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RANDOM FOURIER SERIES ON COMPACT ABELIAN HYPERGROUPS

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Abstract

Random Fourier series are studied for a class of compact abelian hypergroups. The randomizing factors are assumed to be independent and uniformly subgaussian. In the presence of a natural technical hypothesis, an entropy condition of Dudley is shown to be sufficient for almost sure continuity. The classical results on almost sure membership in L^p, where $p < \infty$, are generalized to this setting. As an application, it is shown that a simple condition on the dual object implies that the de Leeuw-Kahane-Katznelson phenomenon occurs. Another application is the analogue of a classical sufficient condition for almost sure continuity. Examples illustrating the general theory are given for the hypergroup of conjugacy classes of $SU(2)$ and for a class of compact countable hypergroups.

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1. Introduction

Random Fourier series have been studied in a variety of settings: on the circle group $[0, 2\pi)$, on compact abelian groups, on compact non-abelian groups, and (under the label "random central Fourier series") on the space of conjugacy classes of certain compact groups. In this paper we work in the generality of a

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compact (metrizable) abelian hypergroup K whose dual $\hat{K}$ is also a hypergroup. The generalizations obtained emphasize the common features of many of the previous settings and raise some new questions.

The space of almost surely continuous functions is studied in Section 3. Recall that by Parseval's relation (see Section 2) we have $f \in L^2(K)$ if and only if $\sum_{n=1}^{\infty} |\hat{f}(\psi_n)|^2 \hat{m}(\psi_n) < \infty$ where $\hat{m}$ denotes the invariant measure on $\hat{K} = \{\psi_1, \psi_2, \ldots\}$. A more restrictive condition, namely $\sum_{n=1}^{\infty} |\hat{f}(\psi_n)|^2 \hat{m}(\psi_n)^2 < \infty$, is necessary for almost sure continuity. This condition imposes itself in several places. An entropy condition of Dudley is shown to be sufficient for almost sure continuity.

Section 4 contains an introduction to random series in $L^p(K)$ and $M(K)$. A simple condition on $\hat{K}$ is given that implies the de Leeuw-Kahane-Katznelson phenomenon: whenever $\sum_{n=1}^{\infty} |b_n|^2 \hat{m}(\psi_n) < \infty$, there exists $f \in C(K)$ such that $|\hat{f}(\psi_n)| \geqslant |b_n|$ for all n. The results in this section also yield the analogue of a classical sufficient condition for almost sure continuity. Multiplier interpretations of some of the material in Section 4 are offered in Section 5.

Examples illustrating the general theory and its limitations are given in Sections 6 and 7. The countable hypergroups in Section 6 are easily understood and are very different from compact abelian groups. The hypergroups of conjugacy classes of compact groups in Section 7 are less bizarre and the analysis on them is more delicate and interesting.

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2. Preliminary results

A topological hypergroup is a generalization of a topological group in which the product of two elements is a probability measure rather than an element. Intuitively, it is a space in which the product of two elements is a variety of elements with various probabilities, rather than a single element for sure. Examples include spaces of conjugacy classes of a compact group, spaces of orbits in a locally compact group induced by a compact group of automorphisms, and double-coset spaces. For many convolution measure algebras, it turns out that the basic underlying structure is a hypergroup. For a survey, with examples, see [Ross (1977)]. Our main technical reference for hypergroups is [Jewett (1975)] where hypergroups are called "convos".

For completeness, we give the definition of a hypergroup. Let K be a locally compact Hausdorff space, and let $M(K)$ denote the space of regular complex Borel measures on K. For each $x \in K$, δ_x represents the point mass at x. The space K is a hypergroup if there is a binary mapping $(x, y) \mapsto \delta_x * \delta_y$ of $K \times K$ into $M(K)$ satisfying the conditions listed below. Note that $\delta_x * \delta_y$ need not equal δ_{xy} since xy need not be defined.

1) The mapping $(\delta_x, \delta_y) \mapsto \delta_x * \delta_y$ extends to a bilinear associative operation $*$ from $M(K) \times M(K)$ into $M(K)$ such that

$$\int_K f \, d\mu * \nu = \int_K \int_K \int_K f \, d(\delta_x * \delta_y) \, d\mu(x) \, d\nu(y)$$

for all f in the space $C_0(K)$ of continuous functions f on K vanishing at infinity.

2) For each $x, y \in K$, the measure $\delta_x * \delta_y$ is a probability measure with compact support.

3) The mapping $(\mu, \nu) \mapsto \mu * \nu$ is continuous from $M^+(K) \times M^+(K)$ into $M^+(K)$, where $M^+(K)$ is the set of nonnegative measures in $M(K)$, and is given the weak topology with respect to the family $C_{00}^+(K) \cup \{1\}$; here $C_{00}^+(K)$ is the set of nonnegative continuous functions on K having compact support.

4) There exists an element e in K such that $\delta_e * \delta_x = \delta_x * \delta_e = \delta_x$ for all $x \in K$.

5) There exists a homeomorphic involution $x \mapsto \check{x}$ of K onto K so that given x, y in K we have $(\delta_x * \delta_y)\check{} = \delta_{\check{y}} * \delta_{\check{x}}$ and also

$$e \in \operatorname{supp}(\delta_x * \delta_y) \quad \text{if and only if } y = \check{x}.$$

6) The map $(x, y) \mapsto \operatorname{supp}(\delta_x * \delta_y)$ is continuous from $K \times K$ into the space $\mathcal{C}(K)$ of compact subsets of K, where $\mathcal{C}(K)$ is given the topology whose subbasis is given by all

$$\mathcal{C}_{U,V} = \{A \in \mathcal{C}(K) : A \cap U \neq \varnothing \text{ and } A \subset V\}$$

where U, V are open subsets of K.

We will study random Fourier series on a compact abelian hypergroup K. Such a hypergroup carries a Haar measure m, such that $m(K) = 1$ and $\delta_x * m = m$ for all x in K. Associated with K is a set $\hat{K}$ of continuous functions on K called characters; they form an orthogonal basis for $L^2(K, m)$. Throughout this paper we make two additional assumptions about K:

(a) $\hat{K}$ is a hypergroup under pointwise operations;

(b) K is metrizable, or equivalently $\hat{K}$ is countable.

Assumption (b) is an inessential convenience, since the Fourier transform $\hat{f}$ has countable support for $f \in L^1(K)$. If K were not metrizable, the study of f could be reduced to the metrizable case by standard methods; see the appendix to [Vrem (1978)]. We always assume for definiteness that K is infinite.

Assumption (a) is essential. It can fail for a 3-element hypergroup, but it appears to hold for most "natural" hypergroups, including spaces of conjugacy classes of compact groups. See [Hartmann-Henrichs-Lasser (1979)] for a general class of hypergroups satisfying (a). Assumption (a) tells us the following: given $\psi, \psi' \in \hat{K}$, the product $\psi\psi'$ is a finite sum $\Sigma_{\phi \in \hat{K}} a_\phi \phi$ where all $a_\phi \geqslant 0$ and $\Sigma_{\phi \in \hat{K}} a_\phi = 1$.

2.1. *Fourier series*. Let m and $\hat{m}$ be the invariant measures on K and $\hat{K}$ normalized so that $m(K) = \hat{m}(\{1\}) = 1$. We write $\hat{K} = \{\psi_1, \psi_2, \dots\}$ and we write $\hat{m}(\psi_n)$ in place of $\hat{m}(\{\psi_n\})$. Observe that

$$(1) \qquad \hat{m}(\psi_n) = \|\psi_n\|_2^{-2} \quad \text{for all } n;$$

see 7.1A of [Jewett (1975)] and 3.6 of [Dunkl (1973)]. For $f \in L^1(K, m)$ its *Fourier series* is

$$f \sim \sum_{n=1}^{\infty} \hat{f}(\psi_n)\hat{m}(\psi_n)\psi_n.$$

Parseval's relation states that $f \in L^2(K, m)$ if and only if $\hat{f} \in l^2(\hat{K}, \hat{m})$, and that

$$\int_K |f(x)|^2 \, dm(x) = \sum_{n=1}^{\infty} |\hat{f}(\psi_n)|^2 \hat{m}(\psi_n).$$

Also, $A(K)$ denotes the Banach space of all functions f such that $\hat{f} \in l^1(\hat{K}, \hat{m})$ with the norm $\|f\|_A = \Sigma_{n=1}^{\infty} |\hat{f}(\psi_n)| \hat{m}(\psi_n)$.

2.2. LEMMA. *Let (h_n) be a sequence in $L^1(K)$ such that* $\sup_n \|h_n\|_1 < \infty$. *The following are equivalent*:

(i) *(h_n) is an approximate unit for $L^1(K)$*;

(ii) $\lim_{n\to\infty} \hat{h}_n(\psi) = 1$ *for all $\psi \in \hat{K}$.*

PROOF. To show (i) $\Rightarrow$ (ii) use the identity $h_n * \psi = \hat{h}_n(\psi)\psi$; see [Jewett (1975), 7.3E]. The reversed implication is established by a standard argument using the fact that trigonometric polynomials are dense in $L^1(K)$; see [Vrem (1979), 2.13].

2.3. LEMMA. *$L^1(K)$ has an approximate unit (h_n) such that*

(i) *each h_n is a trigonometric polynomial*;

(ii) $\|h_n\|_1 = 1$ *for all n*;

(iii) $\hat{h}_n \geqslant 0$ *for all n*;

(iv) $\lim_{n\to\infty} \hat{h}_n(\psi) = 1$ *for all $\psi \in \hat{K}$.*

PROOF. According to [Chilana-Ross (1978), 2.8], $L^1(K)$ has an approximate unit (f_n) consisting of trigonometric polynomials such that $\|f_n\|_1 = 1$ for all n. Let $g_n = f_n * \tilde{f}_n^*$ and $h_n = \|g_n\|_1^{-1} g_n$. Properties (i)–(iii) can be checked directly and a citation of Lemma 2.2 yields (iv).

2.4. PROPOSITION. *If* $f \in L^{\infty}(K)$ *and* $\hat{f} \geqslant 0$, *then* $f \in A(K)$ *and* $\|f\|_A = \|f\|_\infty = f(e)$.

PROOF. Let (h_n) be as in Lemma 2.3. For a finite subset F of $\hat{K}$, Parseval's relation leads to

$$\sum_{\psi \in F} \hat{f}(\psi)\overline{\hat{h}_n(\psi)}\,\hat{m}(\psi) \leqslant \sum_{\psi \in \hat{K}} \hat{f}(\psi)\overline{\hat{h}_n(\psi)}\,\hat{m}(\psi)$$
$$= \int_K f\bar{h}_n\,dm \leqslant \|f\|_\infty \|h_n\|_1 = \|f\|_\infty.$$

Now let $n \to \infty$ and then let F increase to $\hat{K}$ to obtain $\|f\|_A = f(e) \leqslant \|f\|_\infty$. Since $\|f\|_\infty \leqslant \|f\|_A$ in general, the proposition is proved.

We will need a property of $\hat{K}$ that corresponds to "local central Λ_p" in the case of the space of conjugacy classes of a compact group [Rider (1972a)]. The definition can be made quite general.

2.5. DEFINITION. *Let* $\mathfrak{F}$ *be a set of* L^1 *functions on a probability space* (Ω, P). *For* $1 < p \leqslant \infty$, *we say* $\mathfrak{F}$ *has property* λ_p *provided*

$$\sup\{\|f\|_p/\|f\|_1 : f \in \mathfrak{F}\} < \infty. \tag{1}$$

2.6. REMARKS. (a) Of course $\mathfrak{F}$ has property λ_p if and only if $\{|f| : f \in \mathfrak{F}\}$ does.

(b) For $p > 2$ the requirement 2.5(1) is equivalent to

$$\sup\{\|f\|_p/\|f\|_2 : f \in \mathfrak{F}\} < \infty, \tag{1}$$

by an argument using Hölder's inequality; see [López-Ross, page 54].

(c) $\mathfrak{F}$ has property λ_2 if and only if there is a constant $\kappa > 0$ such that

$$P\{|f| \geqslant \tfrac{1}{2}\kappa\|f\|_2\} \geqslant \kappa^2/16 \quad \text{for all } f \in \mathfrak{F}. \tag{2}$$

The necessity of (2) is shown in Lemma 13.7.1 of [Kawata (1972)], and the sufficiency is trivial to verify.

(d) Since $\|\psi\|_\infty = \psi(e) = 1$ for all $\psi \in \hat{K}$, (1) and 2.1(1) show that $\hat{K}$ has property λ_∞ if and only if $\inf_n \|\psi_n\|_2 > 0$ if and only if $\sup_n \hat{m}(\psi_n) < \infty$.

