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CYCLABILITY OF r-REGULAR r-CONNECTED GRAPHS

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For each value of $r \geq 4$, r even, we construct infinitely many r-regular, r-connected graphs whose cyclability is not greater than $6r - 4$ if $r \equiv 0 \pmod 4$ and $8r - 5$ if $r \equiv 2 \pmod 4$.

1. Introduction

We use the terminology and notation in Bondy and Murty [1]. The *cyclability* of a graph G is the largest integer k such that any k vertices of G lie on a common cycle. This notion was introduced and studied by Chvátal [2]. We denote by $f(r)$ the largest integer k such that any k vertices in an r-regular, r-connected graph $(r \geq 3)$ lie on a common cycle. Various authors investigated the function $f(r)$. For $r = 3$, the Petersen graph shows that $f(3) \leq 9$. Holton, McKay, Plummer and Thomassen [4], proved that $f(3) = 9$ and constructed an infinite family of cubic graphs (based on the Petersen graph) with cyclability 9 . Holton [3], proved that $f(r) \geq r + 4$. (This result was also obtained by Kelmans and Lomonosov [7].) Meredith [8], described a construction of non-Hamiltonian r-regular, r-connected graphs for all $r \geq 4$ (thus showing that Nash-Williams' conjecture, that all 4-regular, 4-connected graphs are Hamiltonian is false). Meredith's construction is based on the Petersen graph, and yields the upper bound $f(r) \leq 10r - 11$. It is not obvious how one can obtain from Meredith's construction, for each value of

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r , infinitely many graphs with cyclability not greater than $10r - 11$.

In this paper we modify Meredith's construction, and obtain, for each r , infinitely many r-regular r-connected graphs with cyclabilities $6r - 4$ $(r \equiv 0 \pmod 4)$ and $8r - 5$ $(r \equiv 2 \pmod 4)$. Our method enables us to construct non-Hamiltonian, r-regular, r-connected bipartite graphs. (such a graph on 84 vertices, for $r = 4$ is described.)

2. Meredith's construction

Meredith's construction of r-regular r-connected non-Hamiltonian graphs is based on the following two steps:

(i) in a graph G , add edges parallel to existing edges, so that the resulting multigraph is r-regular and r-edge connected;

(ii) replace each vertex of the multigraph obtained above by a copy of $K_{r,r-1}$, connecting the r $(r-1)$-valant vertices of $K_{r,r-1}$ with the r edges incident with the substituted vertex.

Let G_2 denote the graph obtained from a graph G by the above steps. Meredith proved that G_2 is r-regular, r-connected. G_2 is Hamiltonian, if and only if G is. A graph G , to which step (i) is applicable is called r-good. Meredith showed that the Petersen graph is r-good for all $r \geq 4$. Since the Petersen graph is not Hamiltonian, all r-regular r-connected graphs obtained by the above construction are not Hamiltonian. It is easy to obtain, in each of these graphs, a set of $10r - 10$ vertices that do not lie on a common cycle. Jackson and Parsons [5], [6], in their study of longest cycles in r-regular, r-connected graphs, modified Meredith's construction. They pointed out that any non-Hamiltonian graph G , that is r-good may be used. They proved that all cubic 3-connected graphs are r-good for $r \geq 4$. They also point out, that one is not restricted to use $K_{r-1,r}$ in step (ii). Actually any r-regular, r-connected graph H , for which there is a vertex $h \in V(H)$ such that $H \backslash \{h\}$ cannot be covered by two disjoint paths with endpoints in $N(h)$ can be used in step (ii). All these constructions are based on non-

Hamiltonian graphs, and they do not necessarily yield, infinitely many r-regular r-connected graphs with cyclability $10r - 11$.

3. Construction of r-regular r-connected graphs

In this section, we describe a modification of Meredith's construction. This modification allows a much greater flexibility in choosing the underlying graph (we do not have to start with a non-Hamiltonian graph). We assume that $r \equiv 0 \pmod 2$.

