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A NONLINEAR COMPLEMENTARITY PROBLEM IN BANACH SPACE

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Existence and uniqueness theorems for the nonlinear complementarity problem over closed convex cones in a reflexive real Banach space are established.

Introduction and statement of the theorems

Let B be a reflexive real Banach space and let B^* be its dual. Let the value of $u \in B^*$ at $x \in B$ be denoted by (u, x) . Let C be a closed convex cone in B with vertex at 0 . The polar of C is the cone C^* , defined by

$$C^* = \{u \in B^* : (u, x) \geq 0 \text{ for each } x \in C\} .$$

For each $r \geq 0$ we write

$$D_r = \{x \in C : \|x\| \leq r\} .$$

A mapping $T : C \to B^*$ is said to be monotone if $(Tx-Ty, x-y) \geq 0$ for all $x, y \in C$ and strictly monotone if strict inequality holds whenever $x \neq y$. We way that T is α-monotone if there is a strictly increasing function $\alpha : [0, \infty) \to [0, \infty)$ with $\alpha(0) = 0$ and $\alpha(r) \to \infty$ as $r \to \infty$ such that $(Tx-Ty, x-y) \geq \|x-y\|\alpha(\|x-y\|)$ for all $x, y \in C$. In particular T is strongly monotone if $\alpha(r) = kr$ for some $k > 0$. Note that if T is α-monotone, then it is strictly monotone. T is said to be hemicontinuous on C if for all $x, y \in C$, the map $t \mapsto T\big(ty+(1-t)x\big)$ of $[0, 1]$ to B^* is continuous when B^* is endowed with the weak*

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topology. There are two definitions of boundedness for operators.

DEFINITION 1. $T : C \to B^*$ is said to be bounded if T maps bounded subsets of C into bounded subsets of B^*.

DEFINITION 2. $T : C \to B^*$ is said to be bounded if there is a constant $K > 0$ such that $\|Tx\| \le K\|x\|$ for every $x \in C$.

Boundedness in the sense of Definition 1 has been used in Mosco [8]. It is clear that boundedness in the sense of Definition 2 always implies boundedness in the sense of Definition 1. In the case of linear operators, both these definitions are equivalent.

The purpose of this note is to prove the following existence and uniqueness theorems for the nonlinear complementarity problem.

THEOREM 1. *Let $T : C \to B^*$ be hemicontinuous, strictly monotone and bounded in the sense of Definition* 2. *Then there is a unique x_0 such that*

$$(1) \qquad x_0 \in C , \quad Tx_0 \in C^* \quad \text{and} \quad \left(Tx_0, x_0\right) = 0 .$$

THEOREM 2. *Let $T : C \to B^*$ be hemicontinuous, strictly monotone and bounded in the sense of Definition* 1 *such that $T(0) \in C^*$. Then there is a unique x_0 such that* (1) *holds.*

This work has been motivated by the work of Bazaraa, Goode and Nashed [1] who have proved the same result under the assumption that the operator T is hemicontinuous, α-monotone and bounded in the sense of Definition 1. It is obvious that our results are different from the results obtained in [1]. We have only assumed strict monotonicity in both the theorems instead of α-monotonicity (which is stronger). However, in Theorem 1 we have used the boundedness of T as contained in Definition 2, which is stronger than Definition 1. Similarly, in Theorem 2 we have made a feasibility assumption. Moreover, in this note, we have obtained a stronger result showing that 0 is the unique solution of the nonlinear complementarity problem.

Several authors including Eaves [2], Habetler and Price [3] and Karamardian [4], [5] and [7] have discussed the nonlinear complementarity problem in finite dimensional spaces. Besides the work of Bazaraa, Goode and Nashed [1], the solutions of nonlinear complementarity problems in a general setting can be found in [6] and [11].

Proof of the theorems

The following result, which will be needed in the sequel, is a special case of Theorem A of Mosco [8]. See also Bazaraa, Goode and Nashed [1, Lemma 1].