2.7. EXAMPLES. For ψ in $\hat{G}$, where G is a compact abelian group, all L^p norms are equal to 1 and so $\hat{G}$ trivially has property λ_p for all p. As observed in 7.2, if $\hat{K}$ is the space of conjugacy classes of $SU(2)$, then $\hat{K}$ has property λ_p if and only if $p < 3$, and if $SU(2)$ is replaced by a compact, connected, simply connected Lie group, then $\hat{K}$ has property λ_p for some values of $p > 2$. On the other hand, there are hypergroups K such that $\hat{K}$ has property λ_p for no $p > 1$; see 6.1. We know of no hypergroup $\hat{K}$ for which K has property λ_2 and yet $\hat{K}$ has property λ_p for no $p > 2$.

We finally define the principal object of study in this paper.

2.8. Definition. For $f \in L^2(K)$ we consider *random Fourier series* as follows:

$$X(x, \omega) = \sum_{n=1}^{\infty} \hat{f}(\psi_n)\hat{m}(\psi_n)\psi_n(x)\xi_n(\omega), \qquad (x, \omega) \in K \times \Omega.$$

Here (ξ_n) is a sequence of symmetric independent random variables on a probability space (Ω, P) where $\mathrm{Var}(\xi_n) = 1$ and $E(\xi_n) = 0$ (automatically) for all n. We also assume that (ξ_n) is *uniformly subgaussian*: there exists $\theta > 0$ such that

$$(1) \qquad E(e^{\lambda\xi_n}) \leqslant \exp\left[\frac{\theta\lambda^2}{2}\right] \quad \text{for all } \lambda \in \mathbf{R} \text{ and all } n.$$

Note. If the sequence (ξ_n) is identically distributed, then the adjective "uniformly" is of course superfluous. A gaussian sequence is subgaussian and so is a Rademacher sequence. In fact, any uniformly bounded sequence (ξ_n), with $E\xi_n = 0$ for all n, is uniformly subgaussian. This follows from Lemma 5.3, page 111 in [Jain-Marcus (1978)]; θ can be taken as $4 \cdot \sup_n \|\xi_n\|_\infty^2$.

According to Example 5.5, page 113 in [Jain-Marcus (1978)], our process has subgaussian increments provided $\sum_{n=1}^{\infty} |\hat{f}(\psi_n)\hat{m}(\psi_n)|^2 < \infty$. In Section 3 we will impose this condition on $\hat{f}$ and this forces *all* X_x to be in $L^2(\Omega)$; see 3.7. In any case, for $f \in L^2(K)$ we have $X_x \in L^2(\Omega, P)$ for m-almost all $x \in K$ and $X_\omega \in L^2(K, m)$ for almost all $\omega \in \Omega$.

The following useful result, familiar for Rademacher sequences, holds for our uniformly subgaussian sequence (ξ_n).

2.9. Lemma (*Khintchine's inequality*). *There is a constant* $\kappa > 0$, *depending only on* θ, *such that*

$$(1) \qquad \left\| \sum_{n=1}^{\infty} a_n \xi_n \right\|_p \leqslant \kappa\sqrt{p} \left\| \sum_{n=1}^{\infty} a_n \xi_n \right\|_2$$

for $p > 2$ *and* $(a_n) \in l^2$.

Proof. Let $\xi = \sum_{n=1}^{\infty} a_n \xi_n$, a series that converges a.s. We may suppose that all a_n are real and that $\|\xi\|_2^2 = \sum_{n=1}^{\infty} a_n^2 = 1$. Using the independence of the ξ_n, one shows easily that ξ also satisfies the subgaussian inequality 2.8(1). Inequality (1) now follows from an observation of Kahane (Proposition 9 in [Kahane (1960)] or Exercise 10, Chapter VI, in [Kahane (1968)]).

2.10. REMARKS. (a) Using Hölder's inequality as in [López-Ross (1975), page 55], and 2.9 for $p = 4$, we obtain

$$\left\| \sum_{n=1}^{\infty} a_n \xi_n \right\|_2 \leqslant (2\kappa)^2 \left\| \sum_{n=1}^{\infty} a_n \xi_n \right\|_1 . \tag{1}$$

In particular, we have $\|\xi_n\|_1 \geqslant (2\kappa)^{-2}$ for all n, and so

$$\inf_n \|\xi_n\|_1 > 0. \tag{2}$$

(b) Khintchine's inequality and a slight generalization of (1) together tell us that the set $\{\xi_n\}$ of functions on Ω has property λ_p for all $p > 1$.

In Section 4 we will need a sharpened version of (1) which we base on the following general lemma.

2.11. LEMMA. *Let $\mathfrak{F}$ be a set of functions on a probability space (Ω, P). Suppose that $\mathfrak{F}$ has property λ_2. Then there exist $\varepsilon > 0$ and $c > 0$, depending only on $B = \sup\{\|f\|_2/\|f\|_1 : f \in \mathfrak{F}\}$, such that*

$$P(\Lambda) > 1 - \varepsilon \quad \textit{and} \quad f \in \mathfrak{F} \quad \textit{imply} \quad \|f\|_2 \leqslant c \int_\Lambda |f| dP. \tag{1}$$

PROOF. Select $\varepsilon > 0$ so that $B\sqrt{\varepsilon} < \frac{1}{2}$ and let $c = 2B$. For $P(\Lambda) > 1 - \varepsilon$ and $f \in \mathfrak{F}$, we have

$$\begin{aligned} \int_{\Omega \setminus \Lambda} |f| dP &\leqslant \left(\int_{\Omega \setminus \Lambda} |f|^2 \, dP \right)^{1/2} P(\Omega \setminus \Lambda)^{1/2} \\ &\leqslant \|f\|_2 \sqrt{\varepsilon} \leqslant B\sqrt{\varepsilon}\, \|f\|_1 \leqslant \tfrac{1}{2} \|f\|_1, \end{aligned}$$

hence $\int_\Lambda |f| \, dP \geqslant \frac{1}{2}\|f\|_1$ so that

$$\|f\|_2 \leqslant B\|f\|_1 \leqslant 2B \int_\Lambda |f| dP.$$

2.12. PROPOSITION. *Let (ξ_n) be as in Definition 2.8. There exist $\varepsilon > 0$ and $c > 0$, depending only on θ, such that*

$$P(\Lambda) > 1 - \varepsilon \quad \textit{implies} \quad \left\| \sum_{n=1}^{\infty} a_n \xi_n \right\|_2 \leqslant c \int_\Lambda \left| \sum_{n=1}^{\infty} a_n \xi_n \right| dP$$

for $(a_n) \in l^2$.

PROOF. Apply 2.10(1) and Lemma 2.11 with $B = (2\kappa)^2$.

2.13. REMARK. The same technique shows that in certain discussions of the notation of strict 2-associatedness [Bonami (1970), pages 294–296; López-Ross (1975), 9.3–9.5] the hypothesis that a certain set E be a Λ_q set for some $q > 2$ can be weakened to E being a Λ_2 set.

3. Almost surely continuous functions

We continue to write $\hat{K} = \{\psi_1, \psi_2, \ldots\}$, but we now stress that $\psi_1, \psi_2, \ldots$ endows $\hat{K}$ with an ordering. The next definition, of $U(K)$, depends on the particular ordering of $\hat{K}$.

3.1. DEFINITION. *$U(K)$ consists of all f in $L^1(K)$ such that $(S_N f)_{N=1}^\infty$ converges uniformly where $S_N f = \Sigma_{n=1}^N \hat{f}(\psi_n)\hat{m}(\psi_n)\psi_n$.*

Our first proposition is completely elementary.

3.2. PROPOSITION. *$A(K) \subseteq U(K) \subseteq C(K)$.*

3.3. PROPOSITION. *If $f \in U(K)$, then $\Sigma_{n=1}^\infty \hat{f}(\psi_n)\hat{m}(\psi_n)$ converges.*

PROOF. The series must converge at the identity e and $\psi_n(e) = 1$ for all n.

3.4. REMARKS. (a) The converse to Proposition 3.3 can hold for a particular K and a particular ordering of $\hat{K}$; see 6.2.

(b) Consider a function f such that $f \in U(K)$ for all orderings of $\hat{K}$. Then $\Sigma_{n=1}^\infty \hat{f}(\psi_n)\hat{m}(\psi_n)$ converges for all orderings and so the series converges absolutely. Therefore we must have $f \in A(K)$.

(c) As noted in 6.2, the equality $U(K) = C(K)$ is possible. On the other hand, the equality $U(K) = C(K)$ cannot persist for all orderings of $\hat{K}$, for this would imply $A(K) = C(K)$ by part (b), whereas $A(K) = C(K)$ if and only if K is finite by 2.11 in [Vrem (1978)]; see also 5.4 herein.

(d) It will be shown in 5.4(d) that if $\hat{K}$ has property λ_2, then $U(K) \neq C(K)$ no matter how $\hat{K}$ is ordered.

We next define the space $C_{as}(K)$ of almost surely continuous functions.

3.5. DEFINITION. $C_{as}(K)$ denotes the space of all f in $L^2(K)$ such that

$$X_\omega \sim \sum_{n=1}^{\infty} \hat{f}(\psi_n)\hat{m}(\psi_n)\xi_n(\omega)\psi_n \tag{1}$$

represents a continuous function on K for almost all $\omega \in \Omega$. Similarly, $U_{as}(K)$ is all f in $L^2(K)$ such that the series (1) converges uniformly on K for almost all $\omega \in \Omega$.

The definition of $U_{as}(K)$ appears to depend on the ordering of $\hat{K}$, but the next theorem shows that this appearance is illusory.

3.6. THEOREM. *For any ordering of* $\hat{K}$, *we have* $U_{as}(K) = C_{as}(K)$.

PROOF. We will apply Theorem 1, Chapter II of [Kahane (1968)]. Consider $f \in C_{as}(K)$ and let $X_m(\omega) = \hat{f}(\psi_m)\hat{m}(\psi_m)\xi_m(\omega)\psi_m$; each X_m is a symmetric $C(K)$-valued random variable. Let (h_n) be an approximate unit as in Lemma 2.3, and let $a_{nm} = \hat{h}_n(\psi_m)$. Then $S = (a_{nm})$ is a summation matrix as defined by Kahane. Since $C(K) = L^1(K) * C(K)$ by the Cohen factorization theorem [Hewitt-Ross (1970), 32.22], we have

$$\lim_{n \to \infty} \|g * h_n - g\|_\infty = 0 \quad \text{for all } g \in C(K).$$

In particular, for almost all $\omega \in \Omega$ we have

$$\lim_{n \to \infty} \sum_{m=1}^{\infty} a_{nm} X_m(\omega) = \lim_{n \to \infty} h_n * X_\omega = X_\omega$$

in the Banach space $C(K)$. Thus $\sum_{m=1}^{\infty} X_m$ is a.s. S-summable in $C(K)$, and by Kahane's Theorem 1 this series converges in $C(K)$ a.s. This shows that f must be in $U_{as}(K)$.

3.7. PROPOSITION. *For* $f \in L^2(K)$ *we have* (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) *where*
(i) $f \in C_{as}(K)$;
(ii) $\sum_{n=1}^{\infty} |\hat{f}(\psi_n)|^2 \hat{m}(\psi_n)^2 < \infty$;
(iii) $x \to X_x$ *maps* K *continuously into* $L^2(\Omega)$.

PROOF. By Proposition 3.3, the series $\sum_{n=1}^{\infty} \hat{f}(\psi_n)\hat{m}(\psi_n)\xi_n(\omega)$ converges a.s. Hence (ii) holds by a standard theorem about real-valued random variables; see for example Theorem 13.7.1 in [Kawata (1972)]. Note that $\liminf_n E|\xi_n| > 0$ in view of 2.10(2).

If (ii) holds, then each X_x is in $L^2(\Omega)$ since

$$\|X_x\|_2^2 = \sum_{n=1}^{\infty} \left|\hat{f}(\psi_n)\hat{m}(\psi_n)\psi_n(x)\right|^2$$

and $|\psi_n(x)| \leq 1$ for all n. A routine argument, using the continuity of each ψ_n, shows that $\|X_x - X_y\|_2$ is small provided y is sufficiently close to x.

