\big\langle U(X,Y),Z\big\rangle = \big\langle [Z,X]_{\mathfrak{m}},Y\big\rangle + \big\langle [Z,Y]_{\mathfrak{m}},Z\big\rangle, \quad \forall Z\in\mathfrak{m}.$$

Then we have

$$\begin{aligned}
F &= \alpha+\beta = \sqrt{\langle y,y\rangle} + \langle y,u\rangle,\\
F_{y^k} &= \frac{\langle y,u_k\rangle}{\sqrt{\langle y,y\rangle}} + \langle u,u_k\rangle,\\
\Phi &= \big\langle [y,u],y\big\rangle = -\big\langle U(y,y),u\big\rangle,\\
\Psi &= -cC^0_{0s}(C^0_{ns}+C^s_{n0}) = 2\big\langle U\big(y,U(y,y)\big),u\big\rangle.
\end{aligned}$$

Now we are ready to compute the flag curvature of invariant Randers spaces of Douglas type. By definition, the flag curvature of a Finsler space (M,F) is defined by

$$K(P,y) = \frac{g_y(R_y(v),v)}{g_y(v,v)g_y(y,y)-g_y(v,y)^2},$$

where $P\subset T_xM$ is a tangent plane containing y and $v\in P$ such that $P=\operatorname{span}\{y,v\}$ and

$$g_y(v_1,v_2) = \frac{1}{2}\frac{\partial^2}{\partial s\partial t}\big[F^2(y+sv_1+tv_2)\big]_{s=t=0}. \tag{2.2}$$

Sometimes we will also denote the above flag curvature as $K(y,v,y\wedge v)$. Note that in this case the first vector y is the flag pole.

By a simple computation, one gets

$$\begin{aligned}
g_y(W,V) &= \Big\langle W,\frac{y}{\alpha}+u\Big\rangle\Big\langle V,\frac{y}{\alpha}+u\Big\rangle + \frac{F}{\alpha}\langle W,V\rangle - \frac{F}{\alpha^3}\langle W,y\rangle\langle V,y\rangle.\\
g_y(W,W) &= \Big\langle W,\frac{y}{\alpha}+u\Big\rangle^2 + \frac{F}{\alpha}\langle W,W\rangle - \frac{F}{\alpha^3}\langle W,y\rangle^2,\\
g_y(y,W) &= F\Big\langle W,\frac{y}{\alpha}+u\Big\rangle,\\
g_y(y,y) &= F^2,\\
g_y(W,W)g_y(y,y)-g_y(W,y)^2 &= \frac{F^3}{\alpha^3}\big(\langle W,W\rangle\langle y,y\rangle - \langle W,y\rangle^2\big).
\end{aligned}$$

So by (2.1) we have

$$\begin{aligned}
&g_y(R_y(W),V)\\
&=\Big\langle \bar{R}(W,y)y-\big\langle \bar{R}(W,y)y,u\big\rangle\, y+\widetilde{\Phi}W-\frac{\widetilde{\Phi}}{F}\Big\langle u+\frac{y}{\alpha},W\Big\rangle\, y,\frac{y}{\alpha}+u\Big\rangle\Big\langle V,\frac{y}{\alpha}+u\Big\rangle\\
&\quad+\frac{F}{\alpha}\Big\langle \bar{R}(W,y)y-\big\langle \bar{R}(W,y)y,u\big\rangle\, y+\widetilde{\Phi}W-\frac{\widetilde{\Phi}}{F}\Big\langle u+\frac{y}{\alpha},W\Big\rangle\, y,V\Big\rangle\\
&\quad-\frac{F}{\alpha^3}\Big\langle \bar{R}(W,y)y-\big\langle \bar{R}(W,y)y,u\big\rangle\, y+\widetilde{\Phi}W-\frac{\widetilde{\Phi}}{F}\Big\langle u+\frac{y}{\alpha},W\Big\rangle\, y,y\Big\rangle\langle V,y\rangle\\
&=\frac{F}{\alpha}\big\langle \bar{R}(W,y)y,V\big\rangle+\frac{F}{\alpha^3}\widetilde{\Phi}\big(\langle W,V\rangle\langle y,y\rangle-\langle W,y\rangle\langle V,y\rangle\big),
\end{aligned}$$

where

$$\widetilde{\Phi}=3\Big(\frac{\Phi}{2F}\Big)^2-\frac{\Psi}{2F}=\frac{1}{4F^2}\Big(3\big\langle U(y,y),u\big\rangle^2-4F\cdot\big\langle U(y,U(y,y)),u\big\rangle\Big).$$

