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RESEARCH ARTICLE

Inhomogeneous isotropic quantum spin chain associated with the difference Lamé equation

Jan Felipe van Diejen

Instituto de Matemáticas, Universidad de Talca, Casilla 747, Talca, Chile; E-mail: diejen@utalca.cl.

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Abstract

The spectrum and orthogonal eigenbasis are computed of a tridiagonal matrix encoding a finite-dimensional reduction of the difference Lamé equation at the single-gap integral value of the coupling parameter. This entails the exact solution, in terms of single-gap difference Lamé wave functions, for the spectral problem of a corresponding open inhomogeneous isotropic XY chain with coupling constants built from elliptic integers.

Contents

1. Introduction

The homogeneous XY model is an exactly solvable quantum spin chain going back to the pioneering work of Lieb, Schultz and Mattis [LSM61]. Its inhomogeneous isotropic variants, often referred to as XX chains in the literature, serve as models for the transfer of qubit states through quantum wires [B07, BV17, K10]. The computation of the spectrum and eigenstates of such isotropic XY chains is achieved through the diagonalization of the underlying Jacobi matrix of coupling constants (representing the one-particle Hamiltonian), cf., for example, [A-E04, CV10, HSS12]. It is well known, cf., for example, [W78, Chapter 2], that both the eigenbasis and spectrum of a Jacobi matrix can be conveniently computed by means of orthogonal polynomials and their zero loci. For particular Jacobi matrices stemming from the hypergeometric orthogonal polynomials in Askey's scheme [KLS10] the construction of the spectrum via the zero locus becomes explicit, thus giving rise to a rich family of corresponding inhomogeneous isotropic XY chains for which spectrum and eigenstates can be determined in closed form [A-E04, BV17, CV10, CNV19, FG20, GVZ16, GVZ13, JV10, SV11, VZ12].

Recently, the method of orthogonal polynomials was employed to construct the eigenvectors of a tridiagonal matrix obtained by truncating a discrete variant of the difference Lamé operator [DG21], but detailed insight into the pertinent zero locus yielding the spectrum has unfortunately remained somewhat elusive so far. The difference Lamé operator itself had emerged previously as a rank-one elliptic quantum Ruijsenaars–Schneider Hamiltonian [R90, R99a] that turns out to be deeply connected to both the representation theory of the Sklyanin algebra [KZ95, R04, S83] and to the representation theory of elliptic quantum groups [FV96]. An in-depth spectral analysis of the difference Lamé equation was performed in [R99b] for a dense parameter regime of positive values for the coupling parameter. Like in the case of the classical Lamé differential equation, at the single-gap integral value of the coupling parameter this spectral analysis simplifies and the eigensolutions can be expressed compactly in terms of Jacobi theta functions [R99c]. This opens the way to achieve the main goal of this note: To compute the spectrum of the truncated discrete Lamé equation of [DG21] at the single-gap integral value of the coupling parameter and therewith solve the spectral problem for the isotropic XY chain associated with the corresponding Jacobi matrix.

Specifically, we will consider an open chain of m quantum spins placed on the finite integer lattice $\{1, 2, \ldots, m\}$ that is characterized by a Hamiltonian of the form

$$\boldsymbol{H}^{(m)} = \frac{1}{2}\sum_{l=1}^{m-1} \mathrm{J}_l \Bigl(\sigma_l^{\mathrm{x}}\sigma_{l+1}^{\mathrm{x}} + \sigma_l^{\mathrm{y}}\sigma_{l+1}^{\mathrm{y}}\Bigr) \quad \text{with} \quad \mathrm{J}_l = \sqrt{\frac{\vartheta_1(\frac{\alpha}{2}l;p)\,\vartheta_1(\frac{\alpha}{2}(m-l);p)}{\vartheta_1(\frac{\alpha}{2}(l+g);p)\,\vartheta_1(\frac{\alpha}{2}(m-l+g);p)}},$$