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3.8. COROLLARY. *$C_{as}(K)$ is a Banach space with the norm*

$$\|f\|_{as} = \int_\Omega \left\| \sum_{n=1}^{\infty} \hat{f}(\psi_n)\hat{m}(\psi_n)\xi_n(\omega)\psi_n \right\|_\infty dP(\omega).$$

PROOF. Consider $f \in C_{as}(K)$ and let (X_m) be as in the proof of Theorem 3.6; we need to show that $S = \sum_{m=1}^{\infty} X_m$ is in $L^1(\Omega, C(K))$. By Corollary 3.3 in [Hoffman-Jørgensen (1974)], it suffices to show that $N \in L^1(\Omega)$ where $N(\omega) = \sup_m \|X_m(\omega)\|_\infty$ for $\omega \in \Omega$. Observe that

$$N(\omega)^2 \leqslant \sum_{m=1}^{\infty} \|X_m(\omega)\|_\infty^2 = \sum_{m=1}^{\infty} |\hat{f}(\psi_m)\hat{m}(\psi_m)|^2 \cdot |\xi_m(\omega)|^2.$$

Adding variances, and applying Proposition 3.7, we find

$$E\left[\sum_{m=1}^{\infty} |\hat{f}(\psi_m)\hat{m}(\psi_m)|^2 \cdot |\xi_m(\omega)|^2\right] = \sum_{m=1}^{\infty} |\hat{f}(\psi_m)\hat{m}(\psi_m)|^2 < \infty.$$

It follows that N belongs to $L^2(\Omega)$ and hence to $L^1(\Omega)$.

3.9. REMARKS. (a) The implication (i) $\Rightarrow$ (ii) in Proposition 3.7 can also be proved using Theorem 7, Chapter III in [Kahane (1968)]. This implication does not say anything if $\sup_n \hat{m}(\psi_n) < \infty$, since in this case (ii) holds for all $f \in L^2(K)$.

(b) There are hypergroups K in which (i) and (ii) of Proposition 3.7 are equivalent; see 6.3.

(c) If $\sup_n \hat{m}(\psi_n) = \infty$, then $C_{as}(K) \neq L^2(K)$. To see this, select $f \in L^2(K)$ that violates 3.7(ii).

For emphasis, we remind the reader that K is an infinite, compact, metrizable, abelian hypergroup with hypergroup dual $\hat{K}$.

3.10. THEOREM. (*Dudley's theorem for hypergroups.*) *Consider $f \in L^2(K)$ satisfying*

$$(1) \qquad \sum_{n=1}^{\infty} |\hat{f}(\psi_n)|^2 \hat{m}(\psi_n)^2 < \infty.$$

Let $X(x, \omega)$ be as in 2.8 and define the pseudometric

$$d(x, y) = \|X_x - X_y\|_2 = \left[\sum_{n=1}^{\infty} |\hat{f}(\psi_n)\hat{m}(\psi_n)[\psi_n(x) - \psi_n(y)]|^2\right]^{1/2}$$

on K. Let $N(\varepsilon)$ be the least number of d-balls of radius $\leqslant \varepsilon$ (with centers in K) that cover K. If

$$\int_0^1 \sqrt{\log N(\varepsilon)}\, d\varepsilon < \infty, \tag{2}$$

then $f \in C_{as}(K)$.

Note. Condition (1) is a reasonable hypothesis in view of Proposition 3.7.

PROOF. As noted after Definition 2.8, the process $(X_x)_{x \in K}$ has subgaussian increments. By Proposition 3.7, the pseudometric d is continuous on K and $x \to X_x$ maps K continuously into $L^2(\Omega)$. By Dudley's theorem ([Dudley (1967)], [Dudley (1973)], or Theorem 5.2, page 165 in [Jain-Marcus (1978)]), there is a process $(Y_x)_{x \in K}$ equivalent to $(X_x)_{x \in K}$ so that $(Y_x)_{x \in K}$ has d-continuous sample paths. Such paths are continuous on K and so

(3) each Y_ω is continuous in K;

(4) $P\{X_x \neq Y_x\} = 0$ for all $x \in K$.

It suffices to observe that a.s.

$$Y_\omega \quad \text{has Fourier series} \quad \sum_{n=1}^{\infty} \hat{f}(\psi_n)\hat{m}(\psi_n)\xi_n(\omega)\psi_n.$$

By the uniqueness of Fourier transforms, it suffices to show that a.s.

$$\hat{Y}_\omega(\psi_n) = \hat{f}(\psi_n)\xi_n(\omega) \quad \text{for all } n.$$

Thus it suffices to fix n and verify

$$\hat{Y}_\omega(\psi_n) = \hat{f}(\psi_n)\xi_n(\omega) \quad \text{a.s.} \tag{5}$$

This can be shown by considering the $L^2(\Omega)$-valued integral $Z = \int_K Y_x \overline{\psi_n(x)}\, dm(x)$ and using (4) to show that $Z = \hat{f}(\psi_n)\xi_n$ as elements of $L^2(\Omega)$. Since $Z(\omega) = \hat{Y}_\omega(\psi_n)$ for all ω, (5) holds.

3.11. REMARKS. (a) It is known [Marcus-Pisier (1980), Theorem 2.32] that if K is a compact abelian group, then the entropy condition (2) is also necessary for almost sure continuity of f. We will see in Section 6 that on the countable hypergroups considered there condition (2) is again necessary for f to be almost surely continuous, but we do not know whether this condition is necessary in general.

(b) We give another sufficient condition for almost sure continuity at the end of Section 4.

3.12. DISCUSSION. Suppose that the hypergroup K can be identified with an interval $[0, b]$. Let $\beta(u) = u$ when $0 \leqslant u \leqslant 1$, and $\beta(u) = 1$ when $u \geqslant 1$. Suppose that there are constants $C_n \geqslant 1$ so that

$$(1) \qquad |\psi_n(x) - \psi_n(y)| \leqslant C_n \beta(n|x-y|) \quad \text{for all } n \in \mathbf{N}, \quad x, y \in K.$$

Arguing as in Section IV.3 and applying Lemma IV.5.3 of [Jain-Marcus (1978)], we can show that if

$$(2) \qquad \sum_{n=2}^{\infty} \frac{\left(\sum_{m=n}^{\infty} [\hat{f}(\psi_m)\hat{m}(\psi_m)C_m]^2\right)^{1/2}}{n(\log n)^{1/2}} < \infty$$

then condition 3.10(2) holds, and $f \in C_{as}$. When K is the circle group, condition (1) holds with $C_n = 2$, and condition (2) is the Salem-Zygmund sufficient condition for almost sure continuity. When K is the set of conjugacy classes of $SU(2)$, condition (1) holds with $C_n = C \log n$, and condition (2) is then only a little more restrictive than condition 3.7(ii), which is necessary for almost sure continuity; see 7.6 for more details.

4. Series in $L^p(K)$ and $M(K)$

We begin with a simple "sure" result that shows, among other things, that the requirement $X_\omega \in C(K)$ for *all* ω is a much stronger condition than the requirement $X_\omega \in C(K)$ for almost all ω. A proof can be based on 1.5 in [Edwards (1965)] if desired.

4.1. PROPOSITION. *Suppose that*

$$X_\omega \sim \sum_{n=1}^{\infty} \hat{f}(\psi_n)\hat{m}(\psi_n)\varepsilon_n(\omega)\psi_n$$

represents a function in $L^\infty(K)$ *for all* $\omega \in \Omega = \{-1, 1\}^{\aleph_0}$ *where* ε_n *denotes the nth projection. Then we have* $f \in A(K)$.

We next present a very useful lemma.

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4.2. LEMMA. *Let (X,μ), (Y,ν) be σ-finite measure spaces and let F be $\mu\times\nu$-measurable on $X\times Y$. If $0<q\leqslant p\leqslant\infty$, then*

$$\left\{\int_Y\left(\left[\int_X|F(x,y)|^q\,d\mu(x)\right]^{1/q}\right)^p d\nu(y)\right\}^{1/p}$$

(1)

$$\leqslant\left\{\int_X\left(\left[\int_Y|F(x,y)|^p\,d\nu(y)\right]^{1/p}\right)^q d\mu(x)\right\}^{1/q}.$$

PROOF. This lemma is well known, but we sketch a proof for the readers' convenience. One may assume $F\geqslant 0$ and $p<\infty$. First assume $q=1$ and let $\phi(y)=\int_X F(x,y)\,d\mu(x)$. Then show that

$$\left|\int_Y\phi\psi\,d\nu\right|\leqslant\int_X\left[\int_Y F(x,y)^p\,d\nu(y)\right]^{1/p}d\mu(x)\cdot\|\psi\|_{p'}$$

for $\psi\in L^{p'}(Y,\nu)$. For arbitrary q, apply (1) with F, p and q replaced by F^q, p/q and 1, respectively.

The next lemma is a tiny generalization of Lemma 4 in [Dooley (1980)].

4.3. LEMMA. *If $1\leqslant p<\infty$ and*

$$\int_K\left[\sum_{n=1}^{\infty}|\hat{f}(\psi_n)\hat{m}(\psi_n)\psi_n(x)|^2\right]^{p/2}dm(x)<\infty,$$

then X_ω is a.s. in $L^p(K)$ and

$$\int_\Omega\|X_\omega\|_p^p\,dP(\omega)\leqslant\kappa^p p^{p/2}\int_K\left[\sum_{n=1}^{\infty}|\hat{f}(\psi_n)\hat{m}(\psi_n)\psi_n(x)|^2\right]^{p/2}dm(x).$$

PROOF. By Fubini's theorem and Khintchine's inequality 2.9,

$$\int_\Omega\|X_\omega\|_p^p\,dP(\omega)=\int_K\|X_x\|_p^p\,dm(x)\leqslant\left(\kappa\sqrt{p}\right)^p\int_K\|X_x\|_2^p\,dm(x)$$

$$=\left(\kappa\sqrt{p}\right)^p\int_K\left[\sum_{n=1}^{\infty}|\hat{f}(\psi_n)\hat{m}(\psi_n)\psi_n(x)|^2\right]^{p/2}dm(x).$$

Since $|\psi_n|\leqslant 1$ for all n, the next theorem is obvious from Lemma 4.3.

4.4. THEOREM. *If $\sum_{n=1}^{\infty}|\hat{f}(\psi_n)|^2\hat{m}(\psi_n)^2<\infty$, then X_ω is a.s. in all $L^p(K)$ for $1\leqslant p<\infty$.*

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4.5. THEOREM. *Let* $2 < p < \infty$, $0 < \varepsilon \leqslant 1$, *and* $q = p/\varepsilon$. *Suppose that* $\hat{K}$ *has property* λ_p. *If*

$$\sum_{n=1}^{\infty} |\hat{f}(\psi_n)|^2 \hat{m}(\psi_n)^{2-\varepsilon} < \infty, \tag{1}$$

then X_ω *is a.s. in* $L^q(K)$.

PROOF. By Lemma 4.3 it suffices to show

$$A = \int_K \left[\sum_{n=1}^{\infty} |\hat{f}(\psi_n)\hat{m}(\psi_n)\psi_n(x)|^2 \right]^{q/2} dm(x) < \infty.$$

By hypothesis, there is a constant B such that $\|\psi_n\|_p \leqslant B\|\psi_n\|_2$ for all n. By Hölder's inequality we have

$$\|\psi_n\|_q \leqslant \|\psi_n\|_p^{\varepsilon} \|\psi_n\|_\infty^{1-\varepsilon} = \|\psi_n\|_p^{\varepsilon} \leqslant B^{\varepsilon}\|\psi_n\|_2^{\varepsilon}.$$

Now by Lemma 4.2, we have

$$\begin{aligned} A^{1/q} &\leqslant \left\{ \sum_{n=1}^{\infty} \left(\left[\int_K |\hat{f}(\psi_n)\hat{m}(\psi_n)\psi_n(x)|^q \, dm(x) \right]^{1/q} \right)^2 \right\}^{1/2} \\ &= \left\{ \sum_{n=1}^{\infty} |\hat{f}(\psi_n)\hat{m}(\psi_n)|^2 \|\psi_n\|_q^2 \right\}^{1/2} \\ &\leqslant B^{\varepsilon} \left\{ \sum_{n=1}^{\infty} |\hat{f}(\psi_n)|^2 \hat{m}(\psi_n)^2 \|\psi_n\|_2^{2\varepsilon} \right\}^{1/2} \\ &= B^{\varepsilon} \left\{ \sum_{n=1}^{\infty} |\hat{f}(\psi_n)|^2 \hat{m}(\psi_n)^{2-\varepsilon} \right\}^{1/2} < \infty. \end{aligned}$$

An early version of the next theorem was kindly shown to us by Giancarlo Travaglini.

4.6. THEOREM. *Consider* $2 < p < \infty$.

(a) *If* $\hat{K}$ *has property* λ_p *and* $f \in L^2(K)$, *then* X_ω *is a.s. in* $L^p(K)$.

(b) *If* $\hat{K}$ *does not have property* λ_p, *there exists* f *in* $L^2(K)$ *such that* $X_\omega \notin L^p(K)$ *for almost all* ω.

PROOF. (a) This is just Theorem 4.5 with $\varepsilon = 1$.