Let G be an r-regular, r-connected graph. Let $e_i = (g_{2i-1}, g_i)$, $i = 1, \ldots, r/2$, be disjoint edges of G. Let H_r be any r-regular, r-connected graph, $G \cap H_r = \emptyset$. Let $N(y) = \{y_1, \ldots, y_r\}$, $y \in V(H_r)$ be the neighbors of y in H_r. We say that the graph

$$F = (G \backslash \{e_1, \ldots, e_{r/2}\}) \cup (H_r \backslash \{y\}) \cup \{(g_j, y_j) \mid j = 1, \ldots, r\}$$

is obtained from G by replacing the edges $\{e_1, \ldots, e_{r/2}\}$ by H_r. (Observe that many graphs F can be produced by a fixed choice of $\{e_1, \ldots, e_{r/2}\}$ and H_r, we still have the freedom of choosing $y \in V(H_r)$ and ordering $N(y)$.)

LEMMA 1. *Let G be an r-regular, r-connected graph. Let F be obtained from G by replacing the $r/2$ disjoint edges $\{e_1, \ldots, e_{r/2}\}$ by the r-regular r-connected graph H_r. Then F is an r-regular, r-connected graph.*

Proof. The proof is essentially similar to the proof of Lemma 4 in Rosenfeld [9]; we omit the details.

To obtain our construction, we add to Meredith's construction the following step:

(iii) replace $\{e_1, \ldots, e_{r/2}\}$ by H_r.

Let G and G' be the graphs in Figure 1 (see p. 4). If $r = 4m$, let G_r be the multigraph obtained from G by replacing the edges of the 1-factor A_iB_i, $i = 0, 1, 2$, by $2m$ parallel edges each and all other edges by m parallel edges each. If $r = 4m + 2$, let G_r be the

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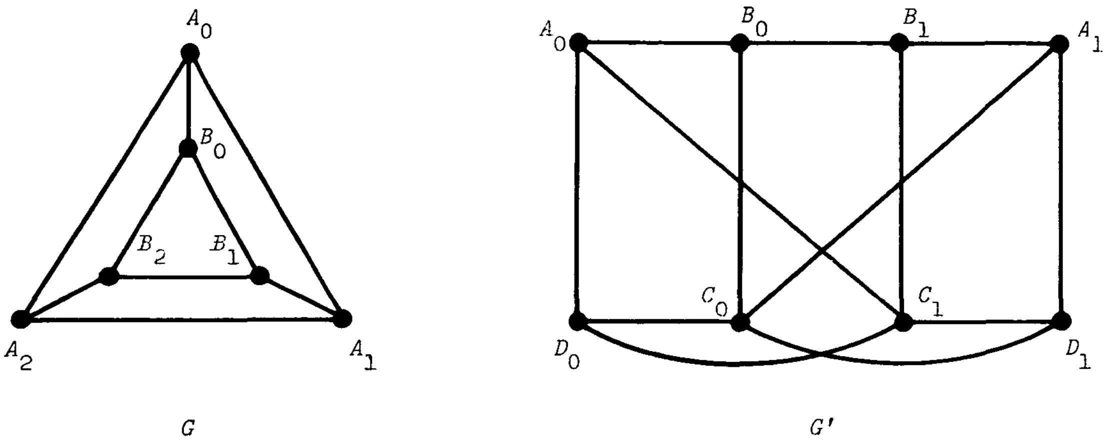

FIGURE 1

multigraph obtained from G' by replacing A_0C_1, A_1C_0, C_0D_1, and C_1D_0 by m parallel edges each, A_0D_0 and A_1D_1 by $m+1$ parallel edges each, B_0B_1 by $2m$ parallel edges and A_iB_i and C_iD_i, $i = 0, 1$ by $2m+1$ parallel edges each.

LEMMA 2. *G_r is r-edge-connected.*

Proof. The lemma is proved by showing that any pair of vertices of G_r are connected by r edge-disjoint paths.

CASE 1. $r = 4m$. The pairs of vertices are:

(1) A_iB_i, $i = 0, 1, 2$,

(2) A_iB_j

(3) A_iA_j, B_iB_j } , $i \neq j$, $i, j = 0, 1, 2$.

By symmetry, it suffices to consider one example from each case.

