Proceedings of the Edinburgh Mathematical Society (2010) **53**, 239–254 ©
DOI:10.1017/S001309150700106X Printed in the United Kingdom

SPECTRAL ANALYSIS OF A CLASS OF HERMITIAN JACOBI MATRICES IN A CRITICAL (DOUBLE ROOT) HYPERBOLIC CASE

SERGUEI NABOKO AND SERGEY SIMONOV

Department of Mathematical Physics, Institute of Physics, St Petersburg University, Ulianovskaia 1, St Petergoff, St Petersburg 198904, Russia
(naboko@snoopy.phys.spbu.ru; sergey_simonov@mail.ru)

(Received 10 September 2007)

Abstract We consider a class of Jacobi matrices with periodically modulated diagonal in a critical hyperbolic ('double root') situation. For the model with 'non-smooth' matrix entries we obtain the asymptotics of generalized eigenvectors and analyse the spectrum. In addition, we reformulate a very helpful theorem from a paper by Janas and Moszynski in its full generality in order to serve the needs of our method.

Keywords: Jacobi matrices; subordinacy theory; asymptotics of generalized eigenvectors

2000 *Mathematics subject classification:* Primary 47B36

1. Introduction

We consider a class of Jacobi matrices with periodically modulated growing diagonal, giving an example of the 'double root' problem. This means a critical situation for asymptotics of the generalized eigenvectors related to the matrix. Such a situation particularly arises in the spectral phase transition phenomenon. If the matrix depends on some parameters, the decomposition of its spectrum into different types (absolutely continuous, singular continuous, pure point, discrete [**1**]) may be independent of these parameters. But if the structure of this decomposition changes under a variation of the parameters, a spectral phase transition occurs. If this change happens by a jump, such a phenomenon is called a spectral phase transition of the first type, whereas if this change is smooth with the change of the parameters when they move across some hypersurface in a space, it is called a spectral phase transition of the second type. The transition in types of spectrum is closely related to the change of form of asymptotics of generalized eigenvectors due to subordinacy theory [**6**], which was generalized to the case of Jacobi matrices in [**12**].

During the last decade the spectral analysis of Jacobi matrices attracted the attention of many specialists in operator theory and mathematical physics.

We consider the Jacobi matrix J with diagonal entries b_n:

$$b_n = \begin{cases} bn^{\alpha} & \text{for odd values of } n, \\ 0 & \text{for even values of } n \end{cases}$$

(where α and b are real parameters $\alpha \in (\frac{2}{3};1)$ and $b \neq 0$), and off-diagonal entries (weights)

$$a_n = n^\alpha.$$

The model demonstrates the situation of the spectral phase transition of the first order corresponding to the 'moment of transition' exactly. We are interested in asymptotics of generalized eigenvectors, i.e. solutions of the spectral recurrence relation

$$a_{n-1}u_{n-1} + b_n u_n + a_n u_{n+1} = \lambda u_n, \quad n \geqslant 2. \tag{1.1}$$

This model has been studied in [**5**] and in particular the following result was obtained: for $\lambda < 0$ there are two solutions u_n^+ and u_n^- of (1.1) with the following asymptotics as $n \to \infty$:

$$u_{2n}^{\pm} \sim (-1)^n n^{-\alpha/4} \exp\left(\pm \sqrt{\frac{b\lambda}{2^\alpha}} \frac{n^{1-(\alpha/2)}}{1-(\alpha/2)} \right),$$
$$u_{2n+1}^{\pm} \sim \pm\sqrt{\frac{\lambda}{2^\alpha b}}\left(1 - \frac{\alpha}{2}\right)(-1)^n n^{-3\alpha/4} \exp\left(\pm \sqrt{\frac{b\lambda}{2^\alpha}} \frac{n^{1-(\alpha/2)}}{1-(\alpha/2)} \right).$$

The problem of determining the asymptotics for $\lambda > 0$ was stated in [**5**], and we show here that the answer has the same form. However, this interesting question is not the main concern of this paper. The principal difficulty in our analysis is that the situation is 'critical hyperbolic', unlike the 'critical elliptic' situation in [**5**], which roughly means that exponents in the answer grow and decay if $\lambda > 0$ and oscillate if $\lambda < 0$.

Let us explain the problem of the critical situation in more detail. Whenever one deals with the three-term recurrence relation (1.1), it is often useful to write it in the vector form, introducing the sequence

$$\boldsymbol{u}_n := \begin{pmatrix} u_{n-1} \\ u_n \end{pmatrix}$$

and the transfer matrix

$$B_n := \begin{pmatrix} 0 & 1 \\ -\dfrac{a_{n-1}}{a_n} & \dfrac{\lambda - b_n}{a_n} \end{pmatrix}.$$

So (1.1) is equivalent to the discrete linear system in $\mathbb{C}^2$:

$$\boldsymbol{u}_{n+1} = B_n \boldsymbol{u}_n, \quad n \geqslant 2. \tag{1.2}$$

The solution for such a system is obtained by taking the chronological product of transfer matrices,

$$\left(\prod_{k=2}^{n} B_k\right)\boldsymbol{u}_2 = B_n \cdots B_2 \boldsymbol{u}_2.$$

The analysis becomes much easier if the matrix of the system is in some sense smooth in n (say, has a limit and asymptotic expansion in inverse powers of n as $n \to \infty$). In

https://doi.org/10.1017/S001309150700106X Published online by Cambridge University Press

our case the transfer matrix is not smooth in this sense because the coefficients of the spectral equation 'jump' all the time. This is the first (simple) problem that we face and it may be solved by taking the product of two consecutive transfer matrices, introducing the new linear system with the sequence of the coefficient matrices

$$M_n := B_{2n}B_{2n-1}, \tag{1.3}$$

which turns out to be smooth in n. It is easy to see that the sequence M_n has a limit $M := \lim_{n\to\infty} M_n$ with $\det M = 1$. The are three possibilities for the eigenvalues of M: they can be

- unimodular and complex conjugate (the elliptic situation),
- real and different (the hyperbolic situation; hence one of them is greater than 1 and another is less than 1 in absolute value), or
- coincide and equal 1 or -1 (the critical situation = the double-root case).