where $g > 0$ and $\alpha = \frac{2\pi}{2g+m-1}$. Here, $\vartheta_1(\cdot;p)$ refers to Jacobi's theta function (2.3) and σ_l^{x}, σ_l^{y} denote the corresponding local spin-$\frac{1}{2}$ operators at site l (cf. Equations (4.3a), (4.3b)). Below, the spectrum and eigenfunctions of this open inhomogeneous isotropic XY model will be computed at the single-gap value $g = 2$ (with $0 < p < 1$) and also in the trigonometric limit $p \to 0$ for arbitrary parameter values $g > 0$. The rational limit ($g \to +\infty$) recovers the spin couplings of the so-called Krawtchouk chain: $\mathrm{J}_l \to \sqrt{l(m-l)}$. The latter spin chain has been thoroughly studied in the literature as a model for perfect state transfer, cf., for example, [A-E04, B07, BV17, CV10, GS18, K10, NPL03, VZ12]. As an isotropic XY Hamiltonian $\boldsymbol{H}^{(m)}$ only exhibits nearest neighbor couplings, which contrasts with the long-range one-dimensional spin models with elliptic couplings found previously by Inozemtsev, cf. [I23, Chapter 3] and [KL22] (and references therein). Let us recall at this point that the two-magnon wave functions for Inozemtsev's spin chain are given by single-gap Lamé functions, while more generally the n-magnon wave functions are given by eigenfunctions of the corresponding elliptic quantum Calogero–Moser model.

The material is organized as follows. In Section 2, the difference Lamé equation is recalled together with the single-gap difference Lamé wave function stemming from [R99c]. At this point, it is pertinent to emphasize that the difference Lamé equation admits various nonequivalent real forms (i.e., Hilbert space formulations) each giving rise to a corresponding spectral theory. In terms of the classification originating from [R90]: Here, we are dealing with an example of the (rank-one) *compactified* elliptic quantum Ruijsenaars–Schneider model whereas [R99c] concentrates rather on the more conventional (but at the same time very intricate) *noncompact* variant(s). As a consequence, the spectral analysis in [R99c] does not apply directly to our situation and needs to be adapted. In Section 3, a finite-dimensional system of real solutions of the single-gap difference Lamé equation is isolated. These solutions are both smooth and periodic on the real axis. Upon scaling the period of the elliptic functions such that the difference Lamé equation reduces to a finite-dimensional tridiagonal eigenvalue problem [DG21], these real solutions provide a complete basis of orthogonal eigenvectors that give rise to explicit formulas

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for the corresponding eigenvalues. In Section 4, the open inhomogeneous isotropic *XY* quantum spin Hamiltonian $\boldsymbol{H}^{(m)}$ associated with the finite Jacobi matrix under consideration is studied. We compute the n-particle Hamiltonian and construct its eigenfunctions in terms of Slater determinants of single-gap difference Lamé wave functions. Finally, the discussion is closed by connecting the results on the quantum spin chain to those in the literature via trigonometric and rational limits. Readers primarily interested in the spin model are invited to skip straight to Section 4 and skim back over Sections 2 and 3 to pick up some notations and further essentials from the difference Lamé theory when needed.

2. Difference Lamé equation

2.1. *Elliptic numbers*

For

$$\boxed{0 < \alpha < \pi \quad \text{and} \quad 0 < p < 1}, \tag{2.1}$$

let us recall the definition of the *elliptic number* associated with $z \in \mathbb{C}$ (cf. [GR04, Section 1.6]):

$$[z] = [z; \alpha, p] := \frac{\vartheta_1(\frac{\alpha z}{2}; p)}{\vartheta_1(\frac{\alpha}{2}; p)}, \tag{2.2}$$

where ϑ_1 represents the Jacobi theta function [D-F23, Chapter 20]

$$\begin{aligned} \vartheta_1(z;p) &= 2\sum_{k=0}^{\infty}(-1)^k p^{\left(k+\frac{1}{2}\right)^2}\sin(2k+1)z \\ &= 2p^{1/4}\sin(z)\prod_{k=1}^{\infty}(1-p^{2k})(1-2p^{2k}\cos(2z)+p^{4k}). \end{aligned} \tag{2.3}$$

All zeros of $[z;\alpha,p]$ are simple, and their locus is given by the period lattice $\frac{2\pi}{\alpha}\left(\mathbb{Z}+\tau\mathbb{Z}\right)$ with $\tau := \frac{\log p}{\mathrm{i}\pi}$. We notice that the elliptic numbers are odd in z and quasi-periodic with respect to translations over the periods:

$$[-z] = -[z], \quad [z + \tfrac{2\pi}{\alpha}] = -[z], \quad [z + \tfrac{2\pi\tau}{\alpha}] = -\tfrac{1}{p}e^{-\mathrm{i}\alpha z}[z]. \tag{2.4}$$