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Case	Edge	Types of Paths	Number of Paths
1	A_0B_0,	A_0B_0, $A_0A_1B_1B_0$, $A_0A_2B_2B_0$	$2m$, m, m
2	A_0B_1	$A_0B_0B_1$, $A_0B_0B_2B_1$, $A_0A_1B_1$, $A_0A_2A_1B_1$	m, m, m, m
3	A_0A_1	$A_0B_0B_1A_1$, $A_0B_0B_2B_1A_1$, $A_0A_2A_1$, A_0A_1	m, m, m, m

CASE 2. $r = 4m + 2$. The pairs of vertices are:

(1) A_iB_i ,
(2) A_iC_{i+1} ,
(3) A_iD_i ,
(4) B_0B_1 ,
(5) B_iC_i ,
(6) C_iD_{i+1} ,
(7) C_iD_i ,
(8) A_iB_{i+1} ,
(9) A_0A_1 ,
(10) A_iC_i ,
(11) A_iD_{i+1} , $i \equiv 0, 1 \pmod 2$,
(12) B_iC_{i+1} ,
(13) B_iD_i ,
(14) B_iD_{i+1} ,
(15) C_0C_1 ,
(16) D_0D_1 .

Again, by symmetry, it suffices to consider one example from each case.

Case	Edge	Types of Paths	Number of Paths
1	A_0B_0	A_0B_0, $A_0D_0C_0A_1B_1B_0$, $A_0C_1D_1A_1B_1B_0$	$2m+1$, m, m
		$A_0D_0C_0B_0$	1
2	A_0C_1	$A_0B_0B_1A_1D_1C_1$, A_0C_1, $A_0D_0C_1$, $A_0B_0B_1A_1C_0D_1C_1$	$m+1$, m, m, $m-1$
		$A_0D_0C_0A_1B_1C_1$, $A_0B_0C_0D_1C_1$	1, 1
3	A_0D_0	A_0D_0, $A_0C_1D_0$, $A_0B_0B_1A_1C_0D_0$	$m+1$, m, m
		$A_0B_0B_1A_1D_1C_0D_0$, $A_0B_0C_0D_0$	m, 1

Case	Edge	Types of Paths	Number of Paths
4	B_0B_1	B_0B_1, $B_0A_0D_0C_0A_1B_1$, $B_0A_0C_1D_1A_1B_1$	$2m$, m, m
		$B_0A_0D_0C_1B_1$, $B_0C_0D_1A_1B_1$	1, 1
5	B_0C_0	$B_0A_0D_0C_0$, $B_0B_1A_1C_0$, $B_0A_0C_1D_0C_0$	m+1, m, m
		$B_0B_1A_1D_1C_0$, B_0C_0	m, 1
6	C_0D_1	$C_0D_0A_0B_0B_1A_1D_1$, $C_0D_0C_1D_1$, C_0D_1	m+1, m, m
		$C_0A_1B_1B_0A_0C_1D_1$, $C_0B_0A_0C_1D_1$, $C_0A_1B_1C_1D_1$	m−1, 1, 1
7	C_0D_0	C_0D_0, $C_0D_1C_1D_0$, $C_0A_1B_1B_0A_0D_0$, $C_0B_0A_0D_0$	$2m$+1, m, m, 1
8	A_0B_1	$A_0B_0B_1$, $A_0C_1D_1A_1B_1$, $A_0D_0C_0A_1B_1$	$2m$, m, m
		$A_0B_0C_0D_1C_1B_1$, $A_0D_0C_0D_1A_1B_1$	1, 1
9	A_0A_1	$A_0B_0B_1A_1$, $A_0D_0C_0A_1$, $A_0C_1D_1A_1$	$2m$, m, m
		$A_0B_0C_0D_1A_1$, $A_0D_0C_1B_1A_1$	1, 1
10	A_0C_0	$A_0D_0C_0$, $A_0B_0B_1A_1C_0$, $A_0B_0B_1A_1D_1C_0$	m+1, m, m
		$A_0C_1D_0C_0$, $A_0B_0C_0$	m, 1
11	A_0D_1	$A_0B_0B_1A_1D_1$, $A_0B_0B_1A_1C_0D_1$, $A_0B_0C_0D_1$	m+1, m−1, 1
		$A_0C_1D_1$, $A_0D_0C_1D_1$, $A_0D_0C_0A_1B_1C_1D_1$	m, m, 1
12	B_0C_1	$B_0B_1A_1D_1C_1$, $B_0B_1A_1C_0D_1C_1$, $B_0A_0C_1$	m+1, m−1, m
		$B_0A_0D_0C_1$, $B_0C_0D_1C_1$, $B_0A_0D_0C_0A_1B_1C_1$	m, 1, 1
13	B_0D_0	$B_0A_0D_0$, $B_0A_0C_1D_0$, $B_0C_0D_0$	m+1, m, 1
		$B_0B_1A_1C_0D_0$, $B_0B_1A_1D_1C_0D_0$	m, m
14	B_0D_1	$B_0B_1A_1D_1$, $B_0B_1A_1C_0D_1$, $B_0A_0C_1D_1$	m+1, m−1, m
		$B_0A_0D_0C_1D_1$, $B_0C_0D_1$, $B_0A_0D_0C_0A_1B_1C_1D_1$	m, 1, 1
15	C_0C_1	$C_0D_1C_1$, $C_0A_1D_1C_1$, $C_0D_0A_0C_1$	m, m, m
		$C_0D_0C_1$, $C_0B_0B_1C_1$, $C_0D_0A_0B_0B_1A_1D_1C_1$	m, 1, 1