In the hyperbolic situation solutions are supposed to grow or decay; in the elliptic situation they are supposed to oscillate, having similar behaviour of their norms $\|\boldsymbol{u}_n\|$. The numerous variants of analogues to the Levinson Theorem (also known as the Benzaid–Lutz Theorem [**2**]) for differential linear systems [**4**] can be applied in these two cases [**9**, **13**].

In the critical situation one can also distinguish the 'critical elliptic' and the 'critical hyperbolic' cases. This separation depends on the lower-order behaviour of M_n as $n \to \infty$, namely on the asymptotic sign of the discriminant $(\operatorname{discr} M_n := (\operatorname{tr} M_n)^2 - 4\det M_n)$ of matrices M_n. In the critical case, the matrix M is similar to the Jordan block, with powers

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix},$$

and the difficulty here is not that the powers of the matrix grow with n (as they do in the hyperbolic situation), but the fact that the large entry is off-diagonal. This mixes upper and lower components of the solution, which makes the system unstable and sensitive to small perturbations. The problem of the 'critical hyperbolic' situation was considered for smooth matrix elements in [**10**]. In the present paper, considering a model of the Jacobi matrix with 'oscillating' diagonal, we intend to greatly simplify the scheme of successive transformations of the matrix system that was used there by making it more general and transparent (see § 3). This approach differs from that of [**5**] for the 'critical elliptic' situation, because in our case we have to deal with growing exponents (see step 3 on p. 247). Moreover, we consider in the appendix a theorem from [**9**] which is necessary for the final step of our method. The problem is that the original formulation of this theorem states asymptotics of only one (principal) solution. Although it is surely the most difficult part of the problem considered, we want to state the result in its full generality.

The method in this paper works for a much wider class of Jacobi matrices. However, our goal is to present a simple formulation of the method and to show by means of an example how it works.

A similar problem (the critical hyperbolic situation) was considered in [**7**] for a 'smooth' model, where the author used a completely different method related to [**11**].

2. Preliminaries

As usual, the operator J in the Hilbert space $l^2(\mathbb{N})$ is first defined (as $\mathcal{J}$) on the linear set of vectors which have only a finite number of non-zero components, $l_{\mathrm{fin}}(\mathbb{N})$, by the rule

$$\begin{aligned}(\mathcal{J}u)_1 &= b_1u_1 + a_1u_2,\\ (\mathcal{J}u)_n &= a_{n-1}u_{n-1} + b_nu_n + a_nu_{n+1} \quad \text{for } n \geqslant 2.\end{aligned}$$

Then its closure $J = \bar{\mathcal{J}}$ is a self-adjoint operator provided that the Carleman condition [**3**]

$$\sum_{n=0}^{\infty} \frac{1}{a_n} = +\infty$$

is satisfied. In the standard basis $\{e_n\}_{n=1}^{\infty}$ (where e_n is the vector in which all the components are zeros except for the nth) the operator J admits the following matrix representation:

$$\begin{pmatrix} b_1 & a_1 & 0 & \cdots \\ a_1 & b_2 & a_2 & \cdots \\ 0 & a_2 & b_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We start with the system

$$\begin{pmatrix} u_{2n} \\ u_{2n+1} \end{pmatrix} = M_n \begin{pmatrix} u_{2n-2} \\ u_{2n-1} \end{pmatrix} \tag{2.1}$$

(see (1.2), (1.3)). One can obtain a solution of this system directly, by taking a product of matrices M_n:

$$\begin{pmatrix} u_{2n} \\ u_{2n+1} \end{pmatrix} = \left[\prod_{k=2}^{n} M_k\right] \begin{pmatrix} u_2 \\ u_3 \end{pmatrix}.$$

Matrices M_n are given by

$$M_n = B_{2n}B_{2n-1} = \begin{pmatrix} -1 & -b \\ 0 & -1 \end{pmatrix} + \frac{\lambda}{(2n)^{\alpha}} \begin{pmatrix} 0 & 1 \\ -1 & -b \end{pmatrix} + \frac{\alpha}{2n} I + O\left(\frac{1}{n^{2\alpha}}\right),$$

having smooth-in-n asymptotics as $n \to \infty$.

The eigenvalues of the limit matrix

$$M := \begin{pmatrix} -1 & -b \\ 0 & -1 \end{pmatrix}$$

https://doi.org/10.1017/S001309150700106X Published online by Cambridge University Press

coincide, so the situation is critical (the double-root case). By an easy calculation, one can see that the discriminants of matrices

$$\operatorname{discr} M_n = \frac{4b\lambda}{(2n)^\alpha} + O\left(\frac{1}{n}\right);$$

hence, $\lambda < 0$ indeed corresponds to the elliptic situation and $\lambda > 0$ corresponds to the hyperbolic situation (both are critical).

Our method is based upon a sequence of transformations which are determined by some anzatz. A similar consideration for the construction of the anzatz may be also found in [**10**].

