EDGE-REALIZABLE GRAPHS WITH UNIVERSAL VERTICES

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All graphs considered in this article are finite connected, without loops and multiple edges. Let G be a graph and x be a vertex. The vertex neighbourhood graph (or v-neighbourhood) of x in G (denoted by $N^v_G(x)$) is the subgraph of G induced by the set of all vertices of G adjacent to x. Analogously if $f = xy$ is any edge of G, the edge neighbourhood graph (or e-neighbourhood) of f in G is the subgraph of G (denoted $N^e_G(f)$ or $N^e_G(xy)$) induced by the set of all vertices of G which are adjacent to at least one vertex of the pair x, y and are different from x, y.

Zelinka [**6**] proposed the edge neighbourhood version of the well-known Zykov's problem [**7**] (concerning v-neighbourhoods) in the following way.

Problem. Characterize the graphs H with the property that there exists a graph G such that $N^e_G(f) \cong H$ for each edge f of G.

A graph H with the property mentioned above is called *e-realizable* and G is called the *e-realization* of H (or *v-realizable* and *v-realization* in the v-neighbourhood version).

Zelinka [**6**] and others ([**1**], [**2**], [**5**]) studied some families of e-realizable graphs. Hell [**4**] proved the following result.

Theorem 1. (P. Hell) *If H has n universal vertices, then H is not v-realizable unless $H = K_n + H'[K_{n+1}]$ for a v-realizable graph H' without universal vertices.*

By a universal vertex of H we mean a vertex which is adjacent to all other vertices of H; $+$ denotes Zykov's sum and $F[G]$ denotes the lexicographic product [**3**, p. 21]. A graph induced by the vertex set $\{x_1, x_2, \ldots, x_n\}$ will be denoted by $\langle x_1, x_2, \ldots, x_n\rangle$.

We will prove the e-neighbourhood version of Theorem 1.

Theorem 2. *Let a graph H with $n \geqq 3$ vertices contain at least one universal vertex, and let G be an e-realization of H. Then each edge of G is incident to a vertex of degree n or $n+1$ and G has exactly $n+2$ vertices.*

Proof. Let $N^e_G(y_1y_2) = \langle x_1, x_2, \ldots, x_n\rangle$ be the e-neighbourhood of an arbitrary edge $e = y_1y_2$. If $x_1, x_2, \ldots, x_k$ $(1 \leqq k \leqq n)$ are universal vertices of $N^e_G(y_1y_2)$ and x_1 is adjacent to y_2 then $N^e_G(x_1y_2) = \langle y_1, x_2, \ldots, x_n\rangle$ contains some universal vertex different from $x_2, \ldots, x_k$. Since the vertices $x_{k+1}, \ldots, x_n$ are not universal in $N^e_G(y_1y_2)$ (if $k < n$) they also cannot be universal in $N^e_G(x_1y_2)$. Thus in any case y_1 is the universal vertex in $N^e_G(x_1y_2)$ and it is of degree at least n in G.

It is clear that x_1 cannot be adjacent to any other vertex z different from $y_1, y_2, x_2, \ldots, x_n$; for in this case $N^e_G(x_1y_2)$ contains at least $n+1$ vertices $y_1, x_2, \ldots, x_n, z$. Analogously, because y_1 is adjacent to $y_2, x_2, \ldots, x_n$, none of these vertices can be adjacent to any other vertex z (for in this case $N^e_G(x_1x_i)$, for some $i \geqq 2$, contains $n+1$ vertices $y_1, y_2, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$, which is a contradiction). Hence G has $n+2$ vertices. □

Let $G = \langle x_1, x_2, \ldots, x_{n+2}\rangle$ and H be graphs as in Theorem 2, with p and q edges, respectively. Then the neighbourhood of any edge x_ix_j contains all other vertices of G and

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q edges, and it is evident that

$$\deg x_i + \deg x_j = p - q + 1 = \text{constant}$$

for each pair of mutually adjacent vertices x_i, x_j. If G contains a vertex x_1 of degree $n+1$ then all the other vertices must be of the same degree r, and, for each pair of mutually adjacent vertices x_i, x_j $(i, j > 1)$ we have

$$2r = \deg x_i + \deg x_j = \deg x_1 + \deg x_i = n + 1 + r.$$

Thus $r = n+1$ and G is isomorphic to K_{n+2}.

If the maximal degree of G is n, and x_1, of degree n, is adjacent to $x_2, \ldots, x_{n+1}$, then $\deg x_2 = \ldots = \deg x_{n+1} = r$. In addition, because x_{n+2} is adjacent to some x_i $(i > 1)$, it is clear that $\deg x_{n+2} = n$.

Now let there exist an edge x_ix_j $(i, j \neq 1, n+2)$. Then by a similar argument to the above, $\deg x_2 = \ldots = \deg x_{n+1} = n$ and G is regular of degree n. Thus n is an even number and G is isomorphic to $K_{n+2} - \dfrac{n+2}{2} K_2$.

Finally if there exists no such edge x_ix_j then G is isomorphic to $K_{2,n}$. Thus we have proved the following result.

THEOREM 3. *If a graph H contains universal vertices then H is not e-realizable unless*

(i) $H \cong K_{1,n}$,

(ii) $H \cong K_{1,1,2,..,2}$

or

(iii) $H \cong K_n$.

Note that, in comparison, the graph K_n is v-realizable while $K_{1,n}$ (for $n > 1$) and $K_{1,1,2,\ldots,2}$ are not v-realizable.

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