(b) Let $h_0 = 0$ and given trigonometric polynomials $h_0, h_1, \ldots, h_{k-1}$ and characters $\psi_1, \ldots, \psi_{n_{k-1}}$, we apply 2.9 in [Chilana-Ross (1978)] to select a trigonometric

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polynomial h_k such that

$$\hat{h}_k = 1 \quad \text{on } \{\psi_1, \ldots, \psi_{n_{k-1}}\}$$

and $\|h_k\|_1 < 2$. We select $\psi_{n_k} \notin \bigcup_{j=1}^{k-1} \operatorname{supp}(\hat{h}_j)$. Since $\hat{K}$ does not have property λ_p, we can also arrange for

$$\|\psi_{n_k}\|_p^p > k^{p+1} \|\psi_{n_k}\|_2^p.$$

Finally, choose $f \in L^2(K)$ so that $\hat{f}(\psi_{n_k})^2 \hat{m}(\psi_{n_k}) = 1/k^2$ and $\hat{f} = 0$ off $\{\psi_{n_1}, \psi_{n_2}, \ldots\}$. We complete the proof of (b) by showing

(1) $$X_\omega \in L^p(X) \quad \text{implies} \quad \sup_k |\xi_{n_k}(\omega)|^p k < \infty$$

and

(2) $$\sup_k |\xi_{n_k}(\omega)|^p k = \infty \quad \text{a.s.}$$

To check (1), note that

$$\left\|\xi_{n_k}(\omega) \hat{f}(\psi_{n_k}) \hat{m}(\psi_{n_k}) \psi_{n_k}\right\|_p^p = \|(h_k - h_{k-1}) * X_\omega\|_p^p \leqslant 4^p \|X_\omega\|_p^p$$

for all k, while

$$\begin{aligned}\left\|\xi_{n_k}(\omega) \hat{f}(\psi_{n_k}) \hat{m}(\psi_{n_k}) \psi_{n_k}\right\|_p^p &= |\xi_{n_k}(\omega)|^p \left[|\hat{f}(\psi_{n_k})|^2 \hat{m}(\psi_{n_k}) \right]^{p/2} \hat{m}(\psi_{n_k})^{p/2} \|\psi_{n_k}\|_p^p \\ &= |\xi_{n_k}(\omega)|^p [k^{-2}]^{p/2} \left[\|\psi_{n_k}\|_2^{-2} \right]^{p/2} \|\psi_{n_k}\|_p^p \\ &> |\xi_{n_k}(\omega)|^p k^{-p} \cdot k^{p+1} = |\xi_{n_k}(\omega)|^p \cdot k.\end{aligned}$$

If (ξ_n) is a Rademacher sequence, for example, we are done since (1) shows that $X_\omega \notin L^p(K)$ for all ω. For the general uniformly subgaussian case, we need to apply Proposition 2.12. So let ε and c be as in 2.12 and let $\Lambda_k = \{\omega \in \Omega: |\xi_{n_k}(\omega)| < 1/c\}$. Then $P(\Lambda_k) \leqslant 1 - \varepsilon$ for all k since otherwise 2.12 leads to

$$1 = \|\xi_{n_k}\|_2 < c \int_{\Lambda_k} |\xi_{n_k}| \, dP < c \int_{\Lambda_k} \frac{1}{c} \, dP \leqslant 1.$$

To verify (2) we show that

(3) $$P\left\{ |\xi_{n_k}|^p k \leqslant B \text{ for all } k \right\} = 0$$

for each constant $B > 0$. In fact, by independence

$$P\left\{ |\xi_{n_k}|^p k \leqslant B \text{ for all } k \right\} = \prod_{k=1}^{\infty} P\left\{ |\xi_{n_k}| \leqslant \left(\frac{B}{k} \right)^{1/p} \right\}.$$

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For large $k, (B/k)^{1/p} < 1/c$ and the corresponding factors in the infinite product are bounded by $1 - \varepsilon$. This proves (3), hence (2).

4.7. REMARKS. (a) Proposition 3.7 and Theorem 4.4 combine to give the comforting implication:

$$\text{if } X_\omega \in C(K) \text{ a.s, then } X_\omega \in L^p(K) \text{ for } p < \infty \text{ a.s.}$$

(b) In general, the converse to Theorem 4.4 fails. Indeed, if $\hat{K}$ has property λ_p for some $p > 2$ but not property λ_∞, then there exists $f \in L^2(K)$ such that $\sum_{n=1}^\infty |\hat{f}(\psi_n)|^2 \hat{m}(\psi_n)^2 = \infty$ and yet $X_\omega \in L^p(K)$ a.s. for all $p < \infty$. To see this, note that $\sup_n \hat{m}(\psi_n) = \infty$, and so we can arrange for 4.5(1) to hold for all $\varepsilon > 0$ and yet $\sum_{n=1}^\infty |\hat{f}(\psi_n)|^2 \hat{m}(\psi_n)^2 = \infty$.

(c) Let K be the set of conjugacy classes of a compact, connected, simply connected, non-abelian Lie group G; let $\varepsilon_G = 2 \operatorname{rank} G/(\dim G - \operatorname{rank} G)$. Then $\hat{K}$ has property λ_p if and only if $p < 2 + \varepsilon_G$; see [Giulini-Soardi-Travaglini (1981)]. This yields a large class of hypergroups to which part (b) above applies.

(d) For the same class of hypergroups, Theorem 4.6(a) is due to Dooley [Dooley (1980), Theorem 2], and 4.6(b) shows that his result cannot be improved.

(e) The proofs of Theorems 4.4, 4.5, and 4.6 easily yield estimates on expected values of certain norms. We record two such estimates for use later in this section:

$$(1) \qquad \int_\Omega \|X_\omega\|_p^p \, dP(\omega) \leq \kappa^p p^{p/2} \left[\sum_{n=1}^\infty \left| \hat{f}(\psi_n) \hat{m}(\psi_n) \right|^2 \right]^{p/2}, \quad \text{in general;}$$

$$(2) \qquad \int_\Omega \|X_\omega\|_p^p \, dP(\omega) \leq \kappa^p p^{p/2} \|f\|_2^p \quad \text{if } \hat{K} \text{ has property } \lambda_p, \quad p > 2.$$

We next study series that are a.s. in $M(K)$.

4.8. THEOREM. *If $\sum_{n=1}^\infty \xi_n(\omega) a_n \hat{m}(\psi_n) \psi_n$ is a.s. a Fourier-Stieltjes series, then*

$$(1) \qquad \int_K \left[\sum_{n=1}^\infty \left| a_n \hat{m}(\psi_n) \psi_n(x) \right|^2 \right]^{1/2} dm(x) < \infty.$$

Moreover,

$$(2) \qquad \left(\sum_{n=1}^\infty |a_n|^2 \hat{m}(\psi_n)^2 \|\psi_n\|_1^2 \right)^{1/2} < \infty.$$

PROOF. For almost all $\omega \in \Omega$, $\sum_{n=1}^\infty \xi_n(\omega) a_n \hat{m}(\psi_n) \psi_n$ is the Fourier-Stieltjes series for a measure μ_ω in $M(K)$. The function $\omega \to \|\mu_\omega\|$ is measurable on Ω, since there exists a countable family $\mathcal{C}$ of trigonometric polynomials f satisfying

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$\|f\|_\infty \leqslant 1$ such that

$$\|\mu_\omega\| = \sup\left\{\left|\int_K \bar{f}\, d\mu_\omega\right| : f \in \mathcal{C}\right\}.$$

Let $\varepsilon > 0$ and $c > 0$ be as in Proposition 2.12. Since $\bigcup_{m=1}^{\infty}\{\omega \in \Omega: \|\mu_\omega\| \leqslant m\}$ has probability 1, there exists a measurable set Λ in Ω and a constant B so that

$$P(\Lambda) > 1 - \varepsilon \text{ and } \|\mu_\omega\| \leqslant B \quad \text{for all } \omega \in \Lambda. \tag{3}$$

Let (h_n) be an approximate unit for $L^1(K)$ as in Lemma 2.3. For each $\omega \in \Lambda$, (3) shows that $\|h_n * \mu_\omega\|_1 \leqslant B$ and so $\int_\Lambda \int_K |h_n * \mu_\omega|\, dm\, dP(\omega) \leqslant BP(\Lambda)$. Since $h_n * \mu_\omega(x)$ is measurable on $K \times \Omega$, we apply Fubini's theorem and obtain

$$\int_K \int_\Lambda |h_n * \mu_\omega(x)|\, dP(\omega)\, dm(x) \leqslant BP(\Lambda). \tag{4}$$

By Proposition 2.12, for each $x \in K$ and n we have

$$\begin{aligned}\left[\sum_{k=1}^{\infty} |\hat{h}_n(\psi_k) a_k \hat{m}(\psi_k)\psi_k(x)|^2\right]^{1/2} &\leqslant c\int_\Lambda \left|\sum_{k=1}^{\infty} \hat{h}_n(\psi_k) a_k \hat{m}(\psi_k)\psi_k(x)\xi_k(\omega)\right| dP(\omega)\\ &= c\int_\Lambda |h_n * \mu_\omega(x)|\, dP(\omega).\end{aligned}$$

Now integrate over K and use (4):

$$\int_K \left[\sum_{k=1}^{\infty} |\hat{h}_n(\psi_k) a_k \hat{m}(\psi_k)\psi_k(x)|^2\right]^{1/2} dm(x) \leqslant cBP(\Lambda).$$

Now let $n \to \infty$ to obtain (1).

To prove (2) we apply Lemma 4.2 and (1) as follows:

$$\begin{aligned}\left\{\sum_{n=1}^{\infty}\left[\int_K |a_n \hat{m}(\psi_n)\psi_n(x)|\, dm(x)\right]^2\right\}^{1/2} &\leqslant \int_K \left[\sum_{n=1}^{\infty} |a_n \hat{m}(\psi_n)\psi_n(x)|^2\right]^{1/2} dm(x) < \infty.\end{aligned}$$

4.9. Remarks. (a) If $\hat{K}$ has property λ_2, then conclusion (2) in Theorem 4.8 implies

$$\sum_{n=1}^{\infty} |a_n|^2 \hat{m}(\psi_n) < \infty \tag{1}$$

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so that there exists $f \in L^2(K)$ with $\hat{f}(\psi_n) = a_n$ for all n. Then $X_\omega \in L^2(K)$ a.s., and a fortiori $X_\omega \in M(K)$ a.s. To obtain (1) from 4.8(2), note that $\hat{m}(\psi_n)\|\psi_n\|_1^2 = (\|\psi_n\|_1/\|\psi_n\|_2)^2$ is bounded away from 0.

(b) Part (a) applies to the hypergroup of conjugacy classes of a compact Lie group G by Theorem 6.2 in Dooley [1979], the connected case of which is due to Price [1975]. In particular, this provides a generalization of Theorem 1 in [Dooley (1980)]. Dooley's Theorem 1 is given for connected G; even for this case our proof is different.

(c) For the hypergroups in 6.1, condition 4.8(2) is equivalent to the inequality $\sum_{n=1}^{\infty} |a_n|^2 < \infty$, which does not imply (1) above.

(d) The proof of Theorem 4.8 shows that the quantities 4.8(1) and 4.8(2) are both majorized by $\kappa\int_\Omega \|\mu_\omega\|_{M(K)}\, dP(\omega)$.

In the case where K is the unit circle group T, the estimate 4.7(1) originates in a classical result of Paley and Zygmund [Paley-Zygmund (1930), Theorem III]. Recently, de Leeuw, Kahane, and Katznelson [1977] used this classical theorem to prove that every l^2-sequence on the integers can be majorized in absolute value by the Fourier coefficients of some continuous function on T. We now use estimate 4.7(2) to prove a generalization of the latter fact.

4.10. THEOREM. *Suppose that $\hat{K}$ has property λ_p for some index $p > 2$. Then for each element b of $l^2(\hat{K}, \hat{m})$ there is a function f in $C(K)$ for which $|\hat{f}(\psi)| \geqslant |b(\psi)|$ for all ψ in $\hat{K}$.*

PROOF. Following [Kizlyakov (1981)], we use an abstract version, due to S. V. Krushchev, of the original method of de Leeuw, Kahane, and Katznelson. Let θ be a nonnegative nonincreasing function on the half-line $(0, \infty)$ such that $\theta(t) \to 0$ as $t \to \infty$. Say that *θ has property* $(*)$ if there exists an increasing, positive sequence $(c_n)_{n=1}^{\infty}$ such that

$$\sum_{n=1}^{\infty} \frac{c_{n+1}}{c_n}\theta(c_n) < \infty.$$

For example, if $\theta(t) = t^a$ for some constant $a < 0$, then the sum above is finite when $c_n = 2^n$ for all n. According to [Kizlyakov (1981), Theorem 1] the conclusion of Theorem 4.10 holds provided that there is a function θ with property $(*)$ so that the following is true: For each function f on K with $\|f\|_2 = 1$, and each number $t > 0$, there is an element ω of Ω for which the function

$$X_\omega = \sum_{n=1}^{\infty} \hat{f}(\psi_n)\hat{m}(\psi_n)\varepsilon_n(\omega)\psi_n$$

can be written as a sum $g_\omega + h_\omega$, where $\|g_\omega\|_\infty \leqslant t$, and $\|h_\omega\|_2 \leqslant \theta(t)$. Here, the functions ε_n are Rademacher functions taking only the values ± 1, that is, projections on $\Omega = \{-1, 1\}^{\aleph_0}$. We note that [Kizlyakov (1981), Theorem 1] is stated for orthonormal rather than orthogonal systems; to deduce the variant given above, just pass to the orthogonal system $(\psi_n/\|\psi_n\|_2)_{n=1}^\infty$.