Case	Edge	Types of Paths	Number of Paths
16	D_0D_1	$D_0C_1D_1$, $D_0C_0D_1$, $D_0C_0A_1D_1$, $D_0A_0C_1D_1$	m, m, m, m
		$D_0C_0B_0B_1A_1D_1$, $D_0A_0B_0B_1C_1D_1$	1, 1

By the terminology of the previous section, G is r-good for all $r \equiv 0 \pmod 4$ and G' is r-good for all $r \equiv 2 \pmod 4$.

THEOREM 1. *For* $r \geq 4$, $r \equiv 0 \pmod 4$ *there are infinitely many r-regular r-connected graphs with cyclability not greater than* $6r - 4$.

Proof. For each value of r we first construct an infinite family of r-regular, r-connected graphs as follows: let G_r be the multigraph described in Lemma 2, Case 1. Let G'_r be the graph obtained from G_r after applying step (ii) of Meredith's construction. By Meredith's theorem, G'_r is an r-regular r-connected graph. Let G^*_r be a graph obtained from G'_r by replacing the $r/2$ disjoint edges in G'_r, corresponding to the edges A_iB_i, $i = 0, 1, 2$, in G, by an arbitrary r-regular, r-connected graph H_r. (Figure 2, p. 8, shows a graph G^*_4, with 54 vertices, obtained from G'_4 by using K_5 for replacing the indicated edges.) By Lemma 1, G^*_r is r-regular and r-connected.

Let $S_r \subseteq V(G^*_r)$ contain the $r - 1$ vertices of the smaller color class of each copy of the six $K_{r-1,r}$'s used in step (ii) and also contain one vertex from each of the three replacement graphs H_r.

CLAIM. No cycle in G^*_r contains S_r. Assume that such a cycle C, does exist. Since C contains all the vertices in $\{S_1, \ldots, S_{r-1}\}$ (Figure 3, p. 9), it must contain all vertices of the corresponding $K_{r-1,r}$. Therefore, C can use exactly two edges from the set $\{e_1, \ldots, e_k, f_1, \ldots, f_l\}$. Similarly, C uses exactly two edges from the set $\{e'_1, \ldots, e'_k, f'_1, \ldots, f'_l\}$.