Remark 2.1. In what follows, we use transformations that are in fact discrete analogues of the variation-of-parameters method. Whenever one deals with the product of matrices, say A_n, one can also consider the matrices $C_n = T_{n+1}^{-1} A_n T_n$ (with some sequence T_n). Due to the cancellation of intermediate terms, the product of matrices A_n equals

$$\prod_{n=n_1}^{n_2} A_n = \prod_{n=n_1}^{n_2} (T_{n+1} C_n T_n^{-1}) = T_{n_2+1}\left(\prod_{n=n_1}^{n_2} C_n\right) T_{n_1}^{-1}.$$

So the study of the linear difference system with coefficient matrices A_n can be completely reduced to the study of the linear difference system with coefficient matrices C_n.

3. Calculation of the asymptotics in the hyperbolic case

In this section, we proceed through several transformations in order to simplify the problem and finally obtain the system which can be treated with the Janas–Moszynski Theorem (which is properly adjusted in the appendix). So we divide this section into four steps. In fact, we have already made a 'zero' step in the first section, which is reduction to the smooth matrix system by taking the product of two transfer matrices. But we do not include this step in § 3 because the double-root problem actually arises only at this stage. Moreover, after grouping the transfer matrices by pairs, one does not need to perform any inverse transformation in order to obtain the answer.

Step 1 (reduction of the meaningful part of M_n to the transfer matrix form). We write the spectral equation in matrix form to settle the problem of periodically modulated coefficients by grouping transfer matrices in pairs. But now it is simpler to consider a smooth three-term recurrence relation which is equivalent to the system. The transfer matrix corresponding to the three-term recurrence relation should have entries 0 and 1 in the upper row. In this step, we find the suitable transformation which makes the coefficient matrix of the system resemble a transfer matrix. This transformation is generated by the matrix sequence T_n of the form

$$T_n := (-1)^n \left[\begin{pmatrix} 1 & -b \\ 1 & 0 \end{pmatrix} + \frac{\lambda}{(2n)^\alpha}\begin{pmatrix} b + \dfrac{1}{2b} & 0 \\ \dfrac{1}{2b} & -\dfrac{1}{2} \end{pmatrix} + \frac{\alpha}{2n}\begin{pmatrix} 0 & 0 \\ -1 & b \end{pmatrix}\right].$$

https://doi.org/10.1017/S001309150700106X Published online by Cambridge University Press

The transformation

$$N_n := T_{n+1} M_n T_n^{-1} \tag{3.1}$$

gives

$$N_n = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} + \frac{b\lambda}{(2n)^\alpha}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\alpha}{n}\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} + O\left(\frac{1}{n^{2\alpha}}\right).$$

The meaningful part (or the part which we expect to be meaningful: the sum of first few terms) is now a matrix with the upper row of the form $0, 1$. The exact form of matrices T_n can be determined from this requirement (but the answer is not unique of course). Indeed, we try to find T_n in the form

$$T_n = (-1)^n \left[\begin{pmatrix} 1 & -b \\ 1 & 0 \end{pmatrix} + \frac{\lambda}{(2n)^\alpha} T^{(1)} + \frac{\alpha}{2n} T^{(2)}\right], \tag{3.2}$$

with unknown matrices $T^{(1)}$ and $T^{(2)}$ independent of n. Define

$$T := \begin{pmatrix} 1 & -b \\ 1 & 0 \end{pmatrix},$$

which is chosen in order to satisfy

$$-T\begin{pmatrix} -1 & -b \\ 0 & -1 \end{pmatrix} T^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} =: N.$$

By substitution of T_n in the form (3.2) into the relation $T_{n+1} M_n = N_n T_n$, looking at the terms of orders $1/n^\alpha$ and $1/n$, one obtains the following linear conditions on the matrices $T^{(1)}$ and $T^{(2)}$:

$$\left[T^{(1)} T^{-1}, \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}\right] = T\begin{pmatrix} 0 & 1 \\ -1 & -b \end{pmatrix} T^{-1} + \begin{pmatrix} 0 & 0 \\ * & * \end{pmatrix}$$

$$\left[T^{(2)} T^{-1}, \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}\right] = I + \begin{pmatrix} 0 & 0 \\ * & * \end{pmatrix}.$$

We denote by asterisks those matrix entries which are allowed to be non-zero. Therefore, the problem may be reduced to the following: for the commutator equation

$$\left[X, \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}\right] = \begin{pmatrix} f_1 & f_2 \\ x_1 & x_2 \end{pmatrix},$$

prove that, for any given values f_1 and f_2, there exist unique values x_1 and x_2 and a 2×2 matrix X (obviously not unique) which satisfy the equation. Multiplying the equality on the left by the matrix $\left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)$ and on the right by $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ (its inverse), we obtain another

https://doi.org/10.1017/S001309150700106X Published online by Cambridge University Press

form of the commutator equation:

$$\left[Y, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right] = \left[Y, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right]$$
$$= \begin{pmatrix} -y_3 & y_1 - y_4 \\ 0 & y_3 \end{pmatrix} = \begin{pmatrix} f_1 + f_2 & f_2 \\ x_1 + x_2 - f_1 - f_2 & x_2 - f_2 \end{pmatrix}.$$

We define

$$Y := \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} := \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} X \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

It follows now that $x_2 = -f_1$ and $x_1 = 2f_1 + f_2$ are uniquely determined by f_1 and f_2, and the matrix Y exists and is unique up to

$$c_1 I + c_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

with any c_1 and c_2.

Therefore, one can reduce the original system (2.1) to the new one, where M_n are replaced by N_n. Note that the exact form of the matrices $T^{(1)}$ and $T^{(2)}$ is not essential for the result of transformation. We need only to prove the existence of such matrices.