Substituting the above quantities into (2.1) and (2.2), we get the following theorem.

Theorem 2.1 *Let $(G/H,\alpha)$ be a homogeneous Riemannian manifold and suppose that the Lie algebra $\mathfrak{g}$ of G has a decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$ with $Ad(h)\mathfrak{m}\subset\mathfrak{m}$. Let F be a G-invariant Randers metric of Douglas type on G/H with the underlying Riemannian metric α, then the flag curvature of F is given by*

$$K(P,y)=\frac{\alpha^2}{F^2}\bar{K}(P)+\frac{1}{4F^4}\Big(3\big\langle U(y,y),u\big\rangle^2-4F\cdot\big\langle U(y,U(y,y)),u\big\rangle\Big),\tag{2.3}$$

where u is the vector in $\mathfrak{m}$ corresponding to the 1-form and $\bar{K}$ is the sectional curvature of α.

Remark The sectional curvature $\bar{K}$ has a very explicit formula. In fact, given an orthonormal basis v_1, v_2 of P, we have

$$\begin{aligned}
\bar{K}(P)&=|U(v_1,v_2)|^2-\big\langle U(v_1,v_1),U(v_2,v_2)\big\rangle-\frac{3}{4}|[v_1,v_2]_{\mathfrak{m}}|^2\\
&\quad-\frac{1}{2}\big\langle [v_1,[v_1,v_2]]_{\mathfrak{m}},v_2\big\rangle-\frac{1}{2}\big\langle [v_2,[v_2,v_1]]_{\mathfrak{m}},v_1\big\rangle.
\end{aligned}$$

See [HE74, BE87].

3 Left Invariant Randers Metrics on Two-step Nilpotent Lie Groups

From now on we will give some applications of formula (2.3). We first consider left invariant Randers metrics on a two-step nilpotent Lie group. Recall that a Lie algebra $\mathfrak{g}$ is called *two-step nilpotent* if $\mathfrak{g}$ is non-abelian and $[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]]=0$. A connected Lie group G is called two-step nilpotent if its Lie algebra is two-step nilpotent. Two-step nilpotent Lie groups are the simplest non-abelian Lie groups, but they admit many interesting phenomena that do not happen in the abelian case. Studies on left invariant Riemannian metrics on such a type of Lie groups have many important merits in differential geometry. For example, A. Kaplan studied left invariant Riemannian metrics on H-type Lie groups (a special kind of two-step nilpotent Lie group) and obtained many examples of commutative Riemannian manifolds [KA83]. J. Lauret constructed the first example of commutative Riemannian manifold which is not weakly symmetric [LA98], which is a left invariant Riemannian metric on certain two-step nilpotent Lie group. See also [BRV98, GO96, WO07, ZI96] for more information on the study of such metrics.

Now we will give a formula for the flag curvature of an arbitrary left invariant Randers metric on a two-step nilpotent Lie group. For this we need an alternative description of Randers metrics, namely the navigation data. It can be shown that a Randers metric $F = \alpha + \beta$ can also be written as

$$F(x, y) = \frac{\sqrt{h(y, W)^2 + \lambda h(y, y)}}{\lambda} - \frac{h(y, W)}{\lambda},$$

where h is a Riemannian metric, W is a vector field on M with $h(W, W) < 1$ and $\lambda = 1 - h(W, W)$. The pair (h, W) is called the navigation data of the Randers metric F. This version of Randers metric was introduced by Z. Shen in [SH03]. If F is a Randers metric with navigation data (h, W), then we say that F solves Zermelo's navigation problem of the Riemannian metric h under the influence of an external vector field W. The navigation data is convenient in handling problems concerning flag curvature and Ricci scalar. For example, using navigation data, Bao and Robles presented a very convenient way to describe Einstein–Randers metrics as well as Randers spaces of constant flag curvature in [BR04].