To verify that such a splitting of X_ω is possible for some ω, fix a function f as above, and apply inequality 4.7(2). Thus

$$\int_\Omega \|X_\omega\|_p^p \, dP(\omega) \leqslant \kappa^p p^{p/2}.$$

Choose ω so that $\|X_\omega\|_p^p \leqslant \kappa^p p^{p/2}$. Given t, let

$$g(x) = \begin{cases} X_\omega(x) & \text{if } |X_\omega(x)| \leqslant t, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$h(x) = \begin{cases} X_\omega(x) & \text{if } |X_\omega(x)| > t, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $\|g\|_\infty \leqslant t$ as required. On the other hand, since all nonzero values of h exceed t in absolute value,

$$\int_K |h(x)|^2 \, dm(x) \leqslant t^{2-p} \int_K |h(x)|^p \, dm(x);$$

hence

$$\begin{aligned} \|h\|_2 &\leqslant t^{1-p/2} \left(\|h\|_p\right)^{p/2} \\ &\leqslant t^{1-p/2} \left(\|X_\omega\|_p\right)^{p/2} \\ &\leqslant \kappa' t^{1-p/2} \\ &= \theta(t) \qquad \text{say.} \end{aligned}$$

Since this function θ has property $(*)$, the proof is complete.

4.11. Remarks. (a) Let us say that the pair $(K, \hat{K})$ has the *d.L.K.K.-property* if the conclusion of Theorem 4.10 holds for K. Observe first that if K has this property, then $\hat{K}$ must have property λ_2. Indeed, suppose that $\hat{K}$ does not have property λ_2; then there is a sequence $\{n_i\}_{i=1}^\infty$ so that

$$\sum_{i=1}^\infty \|\psi_{n_i}\|_1 / \|\psi_{n_i}\|_2 < \infty.$$

Since $|\hat{f}(\psi)| \leqslant \|f\|_\infty \|\psi\|_1$, and since $1/\|\psi\|_2 = \hat{m}(\psi)^{1/2}$, it would follow that

$$\sum_{i=1}^\infty \left|\hat{f}(\psi_{n_i})\right| \hat{m}(\psi_{n_i})^{1/2} < \infty$$

for all f in $C(K)$. Choose b in $l^2(\hat{K}, \hat{m})$ so that

$$\sum_{i=1}^{\infty} |b(\psi_{n_i})| \hat{m}(\psi_{n_i})^{1/2} = \infty.$$

Then there is no function f in $C(K)$ such that $|\hat{f}(\psi_{n_i})| \geqslant |b(\psi_{n_i})|$ for all i.

(b) We do not know whether the condition that $\hat{K}$ have property λ_2 implies that the pair $(K, \hat{K})$ has the d.L.K.K. property, but we will show in 5.4 that a convexified version of the d.L.K.K. phenomenon does occur if $\hat{K}$ has property λ_2.

(c) The hypothesis that $\hat{K}$ have property λ_p for some $p > 2$ is satisfied by many of our examples, but not by the countable hypergroups to be discussed in Section 6. In particular, this hypothesis is satisfied if K is the space of conjugacy classes of a compact simple, simply connected Lie group.

(d) In [Kizlyakov (1981)], S. V. Kizlyakov showed that in the case of the circle group T, the conclusion of Theorem 4.10 holds with the space $C(T)$ replaced by $U(T)$. This seems to be a much deeper fact than Theorem 4.10, because Kizlyakov's proof requires a key estimate in Carleson's proof of the Lusin conjecture.

(e) Propositions 3.3 and 3.7 show that if f belongs to $U(K)$ or $C_{as}(K)$, then $\lim_{n\to\infty} \hat{f}(\psi_n)\hat{m}(\psi_n) = 0$. It follows easily that if $\sup_\psi \hat{m}(\psi) = \infty$, then the space $C(K)$ in Theorem 4.10 cannot be replaced by $U(K)$ or $C_{as}(K)$. It is also worth mentioning that, when $K = T$, well known necessary conditions on $C_{as}(T)$ [Marcus-Pisier (1980), 7.1.3] show that $C(T)$ cannot be replaced by $C_{as}(T)$ in Theorem 4.10.

Finally, we use the estimate 4.7(1) to derive another sufficient condition for almost sure continuity. Given f in $L^1(K)$, let

$$|||f|||_2 = \left(\sum_{n=1}^{\infty} |\hat{f}(\psi_n)\hat{m}(\psi_n)|^2 \right)^{1/2};$$

thus inequality 4.7(1) asserts that

$$\int_\Omega \|X_\omega\|_p^p \, dP(\omega) \leqslant \kappa^p p^{p/2} |||f|||_2^p.$$

For positive values of t, let $\Phi(t) = e^{t^2} - 1$ and $\phi(t) = t(1 + \log(1+t))^{1/2}$; denote the corresponding Orlicz spaces on K by $L_\Phi(K)$ and $L_\phi(K)$.

4.12. LEMMA. *If $f \in L_\Phi$ and $g \in L_\phi$, then $f * g \in C(K)$, and $\|f * g\|_\infty \leqslant \kappa \|f\|_\Phi \|g\|_\phi$.*

PROOF. If g is a trigonometric polynomial, then $f * g \in C(K)$, and the norm estimate follows easily from the duality between L_Φ and L_ϕ. But trigonometric polynomials are dense in L_ϕ.

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4.13. THEOREM. *Suppose that* $|||f|||_2 < \infty$ *and that* $g \in L_\phi(K)$. *Then* $f * g \in C_{as}(K)$, *and* $\|f * g\|_{as} \leqslant \kappa |||f|||_2 \|g\|_\phi$.

PROOF. As in [Marcus-Pisier (1980), Lemma 6.1.3] inequality 4.7(1) implies that

$$\int_\Omega \|X_\omega\|_\Phi \, dP(\omega) \leqslant \kappa \big|||f|||\big|_2.$$

It then follows from Lemma 4.12 that $X_\omega * g \in C(K)$ almost surely, and that

$$\int_\Omega \|X_\omega * g\|_\infty \, dP(\omega) \leqslant \kappa' \big|||f|||\big|_2 \|g\|_\phi,$$

as required.

4.14. REMARKS. (a) For $K = T$, the theorem above goes back to Paley and Zygmund [1930], page 344. The analogue for random Fourier series on compact groups is implicit in [Rider (1977)].

(b) In 7.6, we will use a consequence of Theorem 4.13: If f and g are trigonometric polynomials, with $\|g\|_1 \geqslant 1$, then

$$\text{(1)} \qquad \|f * g\|_{as} \leqslant \kappa \big|||f|||\big|_2 \|g\|_1 \big(\log\big(1 + \|g\|_2\big)\big)^{1/2}.$$

This inequality follows from the theorem, because, in this case, $\|g\|_\phi \leqslant \kappa' \|g\|_1 (\log(1 + \|g\|_2))^{1/2}$.

(c) Another consequence of Theorem 4.13 is that if $(f_n)_{n=1}^\infty$ and $(g_n)_{n=1}^\infty$ are sequences of trigonometric polynomials, with $\sum_{n=1}^\infty |||f_n|||_2 \|g_n\|_\phi < \infty$, then the series $\sum_{n=1}^\infty f_n * g_n$ is norm convergent in the space C_{as}. A theorem of Pisier [Marcus-Pisier (1980), 6.1.1] states that if K is a group then every function in C_{as} can be represented as a sum of such a series. We shall see in 6.9 that the corresponding assertion is false for some hypergroups.

5. Some multiplier results

5.1. DEFINITION. For Banach spaces B and D of functions on K, $\mathfrak{M}(B, D)$ denotes the space of all functions of *multipliers* p on $\hat{K}$ such that $f \in B$ implies $\hat{f}p = \hat{g}$ for some (unique) $g \in D$.

By the closed graph theorem, the multipliers p correspond to bounded linear operators T_p of B into D. Specifically, $T_p(f)\hat{} = \hat{f}p$ for $f \in B$. We will write the operator norm of T_p as $\|T_p\|_{B,D}$ or $\|p\|_{B,D}$.

5.2. REMARK. A function p on $\hat{K}$ belongs to $\mathfrak{M}(L^2, A)$ if and only if $\sum_{n=1}^{\infty} |p(\psi_n)|^2 \hat{m}(\psi_n) < \infty$, in which case $p(\psi_n) = \hat{f}(\psi_n)$ for all n for some $f \in L^2(K)$. Moreover,

$$\|p\|_{L^2,A}^2 = \sum_{n=1}^{\infty} |p(\psi_n)|^2 \hat{m}(\psi_n) = \|f\|_2^2.$$

5.3. PROPOSITION. (a) *We have* $\mathfrak{M}(C, A) = \mathfrak{M}(L^\infty, A)$ *and*

(1) $$\|p\|_{C,A} = \|p\|_{L^\infty,A} \quad \textit{for } p \in \mathfrak{M}(C, A).$$

(b) *For* $p \in \mathfrak{M}(C, A)$ *we have*

(2) $$\sum_{n=1}^{\infty} |p(\psi_n)|^2 \hat{m}(\psi_n)^2 \|\psi_n\|_1^2 \leqslant \kappa \|p\|_{C,A}^2.$$

(c) *If* $\hat{K}$ *has property* λ_2, *then* $\mathfrak{M}(C, A) = \mathfrak{M}(L^2, A)$.

PROOF. (a) Consider $p \in \mathfrak{M}(C, A)$ and $f \in L^\infty(K)$ and let (h_n) be as in Lemma 2.3. Each $f * h_n$ is in $C(K)$ and so

$$\sum_{m=1}^{\infty} |p(\psi_m)| \cdot \hat{h}_n(\psi_m) |\hat{f}(\psi_m)| \hat{m}(\psi_m) \leqslant \|p\|_{C,A} \|f\|_\infty$$

for all n. The proof is easily completed on letting $n \to \infty$.

(b) Consider $p \in \mathfrak{M}(C, A)$ and let (ε_n) be a Rademacher sequence. For $\omega \in \Omega$ and $f \in C(K)$, we have

$$\left| \int_K \bar{f} \sum_{n=1}^{M} \varepsilon_n(\omega) p(\psi_n) \hat{m}(\psi_n) \psi_n \, dm \right| \leqslant \sum_{n=1}^{M} |p(\psi_n)| \hat{m}(\psi_n) |\hat{f}(\psi_n)| \leqslant \|p\|_{C,A} \|f\|_\infty.$$

In other words, the measure μ_ω^M such that $d\mu_\omega^M = \sum_{n=1}^{M} \varepsilon_n(\omega) p(\psi_n) \hat{m}(\psi_n) \psi_n \, dm$ satisfies

$$\left| \int_K f \, d\mu_\omega^M \right| \leqslant \|p\|_{C,A} \|f\|_\infty \quad \text{for } f \in C(K).$$

Thus $\|\mu_\omega^M\| \leqslant \|p\|_{C,A}$ for all M. By Alaoglu's theorem, $(\mu_\omega^M)_{M=1}^{\infty}$ has a weak-$*$ cluster point in $M(K)$ whose Fourier-Stieltjes series is $\sum_{n=1}^{\infty} \varepsilon_n(\omega) p(\psi_n) \hat{m}(\psi_n) \psi_n$. Remark 4.9(d) now applies to complete the proof.

(c) If $\hat{K}$ has property λ_2, then, as noted in Remark 4.9(a), inequality (2) implies that

(3) $$\sum_{n=1}^{\infty} |p(\psi_n)|^2 \hat{m}(\psi_n) \leqslant \kappa' \|p\|_{C,A}^2.$$

To see that $\mathfrak{M}(C, A) = \mathfrak{M}(L^2, A)$ in this case, observe first that $\mathfrak{M}(L^2, A) \subseteq \mathfrak{M}(C, A)$ in any case, and then deduce from Remark 5.2 that if the left side of inequality (3) is finite, then $p \in \mathfrak{M}(L^2, A)$.