For $\alpha \to 0$ and $p \to 0$, the elliptic numbers degenerate to ordinary complex numbers and their q-deformations, respectively:

$$\lim_{\alpha\to 0}[z;\alpha,p] = z \quad \text{and} \quad \lim_{p\to 0}[z;\alpha,p] = \frac{\sin(\frac{\alpha z}{2})}{\sin(\frac{\alpha}{2})} = \frac{q^{\frac{z}{2}} - q^{-\frac{z}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \text{ with } q := e^{\mathrm{i}\alpha}. \tag{2.5}$$

2.2. *Single-gap wave functions*

The difference Lamé equation is an eigenvalue problem for meromorphic functions $\psi(z)$ on $\mathbb{C}$ of the form [FV96, KZ95, R90, R99a, R99b]:

$$\frac{[z+g]}{[z]}\psi(z+1) + \frac{[z-g]}{[z]}\psi(z-1) = \mathrm{E}\psi(z), \tag{2.6}$$

where $g \in \mathbb{C}$ and $\mathrm{E} \in \mathbb{C}$ denote the coupling parameter and the eigenvalue, respectively. The following elementary solution of the difference Lamé equation at $g = 2$ can be readily gleaned from [R99c].

Proposition 2.1 (Wave Functions). *For $-\frac{2\pi}{\alpha} < \xi < \frac{2\pi}{\alpha}$ and $-\frac{\pi\tau}{i\alpha} < x < \frac{\pi\tau}{i\alpha}$, the wave function*

$$\psi(z;\xi,x) := \frac{[ix-z]\exp\left(\frac{iz}{2}\left(\frac{\alpha\xi}{2}-\pi\right)\right)}{[z+1][z][z-1]} \tag{2.7a}$$

provides a meromorphic solution of the difference Lamé equation (2.6) at the single-gap coupling value $g = 2$ with eigenvalue

$$\mathrm{E}(x) := \frac{[2][ix]}{i|[1+ix]|}, \tag{2.7b}$$

provided the position of the imaginary node ix is related to the real-valued spectral parameter ξ via the nonlinear constraint

$$\frac{[1+ix]}{[1-ix]} = \exp\left(\frac{i\alpha\xi}{2}\right). \tag{2.7c}$$

Proof. If the wave function $\psi(z;\xi,x)$ (2.7a) is substituted into the difference Lamé equation (2.6) with $g = 2$, then one arrives at following relation upon dividing by $\psi(z;\xi,x)$:

$$\begin{aligned}\mathrm{E} &= \frac{[z+2]}{[z]}\frac{\psi(z+1;\xi,x)}{\psi(z;\xi,x)} + \frac{[z-2]}{[z]}\frac{\psi(z-1;\xi,x)}{\psi(z;\xi,x)}\\ &= \frac{1}{i[z][ix-z]}\left([z-1][ix-z-1]e^{\frac{i\alpha\xi}{4}} - [z+1][ix-z+1]e^{-\frac{i\alpha\xi}{4}}\right).\end{aligned}$$

The right-hand side is an elliptic function of z with period lattice $\frac{2\pi}{\alpha}(\mathbb{Z}+\tau\mathbb{Z})$ that has at most simple poles congruent to $z = 0$ and $z = ix$. The constraint (2.7c) is seen to guarantee that the residues at $z = 0$ and $z = ix$ vanish, so the singularities are in fact removable and the right-hand side is thus a constant function of z. This verifies that $\psi(z;\xi,x)$ (2.7a) satisfies the Lamé equation with $g = 2$ provided Equation (2.7c) holds. To compute the corresponding eigenvalue it suffices to evaluate the right-hand side under consideration at $z = -1$:

$$\mathrm{E} = \frac{[2][ix]e^{\frac{i\alpha\xi}{4}}}{i[1][1+ix]},$$

which readily passes over into Equation (2.7b) when rewriting the exponential factor in terms of x with the aid of Equation (2.7c). □