In a local coordinate system, the transformation law between the defining form and navigation data can be described as follows. If

$$F = \alpha + \beta = \sqrt{a_{ij} y^i y^j} + b_i y^i,$$

then the navigation data has the form

$$h_{ij} = (1 - \|\beta\|^2)(a_{ij} - b_i b_j), \quad W^i = -\frac{a^{ij} b_j}{1 - \|\beta\|_\alpha^2}. \tag{3.1}$$

Conversely, the defining form can also be expressed by the navigation data using the formula

$$a_{ij} = \frac{h_{ij}}{\lambda} + \frac{W_i}{\lambda}\frac{W_j}{\lambda}, \quad b_i = \frac{-W_i}{\lambda}, \tag{3.2}$$

here $W_i = h_{ij} W^j$ and $\lambda = 1 - W^i W_i = 1 - h(W, W)$. All of these formulas can be found in [BR04].

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Let G be a two-step nilpotent Lie group with Lie algebra $\mathfrak{g}$. Then left invariant Randers metrics on G are in one-to-one correspondence to the pairs $(\langle\cdot,\cdot\rangle, u)$, where $\langle\cdot,\cdot\rangle$ is an inner product on $\mathfrak{g}$ and u is an element in $\mathfrak{g}$ with length less than 1. The Randers metric generated by $\langle\cdot,\cdot\rangle$ and $u\in\mathfrak{g}$ can be written as

$$F(y)=\sqrt{\langle y,y\rangle}+\langle u,y\rangle.$$

By (3.1) and (3.2) it is easy to obtain the navigation data (h,W). The left invariant Riemannian metric h is generated by the inner product $\langle\cdot,\cdot\rangle_h$ on $\mathfrak{g}$ determined by

$$\langle y,y\rangle_h=\lambda\big(\langle y,y\rangle-\langle u,y\rangle^2\big),\quad y\in\mathfrak{g},$$

and the left invariant vector field W is generated by

$$w=-\frac{1}{\lambda}u\in\mathfrak{g},$$

here $\lambda=1-\langle u,u\rangle$.

Since $\mathfrak{g}$ is nilpotent and non-abelian, the center $\mathfrak{z}$ of $\mathfrak{g}$ is non-zero and not equal to $\mathfrak{g}$. Furthermore we have $[\mathfrak{g},\mathfrak{g}]\subset\mathfrak{z}$. Let $\mathfrak{a}$ be the orthogonal complement of $\mathfrak{z}$ in $\mathfrak{g}$ with respect to the inner product $\langle\cdot,\cdot\rangle_h$, and for $x\in\mathfrak{g}$ we denote by $x_{\mathfrak{z}}$ and $x_{\mathfrak{a}}$ the $\mathfrak{z}$-component and $\mathfrak{a}$-component of x, respectively.

Theorem 3.1 *Let G, $\mathfrak{g}$, $\langle\cdot,\cdot\rangle$, u, $\langle\cdot,\cdot\rangle_h$, w, $\mathfrak{z}$, and $\mathfrak{a}$ be as above and let F be the left invariant Randers metric on G generated by the pair $(\langle\cdot,\cdot\rangle,u)$. Denote $u_1=\frac{\mu}{\lambda}u_{\mathfrak{a}}$, where*

$$\mu=1-\frac{1}{\lambda}\big(\langle u_{\mathfrak{a}},u_{\mathfrak{a}}\rangle-\langle u,u_{\mathfrak{a}}\rangle^2\big).$$