LEMMA (Mosco). *Let* $T : C \to B^*$ *be hemicontinuous, strictly monotone and bounded in the sense of Definition* 1 *and let* $\{K_r\}$ *be a family of nonempty closed convex sets in* C . *Then, for each* r , *there is a unique* $x_r \in K_r$ *such that*

$$(Tx_r, x_r) \le (Tx_r, z) \quad \text{for all} \quad z \in K_r .$$

Proof of Theorem 1. Since D_r is a nonempty closed convex set in C , it follows from Mosco's lemma that for each $r \ge 0$, there is a unique $x_r \in D_r$ such that

$$(Tx_r, x_r) \le (Tx_r, z) \quad \text{for all} \quad z \in D_r .$$

Since $0 \in D_r$, it is clear that $(Tx_r, x_r) \le 0$. Define a function $\theta : [0, \infty) \to (-\infty, 0]$ by the rule

$$\theta(r) = (Tx_r, x_r) .$$

Notice that if $r < s$, then $D_r \subset D_s$. Now suppose that $r \ne 0$, $s \ne 0$ and $r < s$. Then there are unique $x_r \in D_r$ and $x_s \in D_s$ such that

$$(Tx_r, x_r) \le (Tx_r, z) \quad \text{for all} \quad z \in D_r \tag{2}$$

and

$$(Tx_s, x_s) \le (Tx_s, z) \quad \text{for all} \quad z \in D_s . \tag{3}$$

Note that $(r/s)x_s \in D_r$ and $(s/r)x_r \in D_s$. Therefore, from (2) and (3), it follows that

$$(Tx_r, x_r) \le (r/s)(Tx_r, x_s) \tag{4}$$

and

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(5) $$\left(Tx_r,\ x_s\right) \le (s/r)\left(Tx_s,\ x_r\right)\ .$$

From (4) and (5) we have

(6) $$\begin{aligned}\left(Tx_r-Tx_s,\ x_r-x_s\right) &= \left(Tx_r,\ x_r\right) + \left(Tx_s,\ x_s\right) - \left(Tx_r,\ x_s\right) - \left(Tx_s,\ x_r\right)\\ &\le \left(Tx_r,\ x_r\right) + \left(Tx_s,\ x_s\right) - (s/r)\left(Tx_r,\ x_r\right) - (r/s)\left(Tx_s,\ x_s\right)\\ &= \left(1-(s/r)\right)\theta(r) + \left(1-(r/s)\right)\theta(s)\\ &= \left((r-s)/r\right)\theta(r) - \left((r-s)/s\right)\theta(s)\\ &= (s-r)\left|\left(\theta(s)/s\right)-\left(\theta(r)/r\right)\right|\ .\end{aligned}$$

Since $s > r$ and T is monotone, it follows from (6) that

$$\theta(s)/s \ge \theta(r)/r\ .$$

Therefore $\theta(r)/r$ is monotonically increasing function on $(0,\ \infty)$. On the other hand, since T is bounded in the sense of Definition 2, it follows from the Cauchy-Schwarz inequality that

$$|\theta(r)| = \left|\left(Tx_r,\ x_r\right)\right| \le \|Tx_r\|\|x_r\| \le K\|x_r\|^2 \le Kr^2\ .$$

Since $\theta(r) \le 0$, we have $|\theta(r)| = -\theta(r)$ and so

$$-\left(\theta(r)/r\right) \le Kr$$

and consequently

(7) $$-Kr \le \theta(r)/r \le 0\ .$$

Since (7) holds for all $r \in (0,\ \infty)$ and $\theta(r)/r$ is monotonically increasing, it follows that $\theta(r)/r = 0$ and hence $\theta(r) = 0$ for all $r \in (0,\ \infty)$. Hence we have $\left(Tx_r,\ z\right) \ge 0$ for all $z \in D_r$ and so $\left(Tx_r,\ z\right) \ge 0$ for all $z \in C$. Therefore, for each $r \in (0,\ \infty)$, x_r is a solution to (1). Since T is strictly monotone, the complementarity problem (1) can have at most one solution, say y . Therefore $y = x_r \in D_r$ for each r and

$$\|y\| = \|x_r\| \le r$$

for each r ; so $y = 0$. This completes the proof.

Proof of Theorem 2. As in Theorem 1 we can show that, for each $r \ge 0$, there is a unique $x_r \in D_r$ such that

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$$(Tx_r, x_r) \le 0 . \tag{8}$$

Since T is monotone we have

$$(Tx_r - T(0), x_r) \ge 0 . \tag{9}$$

Now, from (9) and the fact that $T(0) \in C^*$, we obtain

$$(Tx_r, x_r) \ge 0 . \tag{10}$$

It follows from (8) and (10) that $(Tx_r, x_r) = 0$ for all $r \in (0, \infty)$ and the result is easily deduced following the same kind of arguments as in Theorem 1.

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