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5.4. REMARKS. (a) The analogue of inequality 5.3(3) for orthogonal systems in $L^2[0, 1]$ having property λ_2 is due to Mahmudov [1965]; our proof is similar to his.

(b) As in Remark 4.11(a), the inclusion $\mathfrak{M}(C, A) \subseteq \mathfrak{M}(L^2, A)$ implies that $\hat{K}$ has property λ_2.

(c) A standard duality argument [Caveny (1969), Theorem 3.3] shows that the inclusion $\mathfrak{M}(C, A) \subseteq \mathfrak{M}(L^2, A)$ holds if and only if the following convexified version of the de Leeuw-Kahane-Katznelson phenomenon occurs: For each function b in $l^2(\hat{K}, \hat{m})$ there is a sequence $(f_k)_{k=1}^{\infty}$ of functions in $C(K)$ so that

$$\sum_{k=1}^{\infty} \|f_k\|_\infty \leqslant \kappa \|b\|_2,$$

and

$$\sum_{k=1}^{\infty} |\hat{f}_k(\psi_n)| \geqslant |b_n| \quad \text{for all } n.$$

This suggests the conjecture that if $\hat{K}$ has property λ_2, then the pair $(K, \hat{K})$ has the d.L.K.K. property.

(d) Proposition 5.3(c) is useful in showing that if $\hat{K}$ has property λ_2, then $U(K) \neq C(K)$ no matter how $\hat{K}$ is ordered. Suppose first that $\hat{K}$ has property λ_∞; then, by Remark 2.6(d), the corresponding orthonormal system $\{\psi_n/\|\psi_n\|_2\}_{n=1}^{\infty}$ is uniformly bounded, because $\|[\psi_n/\|\psi_n\|_2]\|_\infty = 1/\|\psi_n\|_2$. Now it is known [Bočkarev (1978), 2.2] that the Littlewood conjecture holds on the average for uniformly bounded orthonormal systems, and it then follows, by the usual uniform boundedness argument, that $C(K) \neq U(K)$. Suppose next that $\hat{K}$ has property λ_2 but not property λ_∞. Again by Remark 2.6(d), $\sup_n \hat{m}(\psi_n) = \infty$. It follows that there are functions p on $\hat{K}$ so that $\sum_{n=1}^{\infty} |p(\psi_n)| < \infty$, but $\sum_{n=1}^{\infty} |p(\psi_n)|^2 \hat{m}(\psi_n) = \infty$. By Proposition 3.3, any such function p belongs to $\mathfrak{M}(U(K), A(K))$, but by Proposition 5.3(c), any such function p does not belong to $\mathfrak{M}(C(K), A(K))$. Hence $C(K) \neq U(K)$.

(e) In the same spirit, Proposition 5.3 can be used to show that the spaces $C(K)$ and $A(K)$ are distinct, even if $\hat{K}$ does not have property λ_2. Indeed, $\|\psi_n\|_2^2 \leqslant \|\psi_n\|_1$, because $\|\psi_n\|_\infty = 1$; therefore $\hat{m}(\psi_n)\|\psi_n\|_1 \geqslant 1$ for all n. In particular, inequality 5.3(2) implies $\sum_{n=1}^{\infty} |p(\psi_n)|^2 < \infty$ for all p in $\mathfrak{M}(C, A)$. On the other hand, clearly $\mathfrak{M}(A, A) = l^\infty(\hat{K})$, so that $\mathfrak{M}(C, A) \neq \mathfrak{M}(A, A)$, and $C(K) \neq A(K)$. The fact that $C(K) \neq A(K)$ was proved earlier by Vrem [1978], by a method that also applies when K is not abelian.

(f) By 5.3(b), the finiteness of the left side of inequality 5.3(2) is a necessary condition for p to be in $\mathfrak{M}(C, A)$. It is natural to ask if this condition is also sufficient for p to be in $\mathfrak{M}(C, A)$. If $\hat{K}$ has property λ_2, then the answer is "yes", because, in that case, the proof of 5.3(c) shows that if the left side of 5.3(2) is finite, then $p \in \mathfrak{M}(L^2, A) \subseteq \mathfrak{M}(C, A)$. We conjecture that when $\hat{K}$ does not have property λ_2 the answer to the question above is always "no". In 6.5, we verify this conjecture for the special countable hypergroups considered in Section 6.

(g) Finally we describe a condition on p that is necessary and sufficient for membership in $\mathfrak{M}(C, A)$ even if $\hat{K}$ does not have property λ_2. Let $p \in \mathfrak{M}(C, A)$ and let μ_ω be the measure obtained as in the proof of 5.3(b) but with the randomizing factors $\varepsilon_n(\omega)$ taken to be Steinhaus rather than Rademacher. It is easy to verify that $\sup_\omega \|\mu_\omega\| = \|p\|_{C,A}$ and, by the symmetry of the ε_n's, that $\sup_{\omega,N} \|S_N \mu_\omega\|_1 = \sup_\omega \|\mu_\omega\|$.

Introduce an isomorphic copy, K' say, of the hypergroup K. The method of [Ørno (1976)] transfers easily to this setting, and yields that there is a sequence $(c_n)_{n=1}^\infty$ with $\sum_{n=1}^\infty |c_n|^2 < \infty$, an orthonormal sequence $(\Psi_n)_{n=1}^\infty$ in $L^2(K \cup K')$, and a function F in $L^2(K \cup K')$ so that for each n the function $p(\psi_n)\hat{m}(\psi_n)\psi_n$ is the restriction to K of the product $c_n F \cdot \Psi_n$. Conversely, it is easy to verify that if $p(\psi_n)\hat{m}(\psi_n)\psi_n$ is representable as above for each n, then measures μ_ω with $\sup_{\omega,N} \|S_N \mu_\omega\|_1 < \infty$ exist, and $p \in \mathfrak{M}(C, A)$.

If p has such a representation, then it follows easily that

$$\int_K \left[\sum_{n=1}^{\infty} |p(\psi_n)\hat{m}(\psi_n)\psi_n(x)|^2 \right]^{1/2} dm(x) < \infty, \tag{2}$$

and hence that the left side of inequality 5.3(2) is finite. These seem to be the simplest conditions on the size of p that follow easily from the representation discussed above. Inequality (2) goes back to Orlicz [1933].

6. Countable hypergroups

Dunkl and Ramirez [1975] investigate an interesting class of countable hypergroups K_a, indexed by a where $0 < a \leqslant 1/2$. When $a = 1/p$ for a prime p, then K_a can be obtained from the group Δ_p of p-adic integers as follows. Let W be the multiplicative group of units in Δ_p and note that each element of W induces an automorphism of Δ_p via multiplication. Then $K_{1/p}$ is exactly the hypergroup of orbits of Δ_p under W as constructed in [Dunkl-Ramirez (1975), Section 3] or [Jewett (1975), Section 8].

6.1. The hypergroup K_a is identified with $\{0, 1, 2, \ldots, \infty\}$. The invariant measure m is given by

$$m(\{k\}) = a^k(1-a) \quad \text{for } k < \infty, \quad \text{and} \quad m(\{\infty\}) = 0.$$

We have $\hat{K}_a = \{\psi_0, \psi_1, \psi_2, \cdots\}$ where $\psi_0 \equiv 1$ and

$$\psi_n(k) = \begin{cases} 1 & \text{for } k \geqslant n, \\ a/(a-1) & \text{for } k = n-1, n \geqslant 1, \\ 0 & \text{for } k < n-1. \end{cases}$$

Direct computation shows that

$$\|\psi_n\|_p^p = a^n\left[1 + \left(\frac{a}{1-a}\right)^{p-1}\right].$$

Hence $\hat{K}_a$ does not have property λ_p for any $p > 1$.

The invariant measure $\hat{m}$ on $\hat{K}_a$ is given by $\hat{m}(\psi_0) = 1$ and $\hat{m}(\psi_n) = (1-a)a^{-n}$ for $n \geqslant 1$.

6.2. PROPOSITION. *The following are equivalent*:

(i) $f \in C(K_a)$;

(ii) $f \in U(K_a)$;

(iii) $\Sigma_{n=1}^{\infty} \hat{f}(\psi_n)\hat{m}(\psi_n)$ *converges*.

PROOF. (i) and (ii) are equivalent by Theorem 6.3 in [Dunkl-Ramirez (1975)], and (ii) $\Rightarrow$ (iii) by Proposition 3.3. If (iii) holds, then $\Sigma_{n=1}^{\infty} \hat{f}(\psi_n)a^{-n}$ converges. Given $\varepsilon > 0$ there exists N_0 such that

$$N \geqslant M > N_0 \quad \text{imply} \quad \left|\sum_{n=M}^{N} \hat{f}(\psi_n)a^{-n}\right| < \varepsilon.$$

Routine estimates then show that

$$\left|\sum_{n=M}^{N} \hat{f}(\psi_n)(1-a)a^{-n}\psi_n(k)\right| < \varepsilon$$

for $N \geqslant M > N_0$ and all $k \in K_a$. Therefore $f \in U(K_a)$.

6.3. COROLLARY. *We have $f \in C_{as}(K_a)$ if and only if*

$$\sum_{n=1}^{\infty} |\hat{f}(\psi_n)|^2 \hat{m}(\psi_n)^2 < \infty. \tag{1}$$

PROOF. The necessity of (1) is proved in Proposition 3.7. If (1) holds, then $\Sigma_{n=1}^{\infty} \hat{f}(\psi_n)\hat{m}(\psi_n)\xi_n(\omega)$ converges a.s. by Theorem 7, Chapter III in [Kahane (1968)]. Hence $f \in C_{as}(K_a)$ by Proposition 6.2.

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6.4. SUMMARY.

(a) $f \in L^2(K_a)$ *if and only if* $\sum_{n=1}^{\infty} |\hat{f}(\psi_n)|^2 a^{-n} < \infty$.

(b) $f \in A(K_a)$ *if and only if* $\sum_{n=1}^{\infty} |\hat{f}(\psi_n)| a^{-n} < \infty$.

(c) $f \in C(K_a)$ *if and only if* $\sum_{n=1}^{\infty} \hat{f}(\psi_n) a^{-n}$ *converges*.

(d) $f \in C_{as}(K_a)$ *if and only if* $\sum_{n=1}^{\infty} |\hat{f}(\psi_n)|^2 a^{-2n} < \infty$.

Note that $f \in C(K_a)$ and $\hat{f} \geqslant 0$ imply $f \in A(K_a)$; this also follows from the general Proposition 2.4.

6.5. REMARK. Since $\hat{K}_a$ does not have property λ_2, the pair $(K_a, \hat{K}_a)$ does not have the d.L.K.K. property (see Remark 4.11(a)); also $\mathfrak{M}(C, A) \neq \mathfrak{M}(L^2, A)$ (see Remark 5.4(b)). It follows easily from the next proposition that there are functions p for which the left side of inequality 5.3 (2) is finite but $p \notin \mathfrak{M}(C, A)$. Thus the conjecture made in Remark 5.4(f) holds for the hypergroups K_a.

6.6. PROPOSITION. *For* $K = K_a$ *we have*

$$(1) \qquad p \in \mathfrak{M}(C, A) \text{ if and only if } \sum_{n=1}^{\infty} |p(\psi_n)| < \infty.$$

PROOF. Let cond denote the set of sequences (b_n) such that Σb_n is (conditionally) convergent. In view of 6.4, assertion (1) is equivalent to the claim that the space of multipliers from cond to l^1 is precisely l^1. This is an elementary fact; note that given (a_n) in c_0 there exists (b_n) in cond such that $|b_n| \geqslant |a_n|$ for all n.

Our next example shows that the converse to Theorem 4.4 fails for these hypergroups.

6.7. EXAMPLE. There exists $f \in L^2(K_a)$ such that

$$(1) \qquad \sum_{n=1}^{\infty} \left|\hat{f}(\psi_n)\right|^2 \hat{m}(\psi_n)^2 = \infty,$$

$$(2) \qquad X_\omega \text{ is a.s. in all } L^p(K_a), \qquad 1 \leqslant p < \infty.$$

To accomplish (2), Lemma 4.3 says it suffices to obtain

$$\int_K \left[\sum_{n=1}^{\infty} \left|\hat{f}(\psi_n)\hat{m}(\psi_n)\psi_n(x)\right|^2 \right]^{p/2} dm(x) < \infty$$

for all $p < \infty$. Since $m(\{k\}) = a^k(1-a)$, $\hat{m}(\psi_n) = (1-a)a^{-n}$, $|\psi_n| \leqslant 1$ and $\psi_n(k) = 0$ for $n \geqslant k+2$, it suffices to arrange

$$(3) \qquad \sum_{k=1}^{\infty}\left[\sum_{n=1}^{k+1}\left|\hat{f}(\psi_n)a^{-n}\right|^2\right]^{p/2} a^k < \infty \quad \text{for } p < \infty.$$

Let $\hat{f}(\psi_n) = a^n$ for all n; then conditions (1) and (3) both hold, and $f \in L^2(K_a)$.