Step 2 (reduction of the main term to the identity matrix). The structure of the main part of the system allows one to write

$$N_n = \begin{pmatrix} 0 & 1 \\ -F_2(n) & -F_1(n) \end{pmatrix} + O\left(\frac{1}{n^{2\alpha}}\right),$$

where the error term is a 2×2 matrix with norm $O(n^{-2\alpha})$ and the matrix entries $F_1(n)$ and $F_2(n)$ are (set $B := \sqrt{b\lambda/2^\alpha}$)

$$F_1(n) = -2 - \frac{B^2}{n^\alpha} + \frac{\alpha}{n}, \qquad F_2(n) = 1 - \frac{\alpha}{n}.$$

Our goal is to use the fact that the system

$$\begin{pmatrix} v_{n+1} \\ w_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -F_2(n) & -F_1(n) \end{pmatrix} \begin{pmatrix} v_n \\ w_n \end{pmatrix}$$

(where the remainder is omitted) is *equivalent* to the three-term recurrence relation

$$u_{n+1} + F_1(n) u_n + F_2(n) u_{n-1} = 0. \tag{3.3}$$

The latter has two 'approximate solutions' of the form

$$z_n^\pm = n^\gamma \exp\{\pm A n^\delta\},$$

https://doi.org/10.1017/S001309150700106X Published online by Cambridge University Press

with $\gamma = -\alpha/4$, $\delta = 1 - \alpha/2$, $A = B/\delta$. This means that

$$z^{\pm}_{n+1} + F_1(n) z^{\pm}_n + F_2(n) z^{\pm}_{n-1} = O(n^{-2\alpha}) z^{\pm}_n, \tag{3.4}$$

which can be verified by a direct calculation. The form of these 'approximate solutions' can be obtained, for instance, by analogy with the Wentzel–Kramers–Brillouin method (see [**10**] for this type of argument). Having this structure of the 'solution' one can determine unknown values of γ, δ and A from the condition of cancellation of all decreasing terms in (3.4) up to the order $O(n^{-2\alpha})$. Equation (3.4) implies that

$$\begin{aligned} \begin{pmatrix} z^{\pm}_n \\ z^{\pm}_{n+1} \end{pmatrix} &= \left(\begin{pmatrix} 0 & 1 \\ -F_2(n) & -F_1(n) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & O(n^{-2\alpha}) \end{pmatrix} \right) \begin{pmatrix} z^{\pm}_{n-1} \\ z^{\pm}_n \end{pmatrix} \\ &= (N_n + O(n^{-2\alpha})) \begin{pmatrix} z^{\pm}_{n-1} \\ z^{\pm}_n \end{pmatrix}. \end{aligned} \tag{3.5}$$

It is useful now to write the last vector equality in a matrix form. Define

$$S_n = \begin{pmatrix} z^{-}_{n-1} & z^{+}_{n-1} \\ z^{-}_n & z^{+}_n \end{pmatrix}.$$

Then, combining (3.5) for both signs, one has

$$S_{n+1} = (N_n + O(n^{-2\alpha})) S_n$$

and hence

$$S^{-1}_{n+1} N_n S_n = I + S^{-1}_{n+1} O(n^{-2\alpha}) S_n.$$

This actually follows *only* from the fact that $z^{\pm}_n$ are 'approximate solutions' of the recurrence relation (3.3), i.e. from (3.4).

Substituting the expression for $z^{\pm}_n = n^{\gamma} \exp\{\pm A n^{\delta}\}$ into the term

$$S^{-1}_{n+1} O(n^{-2\alpha}) S_n = \frac{1}{\det S_{n+1}} \begin{pmatrix} z^{+}_n z^{-}_n O(n^{-2\alpha}) & z^{+2}_n O(n^{-2\alpha}) \\ z^{-2}_n O(n^{-2\alpha}) & z^{+}_n z^{-}_n O(n^{-2\alpha}) \end{pmatrix}$$

and using the fact that $\det S_{n+1} \sim 2A\delta n^{-\alpha}$ as $n \to \infty$ (for calculations, see [**5**]; one can also find this rate of decay from the constancy of the modified Wronskian), one obtains

$$S^{-1}_{n+1} N_n S_n = I + \frac{1}{n^{3\alpha/2}} \begin{pmatrix} O(1) & \mathrm{e}^{2An^{\delta}} O(1) \\ \mathrm{e}^{-2An^{\delta}} O(1) & O(1) \end{pmatrix} =: K_n. \tag{3.6}$$

These calculations enable us to reduce the original system to the system with the coefficient matrix K_n, for which the 'main term' is the identity matrix. The problem is that it contains exponentially increasing anti-diagonal terms. In the next step we show how to overcome this difficulty by the symmetric cancellation of both anti-diagonal entries.

Remark 3.1. The above calculations demonstrate a simple 'geometrical' approach based on reduction to the three-term recurrence relation (3.3). Another approach (a simplification of calculations from [**10**]) may be rather more straightforward: the 'geometrical construction' is replaced by an explicit calculation. Since $\det S_{n+1} = z_n^- z_{n+1}^+ - z_n^+ z_{n+1}^-$, the substitution of matrices S_n gives

$$\begin{aligned} S_{n+1}^{-1} \begin{pmatrix} 0 & 1 \\ -F_2(n) & -F_1(n) \end{pmatrix} S_n &= \frac{1}{z_n^- z_{n+1}^+ - z_n^+ z_{n+1}^-} \\ &\quad \times \begin{pmatrix} z_n^- z_{n+1}^+ + z_n^+(F_1 z_n^- + F_2 z_{n-1}^-) & z_n^+(z_{n+1}^+ + F_1 z_n^+ + F_2 z_{n-1}^+) \\ -z_n^-(z_{n+1}^- + F_1 z_n^- + F_2 z_{n-1}^-) & -z_n^+ z_{n+1}^- - z_n^-(F_1 z_n^+ + F_2 z_{n-1}^+) \end{pmatrix}. \end{aligned}$$