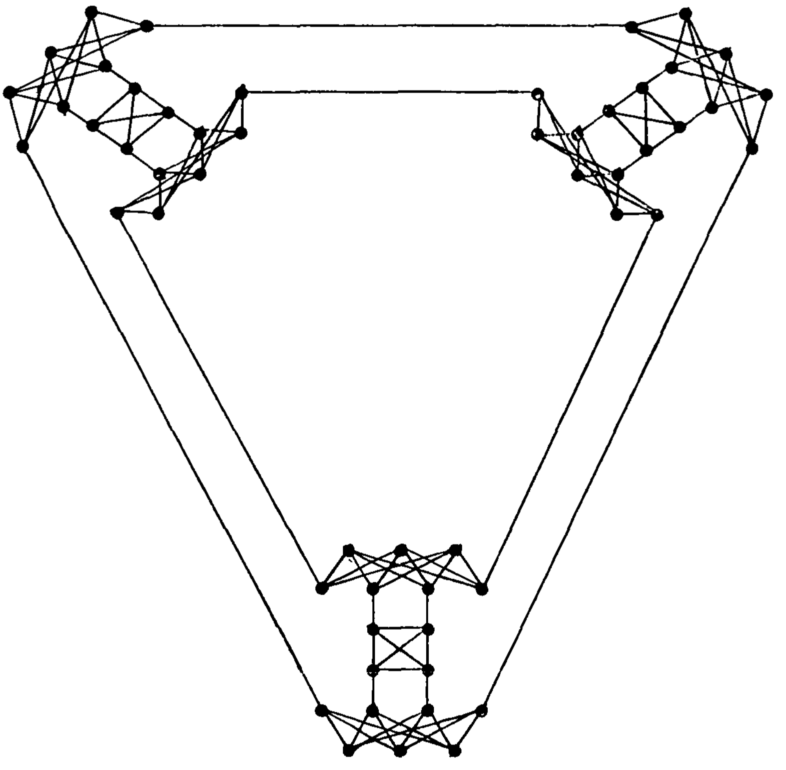

FIGURE 2

Since a vertex of H_r is contained in C, C must contain at least two edges from $\{e_1, \ldots, e_k, e'_1, \ldots, e'_k\}$. If C has two edges from $\{e_1, \ldots, e_k\}$, it cannot contain any edge from $\{f_1, \ldots, f_l\}$. Therefore C has either zero or two edges from $\{e'_1, \ldots, e'_k\}$ and connot contain any edge from the set $\{f'_1, \ldots, f'_l\}$. But then C cannot contain any vertices outside the configuration in Figure 3. It follows that C will have to contain one edge from each of $\{e_1, \ldots, e_k\}$, $\{e'_1, \ldots, e'_k\}$, $\{f_1, \ldots, f_l\}$ and $\{f'_1, \ldots, f'_l\}$.

Contraction of each copy of $K_{r-1,r}$ to a single vertex would yield a cycle in G that contains the three edges A_0B_0, A_1B_1, and A_2B_2. It is easy to see that such a cycle does not exist. This proves our claim. Since $|S_r| = 6(r-1) + 3$, the cyclability of each graph G^*_r thus obtained

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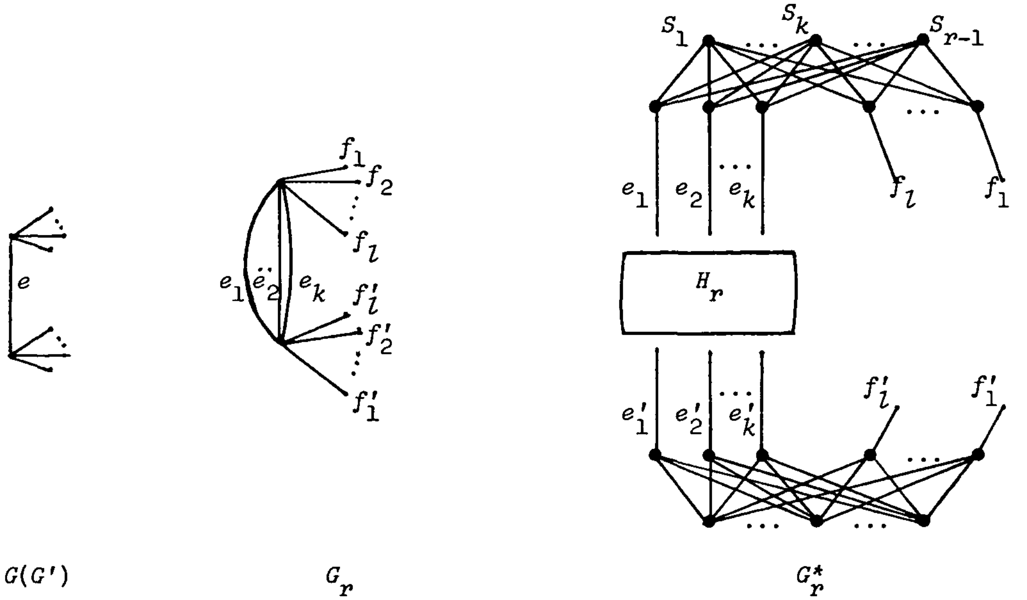

FIGURE 3

is at most $6r - 4$ as claimed.