6.8. EXAMPLE. Here we consider random Fourier series on K_a using a Rademacher sequence. There exists $f \in L^2(K)$ so that

$$(1) \qquad X_\omega \notin L^p(K_a) \quad \text{for all } \omega \in \Omega \text{ and all } p > 2.$$

The proof of Theorem 4.6(b) accomplishes this for each p, so we will not provide details here. We merely state that f can be selected so that $\hat{f}(\psi_n) = a^{n/2}/n$ and that Proposition 7.7 in [Dunkl-Ramirez (1975)] is useful in establishing (1).

Finally, we consider, for the hypergroups K_a, the sufficient conditions for almost sure continuity that were presented at the ends of Sections 3 and 4. We first show that the entropy condition of Section 3 is also necessary in this situation.

6.9. THEOREM. *If $f \in C_{as}(K_a)$, then*

$$(1) \qquad \int_0^1 \sqrt{\log N(\varepsilon)}\, d\varepsilon < \infty.$$

PROOF. First observe that for each nonnegative integer k

$$\psi_n(k) = \begin{cases} 0 & \text{for all } n > k+1, \\ a/(a-1) & \text{for } n = k+1, \\ 1 & \text{for all } n \leqslant k. \end{cases}$$

It follows that the pseudometric d of 3.10 is given by

$$d(j,k) = \left\{\left|\hat{f}(\psi_{j+1})\hat{m}(\psi_{j+1})[a/(a-1) - 1]\right|^2 \right.$$
$$+ \sum_{j+1<n\leqslant k}\left|\hat{f}(\psi_n)\hat{m}(\psi_n)[0-1]\right|^2$$
$$\left. + \left|\hat{f}(\psi_{k+1})\hat{m}(\psi_{k+1})[0 - a/(a-1)]\right|^2\right\}^{1/2},$$

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whenever $0 \leqslant j < k < \infty$. Let d' be the pseudometric on K_a defined by letting

$$d'(j,k) = \left\{ \sum_{n=j+1}^{k+1} \left|\hat{f}(\psi_n)\hat{m}(\psi_n)\right|^2 \right\}^{1/2}$$

whenever $0 \leqslant j < k < \infty$, and

$$d'(j,\infty) = \left\{ \sum_{n=j+1}^{\infty} \left|\hat{f}(\psi_n)\hat{m}(\psi_n)\right|^2 \right\}^{1/2}.$$

It is easy to verify that

$$d' \leqslant d \leqslant 2d'.$$

As in 3.10, let $N'(\varepsilon)$ be the least number of d' balls of radius $\leqslant \varepsilon$ (with centers in K_a) that cover K_a. It follows easily that

$$(2) \qquad \int_0^1 \sqrt{\log N(\varepsilon)}\, d\varepsilon < \infty$$

if and only if

$$(2') \qquad \int_0^1 \sqrt{\log N'(\varepsilon)}\, d\varepsilon < \infty.$$

In view of this and 6.3 it suffices to show that condition 6.3(1) implies condition (2′).

To this end, suppose that f satisfies condition 6.3(1). Fix $\varepsilon > 0$. Define elements $n_1, n_2, \ldots$ of K_a, and d'-balls $B_1, B_2, \ldots$ by the following procedure. Let $n_1 = 0$ and let B_1 be the d'-ball of radius ε centered at n_1. Given $B_1, B_2, \ldots, B_l$, stop at this stage if these balls cover K_a; otherwise, let n_{l+1} be the first point of K_a not covered by $B_1 \cup B_2 \cup \cdots \cup B_l$, and let B_{l+1} be the d'-ball of radius ε centered at n_{l+1}. The idea now is to show that this process stops after most $1 + 4(|||f|||_2/\varepsilon)^2$ steps.

Call n_l *terminal* if the process stops at the lth step. It follows from the definitions above that if n_l is not terminal, and if $n_{l+1} < \infty$, then

$$(3) \qquad \sum_{n=n_l+1}^{n_{l+1}+1} \left|\hat{f}(\psi_n)\hat{m}(\psi_n)\right|^2 \geqslant \varepsilon^2.$$

Similarly, if n_l is not terminal, and if $n_{l+1} = \infty$, then

$$(4) \qquad \sum_{n=n_l+1}^{\infty} \left|\hat{f}(\psi_n)\hat{m}(\psi_n)\right|^2 \geqslant \varepsilon^2.$$

Split the set of integers n_l into two classes by declaring that $n_l \in \mathcal{G}$ if either n_l is not terminal and

$$(5) \qquad \sum_{n=n_l+1}^{n_{l+1}} \left|\hat{f}(\psi_n)\hat{m}(\psi_n)\right|^2 \geqslant \varepsilon^2/2,$$

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or if n_l is terminal and

$$\sum_{n=n_l+1}^{\infty} |\hat{f}(\psi_n)\hat{m}(\psi_n)|^2 \geqslant \varepsilon^2/2, \tag{6}$$

and declaring that $n_l \in \mathcal{P}$ otherwise. Summing the left sides of inequalities (5) and (6) over all n_l in $\mathcal{G}$ yields at most the quantity

$$\sum_{n=1}^{\infty} |\hat{f}(\psi_n)\hat{m}(\psi_n)|^2 = |||f|||_2^2;$$

therefore, the class $\mathcal{G}$ contains at most $|||f|||_2^2/(\varepsilon^2/2)$ points n_l.

Suppose next that $n_l \in \mathcal{P}$ and that n_l is not terminal. Then

$$\sum_{n=n_l+1}^{n_{l+1}} |\hat{f}(\psi_n)\hat{m}(\psi_n)|^2 < \varepsilon^2/2. \tag{7}$$

By inequality (4) it cannot be the case that $n_{l+1} = \infty$. It then follows from (3) and (7) that

$$\left|\hat{f}\left(\psi_{n_{l+1}+1}\right)\hat{m}\left(\psi_{n_{l+1}+1}\right)\right|^2 > \varepsilon^2/2,$$

and hence that $n_{l+1} \in \mathcal{G}$. Therefore the class $\mathcal{P}$ contains at most $1 + |||f|||_2^2/(\varepsilon^2/2)$ points n_l, and the process specified above does indeed stop after at most $1 + 4(|||f|||_2/\varepsilon)^2$ steps. Thus

$$N'(\varepsilon) \leqslant 1 + 4(|||f|||_2/\varepsilon)^2,$$

and inequality (2′) holds as required.

6.10. EXAMPLES. Given a real number $s > 0$, define a function f_s in $L^2(K_a)$ by setting $\hat{f}_s(\psi_n) = n^{-s}/\hat{m}(\psi_n)$ for all n.

(a) First observe that $|||f_s|||_2 < \infty$ if and only if $s > 1/2$. It follows from 6.4(d) that $f_s \in C_{as}$ if and only if $s > 1/2$. By 6.9 the entropy condition holds if and only if $s > 1/2$.

(b) Next we verify that when $s \leqslant 3/4$, the function f_s cannot be represented in the manner discussed in Remark 4.14(c). To do this, we need the fact that if $f \in L_\Phi$ and $|||g|||_2 < \infty$, then

$$\sum_{n=1}^{\infty} \frac{|(f * g)\hat{\ }(\psi_n)|}{n^{1/4}} \hat{m}(\psi_n) \leqslant \kappa \|f\|_\Phi\, |||g|||_2. \tag{3}$$

Assuming this inequality, we see that if f_s could be represented as in Remark 4.14(c), then it would follow that

$$\sum_{n=1}^{\infty} \frac{|\hat{f}_s(\psi_n)|}{n^{1/4}} \hat{m}(\psi_n) < \infty.$$

In fact, however, the sum on the left is infinite when $s \leqslant 3/4$, so that, in this case, f_s cannot be represented in the manner discussed in Remark 4.14(c).

Now inequality (3) follows by Cauchy-Schwarz from the inequality

$$\sum_{n=1}^{\infty} \frac{|\hat{f}(\psi_n)|^2}{n^{1/2}} \leqslant \kappa^2 \|f\|_\Phi^2, \tag{4}$$

and since $\|\hat{f}\|_\infty \leqslant \|f\|_1 \leqslant \|f\|_\Phi$, inequality (4) follows from the inequality

$$\sum_{n=1}^{\infty} \frac{|\hat{f}(\psi_n)|}{n^{1/2}} \leqslant \kappa^2 \|f\|_\Phi. \tag{5}$$

Inequality (5) is the analogue for K_a of an inequality proved for the unit circle group T by C. Bennett [1975], and it follows from Bennett's theorem by a transference argument. To this end, identify $(T, d\theta/2\pi)$ in a measure preserving way with the interval $[0, 1)$. Given a function g on K_a, let G be the function on $[0, 1)$ that is equal to $g(n)$ on the interval $[1 - a^n, 1 - a^{n+1})$; then $\|G\|_\Phi = \|g\|_\Phi$, because each point in K_a has the same measure as the corresponding half-open interval in $[0, 1)$. Denote the Borel field generated by the class of all such functions G by $\mathfrak{F}_a$, and let E_a be the conditional expectation operator $h \to E(h, \mathfrak{F}_a)$; regard $E_a h$ as a function on K_a. In a similar way, let the point ψ_n in $\hat{K}_a$ correspond to the interval $[a^{-n+1}, a^{-n})$ in $[0, \infty)$, and transfer functions b on $\hat{K}_a$ to functions R_b on $[0, \infty)$ by making R_b equal to $b(\psi_n)$ on the interval corresponding to ψ_n. Finally, given any function k on $[0, \infty)$ let rk be its restriction to the set Z^+ of all nonnegative integers.

The proof of Theorem 4.3 of [Bennett (1975)] shows that if a linear operator L is bounded from $L^1(T)$ to $l^\infty(Z^+)$ and from $L^2(T)$ to $l^2(Z^+)$, then there is a constant κ, depending only on the norms of L in the two endpoint cases mentioned above, so that

$$\sum_{j=2}^{\infty} \frac{(Lf)(j)}{j(\log j)^{1/2}} \leqslant \kappa \|f\|_\Phi. \tag{6}$$

We apply this to the operator L defined by the following sequence of operations:

$$L: h \to E_a h \to (E_a h)\hat{} \to rR(E_a h)\hat{}.$$

It is easy to see that L is bounded from $L^1(T)$ to $l^\infty(Z^+)$, with norm 1. Also, the operator E_a is an L^2-contraction, and the operator $E_a h \to R(E_a h)\hat{}$ is an isometry from $L^2(K_a)$ into $L^2[0, \infty)$; finally

$$\|rRb\|_{l^2(Z^+)} \leqslant \sqrt{2}\, \|Rb\|_{L^2[0,\infty)},$$

because $\hat{m}(\psi_n) \geqslant 1$ for all n. So, inequality (6) holds for this operator L.

Let $f \in L_\phi(K_a)$; then $F \in L_\phi[0,1)$, and $\|F\|_\phi = \|f\|_\phi$. Since $E_a F = f$, it follows from inequality (6) that

$$\sum_{j=2}^{\infty} \frac{(rR\hat{f})(j)}{j(\log j)^{1/2}} \leqslant \kappa' \|f\|_\phi.$$

It follows easily from the definitions of the operators R and r that this inequality holds if and only if

$$\sum_{n=1}^{\infty} \frac{|\hat{f}(\psi_n)|}{\sqrt{n}} \leqslant \kappa'' \|f\|_\phi,$$

as required.

It follows that if $1/2 < s \leqslant 3/4$, then $f_s \in C_{as}$, but f_s cannot be represented in the manner discussed in Remark 4.14(c).

(c) Finally, we show that if $s > 1$, then f_s can be represented as in Remark 4.14(c). Consider the Dirichlet kernels

$$D_n = \sum_{j=0}^{n} \hat{m}(\psi_j)\psi_j,$$

and the functions

$$H_n = \sum_{j=0}^{n} \psi_j.$$

Dunkl and Ramirez [1975] show that $\|D_n\|_1 = 1$; on the other hand, by Parseval, $\|D_n\|_2 \leqslant a^{-n/2}$. Also $|||H_n|||_2 = (n+1)^{1/2}$, because $\hat{H}_n(\psi_j) = 1/\hat{m}(\psi_j)$ if $j \leqslant n$, and $\hat{H}_n(\psi_j) = 0$ otherwise. Since $H_n = H_n * D_n$, it follows from inequality 4.14(1) that $\|H_n\|_{as} \leqslant \kappa'(n+1)$. To see that $f_s \in C_{as}$ when $s > 1$, sum the Fourier series for f_s by parts, and use the estimate above for $\|H_n\|_{as}$.