After adding and subtracting terms $z_n^+ z_{n+1}^-$ in the upper-left entry and $z_n^- z_{n+1}^+$ in the lower-right entry for extracting the determinant, the last expression becomes

$$I + \frac{1}{z_n^- z_{n+1}^+ - z_n^+ z_{n+1}^-} \begin{pmatrix} z_n^+(z_{n+1}^- + F_1 z_n^- + F_2 z_{n-1}^-) & z_n^+(z_{n+1}^+ + F_1 z_n^+ + F_2 z_{n-1}^+) \\ -z_n^-(z_{n+1}^- + F_1 z_n^- + F_2 z_{n-1}^-) & -z_n^-(z_{n+1}^+ + F_1 z_n^+ + F_2 z_{n-1}^+) \end{pmatrix}.$$

It is remarkable that the expressions $z_{n+1}^\pm + F_1 z_n^\pm + F_2 z_{n-1}^\pm$ appear in each matrix entry of the second term. Now substituting (3.4) and taking into consideration the fact that $\det S_{n+1} \sim 2A\delta n^{-\alpha}$, one obtains the same expression as in (3.6):

$$\begin{aligned} I + \frac{1}{\det S_{n+1}} \begin{pmatrix} z_n^+ z_n^- O(n^{-2\alpha}) & z_n^{+2} O(n^{-2\alpha}) \\ z_n^{-2} O(n^{-2\alpha}) & z_n^+ z_n^- O(n^{-2\alpha}) \end{pmatrix} \\ = I + \frac{1}{n^{3\alpha/2}} \begin{pmatrix} O(1) & \mathrm{e}^{2An^\delta} O(1) \\ \mathrm{e}^{-2An^\delta} O(1) & O(1) \end{pmatrix}. \end{aligned}$$

The problem now is the growing exponent in the upper-right entry of the matrix. This exponent can be compensated for by the decaying exponent in the lower-left entry, as we show below.

Step 3 (elimination of exponentially increasing off-diagonal terms). In order to produce the elimination we perform yet another transformation generated by the sequence of matrices

$$X_n = \begin{pmatrix} \mathrm{e}^{2An^\delta} & 0 \\ 0 & 1 \end{pmatrix},$$

which yields

$$\begin{aligned} L_n := X_{n+1}^{-1} K_n X_n &= \begin{pmatrix} \mathrm{e}^{2A(n^\delta - (n+1)^\delta)} & 0 \\ 0 & 1 \end{pmatrix} + O\biggl(\frac{1}{n^{3\alpha/2}}\biggr) \\ &= \begin{pmatrix} 1 - \dfrac{2A\delta}{n^{\alpha/2}} + \dfrac{(2A\delta)^2}{2n^\alpha} & 0 \\ 0 & 1 \end{pmatrix} + O\biggl(\frac{1}{n^{3\alpha/2}}\biggr). \end{aligned} \tag{3.7}$$

As a result of all the transformations, the original system is reduced to a new one with coefficient matrices L_n for which the Janas–Moszynski Theorem [**9**] (see our Theorem A 2) is applicable. Take

$$p_n = \frac{2A\delta}{n^{\alpha/2}}, \qquad V_n \equiv V = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$$

and let $R_n = O(n^{-3\alpha/2})$ be the matrix remainder which belongs to l^1 for $\alpha > \frac{2}{3}$. The theorem asserts that the system with coefficient matrices L_n has a basis of solutions of the form $\exp\{-2An^{\delta}\}(\boldsymbol{e}_1 + o(1))$ and $\boldsymbol{e}_2 + o(1)$; we use the notation

$$\boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \boldsymbol{e}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Remark 3.2. Note that we need to apply Theorem A 2 only to ignore the $O(n^{-3\alpha/2})$ remainder term. Without it, the system generated by the matrix (3.7) obviously has two exact solutions $\exp\{-2An^{\delta}\}\boldsymbol{e}_1$ and $\boldsymbol{e}_2$. However, this is not entirely obvious, since the eigenvalues of the limit matrix $\lim_{n\to\infty} L_n$ coincide.

Step 4 (returning to the original system). Returning to the system with coefficient matrices M_n, we recall steps 1–3. We should take into account the matrices that stay in front of the product of matrices L_n. Since

$$M_k = T_{k+1}^{-1} S_{k+1} X_{k+1} L_k X_k^{-1} S_k^{-1} T_k,$$

one has (for every λ we choose a number n_0 such that $\det T_n \neq 0$ for $n \geqslant n_0$)

$$\begin{aligned} \begin{pmatrix} u_{2n} \\ u_{2n+1} \end{pmatrix} &= \left(\prod_{k=n_0}^{n} M_k \right) \begin{pmatrix} u_{n_0} \\ u_{n_0+1} \end{pmatrix} \\ &= (T_{n+1}^{-1} S_{n+1} X_{n+1}) \left(\prod_{k=n_0}^{n} L_k \right) X_{n_0}^{-1} S_{n_0}^{-1} T_{n_0} \begin{pmatrix} u_{n_0} \\ u_{n_0+1} \end{pmatrix} \\ &= (-1)^{n+1} \begin{pmatrix} z_n^+(1+o(1)) & z_n^+(1+o(1)) \\ -\dfrac{A\delta z_n^+}{bn^{\alpha/2}}(1+o(1)) & \dfrac{A\delta z_n^+}{bn^{\alpha/2}}(1+o(1)) \end{pmatrix} \\ &\quad \times \left(\prod_{k=n_0}^{n} L_k \right) X_{n_0}^{-1} S_{n_0}^{-1} T_{n_0} \begin{pmatrix} u_{n_0} \\ u_{n_0+1} \end{pmatrix}. \end{aligned}$$