Define an inner product $\langle\cdot,\cdot\rangle_1$ on $\mathfrak{g}$ by letting

$$|y|_1=\frac{1}{\mu}\sqrt{\lambda\mu\big(|y|^2-\langle y,u\rangle^2\big)+\big(\langle y,u_{\mathfrak{a}}\rangle-\langle u,u_{\mathfrak{a}}\rangle\langle y,u\rangle\big)^2},$$

where $|\cdot|$ and $|\cdot|_1$ denote the length of $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_1$, respectively. Let F_1 be the Randers metrics defined by the pair $(\langle\cdot,\cdot\rangle_1,u_1)$. Then the flag curvature of F can be expressed as

$$\begin{aligned}K(y,v,y\wedge v)&=\frac{\alpha_1^2(\widetilde{y})}{F_1^2(\widetilde{y})}\bar{K}_1(\widetilde{y}\wedge v)\\&\quad+\frac{1}{F_1^4(\widetilde{y})}\Big(3\big\langle U_1(\widetilde{y},\widetilde{y}),u_1\big\rangle_1^2-4F_1(\widetilde{y})\big\langle U_1(\widetilde{y},U_1(\widetilde{y},\widetilde{y})),u_1\big\rangle_1\Big),\end{aligned}$$

where $\bar{K}_1$ denote the sectional curvature of the Riemannian metric α_1, U_1 is a bilinear symmetric map from $\mathfrak{g}\times\mathfrak{g}$ defined by

$$\big\langle U_1(z_1,z_2),z\big\rangle_1=\frac{1}{2}\Big(\big\langle[z,z_1],z_2\big\rangle_1+\big\langle[z,z_2],z_1\big\rangle_1\Big),\quad z,z_1,z_2\in\mathfrak{g},$$

and $\widetilde{y}=y+\frac{F_1(y)}{\lambda}u_{\mathfrak{z}}$.

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Proof Since (h, W) is the navigation data of F, and $\langle\cdot,\cdot\rangle_h$ and w are the corresponding inner product in $\mathfrak{g}$ and vector in $\mathfrak{g}$, respectively, we have $u = -\lambda w$, where $\lambda = 1 - h(w,w) = 1 - |u|^2$ (note that $h(w,w) = \langle u, u\rangle$). It is clear that the Randers metric F can be obtained through two steps of navigation deformation. The first step is to deform the Riemannian metric h under the vector field corresponding to $w_{\mathfrak{a}} = -\frac{1}{\lambda}u_{\mathfrak{a}}$. We denote the resulting Finsler metric by F_1; The second step is to deform the Finsler metric F_1 under the vector field corresponding to $w_{\mathfrak{z}} = -\frac{1}{\lambda}u_{\mathfrak{z}}$, and the resulting Finsler metric is exactly F.

By (3.1) and (3.2), one can easily obtain the defining form of F_1 as follows

$$F_1(y) = \alpha_1(y) + \langle y, u_1\rangle,$$

where $u_1 = \frac{\mu}{\lambda}u_{\mathfrak{a}}$. By the assumption $[\mathfrak{g},\mathfrak{g}] \subset \mathfrak{z}$ we see that $\langle w_{\mathfrak{a}}, [\mathfrak{g},\mathfrak{g}]\rangle_h = 0$. Then we have

$$\begin{aligned}\langle u_1, [\mathfrak{g},\mathfrak{g}]\rangle_1 &= \frac{\mu}{\lambda}\langle u_{\mathfrak{a}}, [\mathfrak{g},\mathfrak{g}]\rangle_1\\ &= \frac{\mu}{\lambda}\Big(\frac{1}{\mu}\big\langle u_{\mathfrak{a}}, [\mathfrak{g},\mathfrak{g}]\big\rangle_h + \frac{1}{\mu^2}\langle w_{\mathfrak{a}}, u_{\mathfrak{a}}\rangle_h \times \big\langle w_{\mathfrak{a}}, [\mathfrak{g},\mathfrak{g}]\big\rangle_h\Big)\\ &= 0.\end{aligned}$$