2.3. *On the spectral parametrization of the node*

The relation stemming from the constraint (2.7c) between the value of the spectral parameter ξ and the position of the node ix (and therewith the eigenvalue $\mathrm{E}(x)$) is determined by the function

$$\mathrm{F}(x) := \frac{2}{i\alpha}\mathrm{Log}\left(\frac{[1+ix]}{[1-ix]}\right), \quad -\frac{\pi\tau}{i\alpha} \le x \le \frac{\pi\tau}{i\alpha}. \tag{2.8}$$

Proposition 2.2 (Monotonicity). *Upon varying the argument x over the interval $-\frac{\pi\tau}{i\alpha} \le x \le \frac{\pi\tau}{i\alpha}$, the functions $\mathrm{F}(x)$ (2.8) and $\mathrm{E}(x)$ (2.7b) strictly increase smoothly from $-\mathrm{F}(\frac{\pi\tau}{i\alpha})$ to $\mathrm{F}(\frac{\pi\tau}{i\alpha})$ and from $-\mathrm{E}(\frac{\pi\tau}{i\alpha})$ to $\mathrm{E}(\frac{\pi\tau}{i\alpha})$, respectively. Moreover, the extremal values are given explicitly by*

$$\boxed{\mathrm{F}\left(\frac{\pi\tau}{i\alpha}\right) = 2\left(\frac{\pi}{\alpha}-1\right)} \tag{2.9a}$$

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and

$$\boxed{\mathrm{E}\left(\tfrac{\pi\tau}{\mathrm{i}\alpha}\right) = \frac{[2]\left[\frac{\pi\tau}{\alpha}\right]}{\mathrm{i}\left|\left[1+\frac{\pi\tau}{\alpha}\right]\right|} = \frac{\mathrm{E}'(0)}{\sqrt{\wp\left(1;\frac{2\pi}{\alpha},\frac{2\pi\tau}{\alpha}\right)-\wp\left(\frac{\pi\tau}{\alpha};\frac{2\pi}{\alpha},\frac{2\pi\tau}{\alpha}\right)}},} \tag{2.9b}$$

with $\mathrm{E}'(0) = \frac{\alpha\vartheta_1(\alpha;p)\vartheta_1'(0;p)}{2\vartheta_1^2(\frac{\alpha}{2};p)}$ *and* $\vartheta_1'(0;p) = 2p^{1/4}\prod_{k=1}^{\infty}(1-p^{2k})^3$, *where* $\wp(z;2\omega,2\tilde{\omega})$ *denotes the Weierstrass* $\wp$*-function associated with half-periods* $\omega = \frac{\pi}{\alpha}$ *and* $\tilde{\omega} = \frac{\pi\tau}{\alpha}$ *[D-F23, Chapter 23].*

Proof. For x on the real axis the quotient $\frac{[1+\mathrm{i}x]}{[1-\mathrm{i}x]}$ belongs to the unit circle, so the function $\mathrm{F}(x)$ is real-valued and odd. The value at $x = \frac{\pi\tau}{\mathrm{i}\alpha}$ can be computed via the quasi-periodicity (2.4):

$$\mathrm{F}\left(\tfrac{\pi\tau}{\mathrm{i}\alpha}\right) = \frac{2}{\mathrm{i}\alpha}\mathrm{Log}\left(\frac{\left[1+\frac{\pi\tau}{\alpha}\right]}{\left[1-\frac{\pi\tau}{\alpha}\right]}\right) = \frac{2}{\mathrm{i}\alpha}\mathrm{Log}\left(-\frac{1}{p}e^{-\mathrm{i}\alpha(1-\frac{\pi\tau}{\alpha})}\right) = \frac{2}{\alpha}(\pi-\alpha).$$

To analyze the monotonicity, we first compute the derivative of $\mathrm{F}(x)$ for $-\frac{\pi\tau}{\mathrm{i}\alpha} < x < \frac{\pi\tau}{\mathrm{i}\alpha}$ in terms of Jacobi theta functions:

$$\mathrm{F}'(x) = \frac{\vartheta_1'\left(\frac{\alpha}{2}+\frac{\mathrm{i}\alpha x}{2};p\right)}{\vartheta_1\left(\frac{\alpha}{2}+\frac{\mathrm{i}\alpha x}{2};p\right)} + \frac{\vartheta_1'\left(\frac{\alpha}{2}-\frac{\mathrm{i}\alpha x}{2};p\right)}{\vartheta_1\left(\frac{\alpha}{2}-\frac{\mathrm{i}\alpha x}{2};p\right)}, \tag{2.10}$$