As one has from the previous step, the vector

$$\left(\prod_{k=n_0}^{n} L_k \right) X_{n_0}^{-1} S_{n_0}^{-1} T_{n_0} \begin{pmatrix} u_{n_0} \\ u_{n_0+1} \end{pmatrix}$$

https://doi.org/10.1017/S001309150700106X Published online by Cambridge University Press

is a linear combination of $\exp\{-2An^{\delta}\}(\boldsymbol{e}_1 + o(1))$ and $(\boldsymbol{e}_2 + o(1))$. Therefore, the vector $\binom{u_{2n}}{u_{2n+1}}$ is a linear combination of

$$(-1)^n \begin{pmatrix} z_n^+(1+o(1)) \\ \dfrac{A\delta z_n^+}{bn^{\alpha/2}}(1+o(1)) \end{pmatrix}.$$

So the following result holds true.

Theorem 3.3. *For any $\lambda > 0$ spectral equation (1.1) has a basis of solutions u_n^+ and u_n^- with asymptotics as $n \to \infty$ of even components*

$$u_{2n}^{\pm} \sim (-1)^n n^{-\alpha/4} \exp\left(\pm \sqrt{\frac{b\lambda}{2^{\alpha}}} \frac{n^{1-(\alpha/2)}}{1-(\alpha/2)} \right)$$

and asymptotics of odd components

$$u_{2n+1}^{\pm} \sim \pm\sqrt{\frac{\lambda}{2^{\alpha}b}} \left(1 - \frac{\alpha}{2}\right)(-1)^n n^{-3\alpha/4} \exp\left(\pm \sqrt{\frac{b\lambda}{2^{\alpha}}} \frac{n^{1-(\alpha/2)}}{1-(\alpha/2)} \right).$$

Regarding the operator J this means (due to the subordinacy theory of Gilbert and Pearson [**6**, **12**]) that the spectrum of the operator on the positive semi-axis is of pure point type. Moreover, if $\lambda \in \sigma_p(J)$, then the solution u_n^- is an eigenvector of J.

Note that the asymptotics of $u_n^{\pm}$ as $n \to \infty$ for $\lambda > 0$ formally coincide with those obtained in [**5**] for $\lambda < 0$, as mentioned above.

Acknowledgements. The authors express their gratitude to Dr Marco Marletta for his useful remarks and to the referee for his/her reading of the manuscript and for making many important remarks.

This work was supported by INTAS Grant no. 05-1000008-7883 and partly by Grant no. RFBR-06-01-00249.

Appendix A. Janas–Moszynski Theorem revisited

Let us turn our attention to a theorem from [**9**], which was not formulated in all details there. By using a complicated technique, the existence of the principal ('smaller') solution for some system was proved. But the existence of the second ('larger') solution was not stated. In what follows we prove this fact.

Remark A 1. For the sequences $\{a_n\}$, in order to avoid any non-essential problems related to vanishing of elements of the sequence, we assume the notation for $\prod_{n=1}^{N} a_n$ as the product of all a_n, $1 \leqslant n \leqslant N$, such that $a_n \neq 0$. We also use the notation l^1 and D^1 for matrix sequences, i.e. the sequence of matrices $\{M_n\}_{n=1}^{\infty}$ belongs to

- l^1 if and only if $\sum_{n=1}^{\infty} \|M_n\| < \infty$,
- D^1 if and only if $\sum_{n=1}^{\infty} \|M_{n+1} - M_n\| < \infty$.

https://doi.org/10.1017/S001309150700106X Published online by Cambridge University Press

Consider the linear difference system in $\mathbb{C}^2$:

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = (I + p_n V_n + R_n) \begin{pmatrix} u_n \\ v_n \end{pmatrix}. \tag{A 1}$$

Suppose it is non-degenerate, i.e. $\det(I + p_n V_n + R_n) \neq 0$ for every n (this means that the system has two linearly independent solutions).

Theorem A 2. *(See [**9**], where the essential part of the theorem was proved.) Let*

- $p_n \to 0$ *be a* positive *sequence such that* $\sum_{n=1}^{\infty} p_n = +\infty$,
- $\{R_n\} \in l^1$ *be a* 2×2 *matrix sequence,*
- $\{V_n\} \in D^1$ *be a* real 2×2 *matrix sequence with* $\operatorname{discr}(\lim_{n\to\infty} V_n) \neq 0$ *(i.e.* $\lim_{n\to\infty} V_n$ *has two different eigenvalues).*

Then the system (A 1) has a basis of solutions $\boldsymbol{u}_n^{(1)}$ *and* $\boldsymbol{u}_n^{(2)}$ *with the following asymptotics as* $n \to \infty$*:*

$$\boldsymbol{u}_n^{(1,2)} = \left(\prod_{k=1}^{n} [1 + p_k \mu_{1,2}(k)] \right) (\boldsymbol{x}_{1,2} + o(1)),$$

where μ_1 *and* μ_2 ($\operatorname{Re}\mu_1 \leqslant \operatorname{Re}\mu_2$) *are the eigenvalues of matrix* $V := \lim_{n\to\infty} V_n$, $\boldsymbol{x}_1$ *and* $\boldsymbol{x}_2$ *are the corresponding eigenvectors,* $\mu_1(n)$ *and* $\mu_2(n)$ *are the eigenvalues of matrices* V_n *(chosen in a way such that* $\mu_1(n) \to \mu_1$ *and* $\mu_2(n) \to \mu_2$ *as* $n \to \infty$*).*