Therefore, u_1 is orthogonal to $[\mathfrak{g},\mathfrak{g}]$ with respect to the inner product $\langle\cdot,\cdot\rangle_1$. Hence F_1 is a Douglas metric. Applying Theorem 2.1, the formula of the flag curvature of F_1 is

$$\begin{aligned}K_1(y, v, y\wedge v) &= \frac{\alpha_1^2(y)}{F_1^2(y)}\bar{K}_1(y\wedge v)\\ &\quad + \frac{1}{F_1^4(\widetilde{y})}\Big(3\big\langle U_1(y,y), u_1\big\rangle_1^2 - 4F_1(\widetilde{y})\big\langle U_1(y, U_1(y,y)), u_1\big\rangle_1\Big),\end{aligned}$$

where $\bar{K}_1$ denotes the sectional curvature of α_1.

Now we consider the second navigation deformation. We first assert that the vector field generated by $w_{\mathfrak{z}}$, denoted by $W_{\mathfrak{z}}$, is a Killing vector field of the Finsler metric F_1. Since both F_1 and $W_{\mathfrak{z}}$ are left invariant, we only need to prove this at the identity element e. The one-parameter transformation group of G generated by $W_{\mathfrak{z}}$ is

$$\varphi_t\colon g \longmapsto g\exp tw_{\mathfrak{z}}.$$

So $W_{\mathfrak{z}}$ is a Killing vector field of F_1 if and only if φ_t consists of isometries of F_1. Now for $v \in T_e(G) = \mathfrak{g}$, we have

$$F_1(d\varphi_t(v)) = F_1\Big(\Big[\frac{d}{ds}\exp(sv)\exp(tw_{\mathfrak{z}})\Big]\Big|_{s=0}\Big).$$

Since $w_{\mathfrak{z}}$ lies in the center of $\mathfrak{g}$, v, $w_{\mathfrak{z}}$ are commutative. Therefore we have

$$\exp(sv)\exp(tw_{\mathfrak{z}}) = \exp(sv + tw_{\mathfrak{z}}).$$

This implies that

$$F_1\big(d\varphi_t(v)\big) = F_1\Big(\Big[\frac{d}{ds}\exp(sv + tw_3)\Big]_{s=0}\Big) = F_1(v).$$

This proves our assertion. Now by a theorem of Huang–Mo ([HM07]), the flag curvature of F, which is obtained by a navigation deformation of F_1 through a Killing vector field, can be expressed as

$$K(y, v, y \wedge v) = K_1(\widetilde{y}, v, \widetilde{y} \wedge v),$$

where $\widetilde{y} = y - F_1(y)w_3 = y + \frac{F_1(y)}{\lambda}u_3$. This completes the proof of the theorem. ■

4 Negatively Curved Spaces

In this section, we use formula (2.3) to give a complete description of all the homogeneous Randers metrics of Douglas type whose flag curvature is negative. We first prove the following theorem.

Theorem 4.1 *Let $F = \alpha + \beta$ be a G-invariant Randers metric of Douglas type on the homogeneous manifold G/H. If the flag curvature of F is negative, then the sectional curvature of α is negative.*

Proof We only need to prove the theorem at the origin o. Assume $\beta \neq 0$. Let P be a tangent plane at o. Note that the orthogonal complement (with respect to the inner product corresponding to α) $u^{\perp}$ of $\mathbb{R}u$ has dimension $\dim\mathfrak{g} - 1$. Thus $\dim(P \cap u^{\perp}) \geq 1$. Therefore we can select $y_0 \neq 0$ such that $y_0 \in P \cap u^{\perp}$. Then

$$F(y_0) = F(-y_0) = \alpha(y_0).$$

On the other hand, by the bi-linearity, we have $U(-y_0, -y_0) = U(y_0, y_0)$ and $U(-y_0, U(-y_0, -y_0)) = -U(y_0, U(y_0, y_0))$. Applying (2.3) to the flags (P, y_0) and $(P, -y_0)$ we get