THEOREM 2. *For* $r \geq 6$, $r \equiv 2 \pmod 4$ *there are infinitely many* r*-regular* r*-connected graphs with cyclability not greater than* $8r - 5$.

Proof. Let G_r be the multigraph described in Lemma 2, case 2. Let G_r' be obtained from G_r by applying step (ii) of Meredith's construction to it, and let G_r^* be obtained from G_r' by replacing the $r/2$ disjoint edges of G_r' , corresponding to the edges A_0B_0, A_1B_1, C_0D_0 , and C_1D_1 in G' by arbitrary copies of r-regular, r-connected graphs H_r . For the set $S_r \subseteq V(G_r^*)$, containing the $8(r-1)$ vertices of the smaller color class of each $K_{r-1,r}$ and a single vertex from each of the four graphs H_r , an identical argument to the proof of Theorem 1, shows that any cycle in G_r^* containing S_r , will yield a cycle in G' containing the edges A_0B_0, A_1B_1, C_0D_0 , and C_1D_1 . Since such a cycle does not exist, S_r is not contained in a cycle. Hence the cyclability of G_r^* is at most $8r - 5$ as claimed.

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4. Concluding remarks

The modification of Meredith's construction enables us to construct many r-regular, r-connected graphs with prescribed properties. The basic idea is to start with an r-good graph in which some edges are not contained in a cycle. In Figure 4 a 4-connected, 4-regular non-Hamiltonian bipartite graph with 84 vertices, is described. This graph, based on the Möbius ladder, uses the fact that no cycle of the Möbius 4-ladder uses the four "vertical edges".

We believe that the upper bound for the cyclability of r-regular r-connected graphs can be further improved by other choices of graphs. It seems though that another idea for odd r is needed.

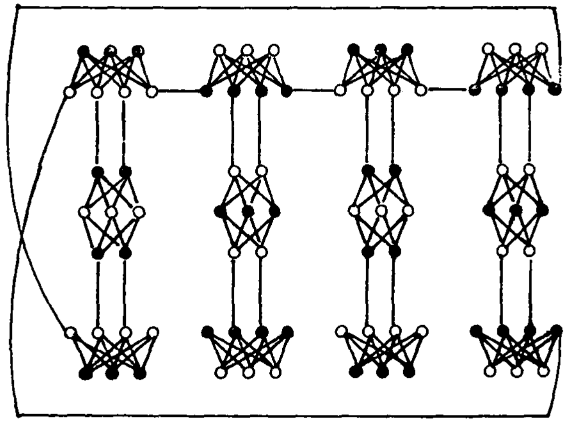

FIGURE 4

References

[1] J.A. Bondy and U.S.R. Murty, *Graph theory with applications* (American Elsevier, New York, 1976).

[2] Vaclav Chvátal, "New directions in Hamiltonian graph theory", *New directions in the theory of graphs*, 65-95 (Proc. Third Ann Arbor Conf. on Graph Theory, University of Michigan, Ann Arbor, Michigan, 1971. Academic Press, New York, 1973).

[3] D.A. Holton, "Cycles through specified vertices in k-connected regular graphs", *Ars Combin.* **13** (1982), 129-143.

[4] D.A. Holton, B.D. McKay, M.D. Plummer and C. Thomassen, "A nine point theorem for 3-connected graphs", *Combinatoria* **2** (1982), 53-62.

[5] Brad Jackson and T.D. Parsons, "On r-regular r-connected non-Hamiltonian graphs", *Bull. Austral. Math. Soc.* **24** (1981), 205-220.

[6] Brad Jackson and T.D. Parsons, "A shortness exponent for r-regular r-connected graphs", *J. Graph Theory* **6** (1982), 169-176.

[7] A.K. Kelmans and M.V. Lomonosov, "When m vertices in a k-connected graph cannot be walked round along a simple cycle", *Discrete Math.* **38** (1982), 317-322.

[8] G.H.J. Meredith, "Regular n-valent n-connected non-Hamiltonian non-n-edge colourable graphs", *J. Combin. Theory Ser. B* **14** (1973), 55-60.

[9] M. Rosenfeld, "Are all simple 4-polytopes Hamiltonian?", *Israel J. Math.* (to appear).

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