Proof. The elliptic case of $\operatorname{discr} V < 0$ (a relatively straightforward one) is a special case of the Janas–Moszynski Theorem proved in [**9**]. In the hyperbolic case of $\operatorname{discr} V > 0$, the existence of the 'smaller' solution $\boldsymbol{u}_n^{(1)}$ is also guaranteed by this theorem. So we need to prove only the existence of the second ('larger') solution $\boldsymbol{u}_n^{(2)}$ (the solution corresponding to the eigenvalue μ_2 with the largest real part) in the hyperbolic case. We emphasize that we do not give a new proof of the result from [**9**], but only add one extra (and rather simple) assertion to it.

Let us reduce the situation to its simpler subcase, i.e. the system of the special form

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \left(I - \begin{pmatrix} p_n & 0 \\ 0 & 0 \end{pmatrix} + R_n \right) \begin{pmatrix} u_n \\ v_n \end{pmatrix}. \tag{A 2}$$

For such a system,

$$V_n \equiv V = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Remark A 3. In fact we deal with a system of this type in § 3, step 3.

https://doi.org/10.1017/S001309150700106X Published online by Cambridge University Press

The following statement holds.

Lemma A 4. *Let*

(i) $p_n \to 0$ *be a positive for sufficiently large* n *sequence such that* $\sum_{n=1}^{\infty} p_n = +\infty$,

(ii) $\{R_n\} \in l^1$ *be a* 2×2 *matrix sequence.*

Then there exists a solution to the system (A 2) of the form

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \boldsymbol{e}_2 + o(1), \qquad \boldsymbol{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

as $n \to \infty$.

Proof. Without loss of generality one can assume that $0 < p_n < 1$ for every n. Then, for any two natural numbers $n_1 < n_2$,

$$\begin{aligned}\left\| \prod_{n=n_1}^{n_2} \left[I - \begin{pmatrix} p_n & 0 \\ 0 & 0 \end{pmatrix} + R_n \right] \right\| &\leqslant \prod_{n=n_1}^{n_2} \left[\left\| I - \begin{pmatrix} p_n & 0 \\ 0 & 0 \end{pmatrix} \right\| + \|R_n\| \right] \\ &\leqslant \prod_{n=n_1}^{n_2} [1 + \|R_n\|] < \infty.\end{aligned}$$

Therefore, every solution of the system (A 2) is bounded and there exists a universal constant C such that, for any natural numbers $n_1 < n_2$ and any solution u_n,

$$\left\| \begin{pmatrix} u_{n_2} \\ v_{n_2} \end{pmatrix} \right\| < C \left\| \begin{pmatrix} u_{n_1} \\ v_{n_1} \end{pmatrix} \right\|. \tag{A 3}$$

Using the variation-of-parameters method one can rewrite the system (A 2) in the following way:

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} \prod_{k=n_0}^{n} (1 - p_k) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_{n_0} \\ v_{n_0} \end{pmatrix} + \sum_{k=n_0}^{n} \begin{pmatrix} \prod_{l=k+1}^{n} (1 - p_l) & 0 \\ 0 & 1 \end{pmatrix} R_k \begin{pmatrix} u_k \\ v_k \end{pmatrix}. \tag{A 4}$$

The equivalence of the two systems (A 1) and (A 4) follows from elementary calculations. Let us take

$$\begin{pmatrix} u_{n_0} \\ v_{n_0} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{k=n_0}^{n} \begin{pmatrix} \prod_{l=k+1}^{n} (1 - p_l) & 0 \\ 0 & 1 \end{pmatrix} R_k \begin{pmatrix} u_k \\ v_k \end{pmatrix}.$$

Note that $\prod_{l=k+1}^{n} (1 - p_l) \to 0$ as $n \to \infty$ for every k due to properties of the sequence $\{p_n\}$. Then, since

$$\left\| \begin{pmatrix} u_n \\ v_n \end{pmatrix} \right\| < C \quad \text{for } n \geqslant n_0$$

and $\{R_n\} \in l^1$, by the Weierstrass Theorem one has

$$\sum_{k=n_0}^{n} \begin{pmatrix} \prod_{l=k+1}^{n}(1-p_l) & 0 \\ 0 & 1 \end{pmatrix} R_k \begin{pmatrix} u_k \\ v_k \end{pmatrix} \to \sum_{k=n_0}^{\infty} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} R_k \begin{pmatrix} u_k \\ v_k \end{pmatrix}.$$

Hence,

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{k=n_0}^{\infty} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} R_k \begin{pmatrix} u_k \\ v_k \end{pmatrix}$$

as $n \to \infty$. The second component of the limit vector in the last expression is non-zero provided that n_0 is sufficiently large (due to (A 3) and the second condition of Lemma A 4). □

Reduction of system (A 1) to the special case of system (A 2) can be done following the standard strategy [**4**, **9**] by using the fact of D^1-diagonalizability of every D^1 sequence of matrices with invertible limit [**8**] (see also [**9**, Lemma 1.3]; this is a discrete version of the result from [**4**]).