$$K(P, y_0) = \frac{\alpha^2(y_0)}{F^2(y_0)}\bar{K}_P + \frac{1}{4F^4(y_0)}\Big(3\langle U(y_0, y_0), u\rangle^2 - 4F(y_0)\cdot\langle U(y_0, U(y_0, y_0)), u\rangle\Big), \tag{4.1}$$

and

$$K(P, -y_0) = \frac{\alpha^2(y_0)}{F^2(y_0)}\bar{K}_P + \frac{1}{4F^4(y_0)}\Big(3\langle U(y_0, y_0), u\rangle^2 + 4F(y_0)\cdot\langle U(y_0, U(y_0, y_0)), u\rangle\Big). \tag{4.2}$$

Taking the summation of both sides of (4.1) and (4.2) we obtain

$$K(P, y_0) + K(P, -y_0) = 2\frac{\alpha^2(y_0)}{F^2(y_0)}\bar{K}_P + \frac{3}{2F^4(y_0)}\big\langle U(y_0, y_0), u\big\rangle^2.$$

By assumption, the left side of the above equation is negative. Therefore $\bar{K}(P) < 0$. This completes the proof of the theorem. ■

Now suppose (M, F) is a connected Randers space of Douglas type with negative flag curvature. Theorem 4.1 asserts that the underlying Riemannian metric α also has negative sectional curvature. Let G be a connected transitive group of isometries of F. Then G is a subgroup of the full group of isometries of the Riemannian manifold (M, α). A Theorem of J. A. Wolf then asserts that G contains a solvable subgroup S which is also transitive on M (see the proof of [WO64, Corollary 1(c)]). Applying the argument of E. Heintze in [HE74] and taking into account Theorem 4.1, we get the following theorem.

Theorem 4.2 *Let (M, F) be a homogeneous Randers space of Douglas type with negative flag curvature. Then F can be viewed as a left invariant Randers metric on a connected simply connected solvable Lie group G. Furthermore, the underlying Riemannian metric α, which is a left invariant metric on G, also has negative sectional curvature.*

Now we give a description of all the homogeneous Randers spaces of Douglas type with negative flag curvature. By the above theorem, any such a space can be written as a pair (G, F), where G is a connected solvable Lie group and F is a left invariant Randers metric on G. Since F has negative curvature, G is necessarily simply connected ([DH07]). Hence the pair (G, F) is uniquely determined by a Randers norm on the Lie algebra $\mathfrak{g}$ of G. In the following we will denote the Randers space (G, F) as a pair $(\mathfrak{g}, F_0)$, where F_0 is the corresponding Minkowski norm on $\mathfrak{g}$. Let $\langle \cdot, \cdot \rangle$ be the corresponding inner product on $\mathfrak{g}$ and let u be the corresponding vector in $\mathfrak{g}$. By Theorem 4.1, the pair $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ also has negative sectional curvature.

Theorem 4.3 *Let $\mathfrak{g}$ be a real solvable Lie algebra. Then the following two conditions are equivalent:*

(i) $\mathfrak{g}$ *admits a Randers norm of Douglas type with negative flag curvature;*

(ii) $\dim \mathfrak{g} - \dim \mathfrak{g}' = 1$, *where* $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ *is the derived subalgebra of* $\mathfrak{g}$, *and there exists* $A_0 \in \mathfrak{g}$ *such that the eigenvalues of the linear map* $ad(A_0)$ *on* $\mathfrak{g}'$ *have positive real part.*

Furthermore, on any real solvable Lie algebra satisfying condition (ii), *there exists non-euclidean Randers norms of Douglas type with negative flag curvature.*