6.11. Discussion. Consider three conditions on a function f with $|||f|||_2 < \infty$:

(i) $f \in C_{as}(K)$.

(ii) the entropy condition 3.10 holds.

(iii) f can be represented as in Remark 4.14(c).

When K is a compact abelian group, these conditions are known [Marcus-Pisier (1980)] to be equivalent. We have shown, in Sections 3 and 4, that the implications (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i) hold for all compact abelian hypergroups K. By 6.9 and 6.10, the implications (i) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iii) are false for each hypergroup K_a. Finally, the implications (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (ii) hold for the hypergroups K_a, but we do not know whether they hold for all compact abelian hypergroups K.

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7. Conjugacy classes of compact Lie groups

7.1. Let G be a compact non-abelian group and K the hypergroup of conjugacy classes. The dual object Σ consists of the equivalence classes σ of continuous irreducible unitary representations of G. Each σ in Σ has finite dimension d_σ and trace χ_σ. The functions χ_σ are called characters but the *hypergroup characters* are normalized by dividing χ_σ by d_σ. More precisely, if $\pi\colon G \to K$ is the natural map, then ψ_σ on K is defined by the formula $\psi_\sigma \circ \pi = d_\sigma^{-1}\chi_\sigma$ and $\hat{K} = \{\psi_\sigma\colon \sigma \in \Sigma\}$. The invariant measure m on K is induced from Haar measure on G via π. The invariant measure $\hat{m}$ on $\hat{K}$ is given by

$$\hat{m}(\psi_\sigma) = d_\sigma^2;$$

see 2.1(1) and 27.31 in [Hewitt-Ross (1970)].

7.2. Observe that $\hat{K}$ has property λ_p, as defined in 2.5, precisely when the dual object Σ is a local central Λ_p set [Rider (1972a)]. Dooley [1979, 6.2] proved that $\hat{K}$ has property λ_2 whenever G is a compact Lie group. Rider [1972b] showed that if $G = U(n)$ or $SU(n)$, then $\hat{K}$ has property λ_p for $p < 2 + 2/n$ but not for $p = 3$. Rider's results have been generalized as follows. Let G be a compact connected Lie group. Then $\hat{K}$ has property λ_p for $p < 2 + \varepsilon_G$ where

$$\varepsilon_G = \frac{2\,\mathrm{rank}\, G}{\dim G - \mathrm{rank}\, G};$$

see Clerc [1976] and Dooley [1979]. In [Giulini-Soardi-Travaglini (1981)] it is shown that $\hat{K}$ does not have property λ_3, and, when G is a connected, simple, simply connected Lie group, $\hat{K}$ does not have property λ_p for $p \geqslant 2 + \varepsilon_G$.

7.3. *Conjugacy classes of $SU(2)$*. For the remainder of this paper, K will denote the hypergroup of conjugacy classes of $SU(2)$. We identify K with $[0, \pi]$ where θ in $[\theta, \pi]$ corresponds to the conjugacy class containing the matrix

$$\begin{pmatrix} \exp(i\theta) & 0 \\ 0 & \exp(-i\theta) \end{pmatrix};$$

see 15.4 in [Jewett (1975)]. For each $n = 1, 2, 3, \ldots,$ the dual subject of $SU(2)$ contains exactly one element of dimension n. We write its character as χ_n and we write ψ_n for the corresponding normalized hypergroup character on $K = [0, \pi]$.

Then

$$\psi_n(\theta) = \frac{\sin n\theta}{n \sin\theta} \quad \text{for } \theta \in (0, \pi).$$

The invariant measure m on K is given by

$$\int_K f\, dm = \frac{2}{\pi}\int_0^\pi f(\theta)\sin^2\theta\, d\theta.$$

As noted in 7.1 we have

$$\hat{m}(\psi_n) = n^2 \text{ and } \|\psi_n\|_2 = \frac{1}{n} \quad \text{for all } n.$$

Thus

(1) $$f \in L^2(K) \quad \text{if and only if} \quad \sum_{n=1}^{\infty} \left|\hat{f}(\psi_n)\right|^2 n^2 < \infty;$$

(2) $$f \in A(K) \quad \text{if and only if} \quad \sum_{n=1}^{\infty} \left|\hat{f}(\psi_n)\right| n^2 < \infty.$$

Propositions 3.3 and 3.7 tells us

(3) $$f \in U(K) \quad \text{implies} \quad \sum_{n=1}^{\infty} \hat{f}(\psi_n) n^2 \text{ converges;}$$

(4) $$f \in C_{as}(K) \quad \text{implies} \quad \sum_{n=1}^{\infty} \left|\hat{f}(\psi_n)\right|^2 n^4 < \infty.$$

A technique of R. A. Mayer [1967] allows us to give some specific examples for this interesting hypergroup.

7.4. *Mayer's examples.* Let (n_k) be a sequence in $\mathbf{N}$ such that $n_1 > 1$ and $n_{k+1} > n_k + 2$ for all k, and let (a_k) be a sequence in l^2. There exists an f in $L^2(K)$ such that

$$\hat{f}\left(\psi_{n_k+1}\right) = \frac{a_k}{n_k + 1}, \qquad \hat{f}\left(\psi_{n_k-1}\right) = \frac{-a_k}{n_k - 1},$$

and $\hat{f}(\psi_n) = 0$ elsewhere. We claim

(1) $f \in A(K)$ if and only if $\sum_{k=1}^{\infty} |a_k| n_k < \infty$;

(2) if $\sum_{k=1}^{\infty} |a_k| < \infty$, then $f \in C(K)$;

(3) if $f \in U(K)$, then $\lim_{k\to\infty} a_k n_k = 0$;

(4) if $\sum_{k=1}^{\infty} |a_k| < \infty$ and $\lim_{k\to\infty} a_k n_k = 0$, then $f \in U(K)$;

(5) if $f \in C_{as}(K)$, then $\sum_{k=1}^{\infty} |a_k|^2 n_k^2 < \infty$.

Claim (1) is trivial. For claim (2), we calculate, as in [Mayer (1967)], the partial sums

$$s_{n_m+1} f(\theta) = 2 \sum_{k=1}^{m} a_k \cos n_k \theta.$$

If $\sum |a_k| < \infty$ these partial sums converge uniformly to a continuous function which has to agree a.e. with f. Claims (3) and (5) follow from Propositions 3.3 and 3.7. Claim (4) is verified by comparing any partial sum with a suitable $s_{n_m+1} f$.

(a) In Mayer's original example, $n_k = k^3$ and $a_k = 1/k^2$. In this case, $f \in C(K)$, $f \notin C_{as}(K)$ and $f \notin U(K)$.

(b) Let $n_k = 3k$ and $a_k = k^{-3/2}$. Then $f \in U(K)$ but $f \notin C_{as}(K)$.

(c) Let $n_k = 3k$ and $a_k = k^{-2}$. Then $f \in U(K)$ and $f \notin A(K)$. Since $\Sigma |a_k|^2 n_k^2 < \infty$, (5) does not tell us whether $f \in C_{as}(K)$. We will see in 7.6 that f does belong to $C_{as}(K)$.

7.5. Remark. We have already spelled out in 4.7 results concerning a.s. membership in $L^p(K)$. Note that Theorem 4.5 implies that if

$$\sum_{n=1}^{\infty} |\hat{f}(\psi_n)|^2 n^{4-\varepsilon} < \infty$$

for all $\varepsilon > 0$, then X_ω is a.s. in all $L^p(K)$, $p < \infty$.

7.6. *Almost sure continuity again*. (a) We now indicate how Discussion 3.12 applies to the hypergroup $K = [0, \pi]$ of conjugacy classes of $SU(2)$. An induction argument involving some elementary trigonometry shows that there is an absolute constant C such that

$$|\psi_n(x) - \psi_n(y)| \leqslant C \log n\, \beta(n|x-y|).$$

Here (ψ_n) is as specified in 7.3 and β is the function defined in 3.12. In this case, condition 3.12(2) is equivalent to

$$\sum_{n=2}^{\infty} \frac{\left(\sum_{m=n}^{\infty} a_m^2 m^4 \log^2 m\right)^{1/2}}{n(\log n)^{1/2}} < \infty. \tag{1}$$

It follows that if (1) holds for $a_n = \hat{f}(\psi_n)$, then f is in $C_{as}(K)$. Related generalizations of the Salem-Zygmund theorem appear in [Ragozin (1976)] and [Rider (1977)].

Condition (1) holds whenever

$$\sum_{n=2}^{\infty} a_n^2 n^4 \log^{3+\varepsilon} n < \infty \tag{2}$$

for some $\varepsilon > 0$; see, for example, page 608 in [Kawata (1972)]. Compare with 7.3(4). As an example, it can be shown that (2), and hence (1), hold for $a_n = \hat{f}(\psi_n)$ where f is defined in 7.4(c). This function f belongs to $C_{as}(K)$.

(b) Finally, we indicate how the methods of Section 4.14 apply in this situation. These methods can be used to show that the condition obtained from (1) above by omitting the term $\log^2 m$ is still sufficient for almost sure continuity (see Marcus [1973], Appendix (iv)); we will just show, however, that the condition

$$\sum_{n=2}^{\infty} a_n^2 n^4 \log^{1+\varepsilon} n < \infty \tag{3}$$

is sufficient for almost sure continuity.

Assume without loss of generality that $\hat{f}(\psi_1) = 0$. For $n \geqslant 2$, let $c_n = a_n(\log n)^{(1+\varepsilon)/2}$, and let $H_n = \sum_{k=2}^{n} c_k k^2 \psi_k$. If condition (3) holds, then $\sup_n |||H_n|||_2 < \infty$. As in 6.12(c), the idea now is to represent H_n as $H_n * V_n$ for a suitable function V_n and thereby estimate $\|H_n\|_{as}$.

As the notation suggests, the kernels V_n to be used have properties like those of the classical de la Vallée-Poussin kernels on the unit circle. Recall that

$$\chi_n(\theta) = n\psi_n(\theta) = \frac{\sin(n\theta)}{\sin(\theta)}.$$

Let $h_n = (\chi_1 + \chi_2 + \cdots + \chi_n)^2$. In analysing these functions it is helpful to consider their differences

$$g_n = h_n - h_{n-1} = \chi_n^2 + 2\chi_n(\chi_1 + \chi_2 + \cdots + \chi_{n-1}).$$

Adopt the convention that $h_0 = 0$, so that $g_1 = h_1 = \chi_1 = \psi_1$. The reader may verify that

$$g_n = \chi_1 + 2\chi_2 + \cdots + n\chi_n + (n-1)\chi_{n+1} + \cdots + \chi_{2n-1}$$

for all $n \geqslant 2$. It follows that $\hat{g}_n(\psi_k) = 1$ for $k = 1, 2, \ldots, n$, that $\hat{g}_n(\psi_k) = 0$ for $k \geqslant 2n$, and that $\hat{g}_n(\psi_k)$ decreases as k increases from n to $2n$. Hence by induction on n, $\hat{h}_n(\psi_1) = n$, and $\hat{h}_n(\psi_k)$ is a nonincreasing function of k. Moreover, since $h_n \geqslant 0$, $\|h_n\|_1 = \hat{h}_n(\psi_1) = n$. The kernels $F_n = h_n/n$ are like the classical Féjer kernels of order $2n$ in that $F_n \geqslant 0$, $\|F_n\|_1 = 1$, $\hat{F}_n(\psi_k)$ is a nonincreasing function of k, and $\hat{F}_n(\psi_k) = 0$ for all $k \geqslant 2n$.

Let

$$V_n = 2F_{2n} - F_n = (h_{2n} - h_n)/n = (g_{n+1} + g_{n+2} + \cdots + g_{2n})/n.$$

It is easy to verify that $\|V_n\|_1 \leqslant 3$, that $\hat{V}_n(\psi_k) = 1$ for all $k \leqslant n$, that $\hat{V}_n(\psi_k) = 0$ for all $k \geqslant 4n$, and that $\hat{V}_n(\psi_k)$ is a nonincreasing function of k. Since $\hat{m}(\psi_k) = k^2$, the Plancherel formula yields the estimate $\|V_n\|_2 \leqslant \kappa n^{3/2}$.

Clearly $H_n = H_n * V_n$, so that, by Theorem 4.13, $\|H_n\|_{as} \leqslant \kappa'(\log n)^{1/2}$. To see that $f \in C_{as}$, write its Fourier series as

$$\sum_{n=2}^{\infty} (\log n)^{-(1+\varepsilon)/2}(H_n - H_{n-1}),$$

and sum by parts.

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