Proposition A 5 (Janas and Moszynski [8]). *Let $\{V_n\}$ be a complex 2×2 matrix sequence such that $\{V_n\} \in D^1$ and $\operatorname{discr}(\lim_{n\to\infty} V_n) \neq 0$. Then the sequence $\{V_n\}$ is D^1-diagonalizable, i.e. there exists such a matrix sequence $\{T_n\} \in D^1$ with invertible limit such that, for sufficiently large values of n,*

$$V_n = T_n \begin{pmatrix} \mu_1(n) & 0 \\ 0 & \mu_2(n) \end{pmatrix} T_n^{-1}.$$

Let us return to the proof of Theorem A 2. By Proposition A 5, for n sufficiently large the corresponding matrices T_n diagonalize V_n. To avoid tedious notation, without loss of generality, all the calculations start with $n = 1$. An explicit calculation shows that

$$\begin{aligned} T_{n+1}^{-1}(I + p_n V_n + R_n)T_n &= (T_{n+1}^{-1}T_n)\left(I + p_n \begin{pmatrix} \mu_1(n) & 0 \\ 0 & \mu_2(n) \end{pmatrix}\right) + T_{n+1}^{-1}R_nT_n \\ &= I + p_n \begin{pmatrix} \mu_1(n) & 0 \\ 0 & \mu_2(n) \end{pmatrix} + Q_n, \end{aligned}$$

where

$$Q_n := T_{n+1}^{-1}(T_n - T_{n+1})\left[I + p_n \begin{pmatrix} \mu_1(n) & 0 \\ 0 & \mu_2(n) \end{pmatrix}\right] + T_{n+1}^{-1}R_nT_n.$$

Further,

$$\begin{aligned} \prod_{k=1}^{n}(I + p_kV_k + R_k) &= T_{n+1}\left(\prod_{k=1}^{n}\left[I + p_k \begin{pmatrix} \mu_1(k) & 0 \\ 0 & \mu_2(k) \end{pmatrix} + Q_k\right]\right) T_1^{-1} \\ &= \left(\prod_{k=1}^{n}(1 + p_k\mu_2(k))\right) T_{n+1}\left(\prod_{k=1}^{n}\left[I - \begin{pmatrix} \tilde{p}_k & 0 \\ 0 & 0 \end{pmatrix} + \tilde{R}_k\right]\right) T_1, \end{aligned}$$

where

$$\tilde{p}_n := p_n \frac{\mu_2(n) - \mu_1(n)}{1 + p_n \mu_2(n)},$$
$$\tilde{R}_n := \frac{1}{1 + p_n \mu_2(n)} Q_n.$$

The properties of the sequence $\{T_n\}$ guarantee that $\{Q_n\} \in l^1$, as well as $\{\tilde{R}_n\} \in l^1$. Obviously, $\tilde{p}_n > 0$ for large values of n, so the system

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \left(I - \begin{pmatrix} \tilde{p}_n & 0 \\ 0 & 0 \end{pmatrix} + \tilde{R}_n \right) \begin{pmatrix} u_n \\ v_n \end{pmatrix}$$

satisfies all the conditions of Lemma A 4. Therefore, it has a solution

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \boldsymbol{e}_2 + o(1).$$

Let $T := \lim_{n\to\infty} T_n$. One has that $T\boldsymbol{e}_2 = \boldsymbol{x}_2$ is an eigenvector of the matrix V corresponding to the eigenvalue μ_2. Then the system (A 1) has a solution equal to

$$\left(\prod_{k=1}^{n} [1 + p_k \mu_2(k)] \right) (\boldsymbol{x}_2 + o(1)) =: \boldsymbol{u}_n^{(2)},$$

which completes the proof. □

References

1. N. Akhiezer and I. Glazman, *Theory of linear operators in Hilbert space*, 2nd edn (Dover, New York, 1993).
2. Z. Benzaid and D. Lutz, Asymptotic representation of solutions of perturbed systems of linear difference equations, *Stud. Appl. Math.* **77** (1987), 195–221.
3. Y. Berezanskii, Expansions in eigenfunctions of selfadjoint operators (Naukova Dumka, Kiev, 1965; in Russian).
4. E. Coddington and N. Levinson, *Theory of ordinary differential equations* (McGraw-Hill, New York, 1955).
5. D. Damanik and S. Naboko, A first-order phase transition in a class of unbounded Jacobi matrices: critical coupling, *J. Approx. Theory* **145**(2) (2007), 221–236.
6. D. Gilbert and D. Pearson, On subordinacy and analysis of the spectrum of one-dimensional Schrödinger operators, *J. Math. Analysis Applic.* **128** (1987), 30–56.
7. J. Janas, The asymptotic analysis of generalized eigenvectors of some Jacobi operators: Jordan box case, *J. Diff. Eqns Applic.* **12**(6) (2006), 597–618.
8. J. Janas and M. Moszynski, Alternative approaches to the absolute continuity of Jacobi matrices with monotonic weights, *Integ. Eqns Operat. Theory* **43** (2002), 397–416.
9. J. Janas and M. Moszynski, Spectral properties of Jacobi matrices by asymptotic analysis, *J. Approx. Theory* **120** (2003), 309–336.
10. J. Janas, S. Naboko and E. Sheronova, Asymptotic behavior of generalized eigenvectors of Jacobi in the critical ('double root') case, preprint mp_arc 07-41, available at http://www.ma.utexas.edu.

11. W. Kelley, Asymptotic analysis of solutions in the ‘double root’ case, *Comput. Math. Appl.* **28** (1994), 167–173.
12. S. Khan and D. Pearson, Subordinacy and spectral theory for infinite matrices, *Helv. Phys. Acta* **65** (1992), 505–527.
13. L. O. Silva, Uniform Levinson type theorems for discrete linear systems, in *Spectral methods for operators of mathematical physics*, Operator Theory: Advances and Applications, Volume 154, pp. 203–218 (Birkhäuser, 2004).