Proof Theorem 3 of [HE74] asserts that condition (ii) is equivalent to the fact that there exists an inner product on $\mathfrak{g}$ with negative sectional curvature. Combining this with Theorem 4.1, we prove the equivalence of conditions (i) and (ii). Now we prove that on any real solvable Lie algebra satisfying (ii) there exists a non-euclidean Randers norm of Douglas type with negative flag curvature. Let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathfrak{g}$ with negative sectional curvature. The set of all tangent planes at the origin is compact, hence there exists a positive number M such that $\bar{K}(P) < -M$, for any tangent plane P at the origin. On the other hand, the map $U : \mathfrak{g} \times \mathfrak{g}$ is smooth, hence the set $\{U(y, y) | \langle y, y \rangle = 1\}$ must be compact. This then implies that the set

$$\left\{ U\left(y, U(y, y)\right) | \langle y, y \rangle = 1 \right\}$$

is also compact. Now we consider the vector cA_0, where c is a constant. Without loss of generality we can assume A_0 has length 1. Further, we can also assume that A_0 is orthogonal to $\mathfrak{g}'$ (see [HE74, Proposition 2]). Then the Randers norm F_c generated by the pair $(\langle \cdot, \cdot \rangle, cA_0)$ is of Douglas type. By the above argument we see from (2.3) that there is a $\varepsilon > 0$ such that for any c satisfying $|c| < \varepsilon$ and any tangent plane P in $\mathfrak{g}$, the flag curvature $K_c(P, y) < 0$, for any $y \in P$ with $\langle y, y \rangle = 1$. This proves the theorem. ∎

Remark Heintze pointed out in [HE74] that there are many examples of solvable Lie algebras satisfying Theorem 4.2(ii). This results in a large numbers of examples of non-Riemannian homogeneous Randers spaces of Douglas type with negative flag curvature. In [HD11], we proved that a homogeneous Randers space with almost isotropic S-curvature and negative Ricci scalar must be Riemannian. The above argument shows that the restriction in S-curvature cannot be dropped.

5 A Rigidity Result on Berwald Spaces

In this section we will prove a rigidity result on homogeneous Randers spaces of Berwald type. Recall that the Randers metric $(G/H, F)$ in Theorem 3.1 is of Berwald type if and only if the vector field generated by u is parallel with respect to α. In [DE08], the second author proved that this is equivalent to the condition that it is of Douglas type and satisfies the following condition:

$$\big\langle [u, v_1]_{\mathfrak{m}}, v_2 \big\rangle + \big\langle v_1, [u, v_2]_{\mathfrak{m}} \big\rangle = 0, \quad \forall v_1, v_2 \in \mathfrak{m}. \tag{5.1}$$

By the definition of the map U, the above condition is just

$$\langle u, U(v_1, v_2) \rangle = 0, \quad \forall v_1, v_2 \in \mathfrak{m}.$$

Therefore, for an invariant Randers metric of Berwald type, the formula of the flag curvature becomes

$$K(P, y) = \frac{\alpha^2(y)}{F^2(y)} \bar{K}(P). \tag{5.2}$$

Now we can prove the following theorem.

Theorem 5.1 *Let (M, F) be a homogeneous Randers space of Berwald type. If the sectional curvature of (M, F) is everywhere non-zero, then F is a Riemannian metric.*

Proof Without loss of generality, we can assume that M is simply connected. Since the flag curvature $K(y, v, y \wedge v)$ is a continuous function of y and v, the condition that K is everywhere nonzero means that K is either everywhere positive or everywhere negative. We now prove the theorem case by case.

First assume that the flag curvature K is everywhere positive. Then by (5.2), the underlying Riemannian metric has positive sectional curvature. Connected, simply connected, homogenous Riemannian manifolds have been classified by N. Wallach ([WA72]) and L. Bérard Bergery ([BB76]). Their results are summarized in the following table:

Even dimensions	isotropy repr.
$S^{2n} = SO(2n+1)/SO(2n)$	irred.
$\mathbb{C}P^m = SU(m+1)/U(m)$	irred.
$\mathbb{H}P^k = Sp(k+1)/(Sp(k) \times Sp(1))$	irred.
$CaP^2 = F_4/Spin(9)$	irred.
$F^6 = SU(3)/T^2$	$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$
$F^{12} = Sp(3)/(Sp(1) \times Sp(1) \times Sp(1))$	$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$
$F^{24} = F_4/Spin(8)$	$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$
Odd dimensions	isotropy repr.
$S^{2n+1} = SO(2n+2)/SO(2n+1)$	irred.
$M^7 = SO(5)/SO(3)$	irred.
$M^{13} = SU(5)/(Sp(2) \times_{\mathbb{Z}_2} S^1)$	$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$
$N_{1,1} = SU(3) \times SO(3)/U^*(2)$	$\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$
$N_{k,l} = SU(3)/S^1_{k,l}$, gcd(k,l)=1, $kl(k+l) \neq 0$	$\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$

Table 5.1: Compact homogeneous Riemannian manifolds with positive curvature

Note that besides the rank one Riemannian symmetric spaces, which can also be written as the coset of other type of Lie groups, for any other space, when we write it as the coset space G/H, the Lie group G is semisimple. Now a theorem of An-Deng ([AD08]) asserts that any invariant Randers metrics of Douglas type on a coset space G/H, such that the Lie algebra $\mathfrak{g}$ of G is perfect (that is, $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$), must be Riemannian. Since a semisimple Lie algebra must be perfect, any invariant Randers space of Berwald type on the above homogeneous manifolds must be Riemannian.

It remains to consider the rank one compact symmetric spaces. These spaces include the spheres S^n, the real projective spaces $\mathbb{R}P^n$, the complex projective spaces $\mathbb{C}P^n$, the quarternion projective spaces $\mathbb{H}P^n$, and the Cayley projective plane CayP^2. Besides the standard transitive Lie groups actions, there are some other Lie groups that act transitively on such spaces. These Lie groups have been classified by Montgomery–Samelson [MS43], A. Borel [BO49] and A. L. Oniščik [ON63]. The list of compact Lie groups that have a transitive and effective action on the spheres can be found in [BE87, p. 179]. Besides the spheres, the only rank one symmetric spaces that admit transitive and effective actions of Lie groups other than the standard Lie groups are $\mathbb{C}P^{2n-1}$, which admits an action of $Sp(n)$ through the identification $\mathbb{H}^n = \mathbb{C}^{2n}$. These results imply that if we write a rank one compact symmetric space as a coset space G/H, then G is semisimple with only two exceptions: $S^{2n-1} = U(n)/U(n-1)$ and $S^{4n-1} = Sp(n)U(1)/Sp(n-1)U(1)$. However, $U(n)$ contains the simple Lie groups $SU(n)$, which is transitive on S^{2n-1}; $Sp(n)U(1)$ contains the simple Lie group $Sp(n)$, which is also transitive on S^{4n-1}. From this we conclude that any homogeneous Randers metric on a rank one compact symmetric space can be viewed as a G-invariant Randers metric on a coset space G/H, where G is a semisimple Lie group. Hence by the above argument we conclude that any homogeneous Randers metric of Douglas type on rank one compact symmetric space must be Riemannian. This assertion holds in particular, for Berwald metrics. This completes the proof of the positive case .

Now we consider the negative case. In this case the underlying Riemannian metric α has negative sectional curvature. Hence by Theorem 4.3, (M, F) can be written as a left invariant Randers metric on a solvable Lie algebra $\mathfrak{g}$, such that the derived subalgebra $\mathfrak{g}'$ has dimension $\dim \mathfrak{g} - 1$. Moreover, the Randers metric is generated by a pair $(\langle\cdot,\cdot\rangle, cA_0)$, where A_0 is orthogonal to $\mathfrak{g}'$, $\langle A_0, A_0\rangle = 1$, c is a constant, and the eigenvalues of $\operatorname{ad} A_0$ on $\mathfrak{g}'$ has positive real part. The last condition implies that cA_0 cannot satisfy condition (5.1) unless $c = 0$, since otherwise $\operatorname{ad} A_0|_{\mathfrak{g}'}$ must be a skew-symmetric map with respect to $\langle\cdot,\cdot\rangle$, hence all its eigenvalues must be either 0 or pure imaginary. This completes the proof of the theorem. ■

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