which is smooth because the zero loci of the denominators are avoided for x on the real axis. The product representation of the Jacobi theta function (2.3) entails the following series for the logarithmic derivative of the Jacobi theta function [D-F23, (20.5.10)]

$$\frac{\vartheta_1'(z;p)}{\vartheta_1(z;p)} = \cot(z) + 4\sin(2z)\sum_{k=1}^{\infty}\frac{p^{2k}}{1-2p^{2k}\cos(2z)+p^{4k}}, \tag{2.11}$$

which converges uniformly on compacts within the strip $|\mathrm{Im}(z)| < \pi\,\mathrm{Im}(\tau) = -\mathrm{Log}\,(p)$. Upon plugging Equation (2.11) into Equation (2.10), one sees that the derivative of F remains positive within the interval $-\frac{\pi\tau}{\mathrm{i}\alpha} < x < \frac{\pi\tau}{\mathrm{i}\alpha}$:

$$\mathrm{F}'(x) = \frac{\sin\alpha}{|\sin\frac{\alpha}{2}(1+\mathrm{i}x)|^2} + 8\sin\alpha\sum_{k=1}^{\infty}\frac{p^{2k}\left((1+p^{4k})\cosh(\alpha x)-2p^{2k}\cos\alpha\right)}{|1-2p^{2k}\cos\alpha(x+\mathrm{i}x)+p^{4k}|^2} > 0$$

because $(1+p^{4k})\cosh(\alpha x)-2p^{2k}\cos\alpha \geq 1+p^{4k}-2p^{2k} = (1-p^{2k})^2 > 0$ and $\sin\alpha > 0$ for $0<\alpha<\pi$.

Regarding the eigenvalue, it is clear that $\mathrm{E}(x)$ (2.7b) constitutes a smooth real-valued function of $x \in \mathbb{R}$ that is odd, while at the same time being strictly increasing in a neighborhood of $x = 0$:

$$\mathrm{E}'(0) = \lim_{x\to 0}\frac{\mathrm{E}(x)}{x} = [2]\lim_{x\to 0}\frac{[\mathrm{i}x]}{\mathrm{i}x} = \frac{\alpha\vartheta_1(\alpha;p)\vartheta_1'(0;p)}{2\vartheta_1^2(\frac{\alpha}{2};p)} > 0.$$

To justify the asserted monotonicity globally in the interval $-\frac{\pi\tau}{\mathrm{i}\alpha} < x < \frac{\pi\tau}{\mathrm{i}\alpha}$, it therefore suffices to check that

$$\frac{1}{\mathrm{E}^2(x)} = \frac{1}{[2]^2}\frac{[1+\mathrm{i}x][1-\mathrm{i}x]}{[\mathrm{i}x][-\mathrm{i}x]}$$

is strictly decreasing for $0 < x < \frac{\pi\tau}{\mathrm{i}\alpha}$. To this end, we notice that

$$\frac{[1+z][1-z]}{[z][-z]} = \left(\frac{2\vartheta_1(\frac{\alpha}{2};p)}{\alpha\vartheta_1'(0;p)}\right)^2\left(\wp\left(1;\tfrac{2\pi}{\alpha},\tfrac{2\pi\tau}{\alpha}\right)-\wp\left(z;\tfrac{2\pi}{\alpha},\tfrac{2\pi\tau}{\alpha}\right)\right). \tag{2.12}$$

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Indeed, the left-hand side of Equation (2.12) defines an elliptic function of order two in z with period lattice $\frac{2\pi}{\alpha}(\mathbb{Z}+\tau\mathbb{Z})$; it has zeros congruent to $z=1$ and $z=-1$ and a double pole congruent to $z=0$ with $\lim_{z\to 0} z^2\frac{[1+z][1-z]}{[z][-z]} = -\Big(\frac{2\vartheta_1(\frac{\alpha}{2};p)}{\alpha\vartheta_1'(0;p)}\Big)^2$. When z moves from $z=0$ to $z=\frac{\pi\tau}{\alpha}$ along the imaginary axis, the Weierstrass $\wp$-function increases monotonically from $-\infty$ to $\wp(\frac{\pi\tau}{\alpha};\frac{2\pi}{\alpha},\frac{2\pi\tau}{\alpha}) < \wp(\frac{\pi}{\alpha};\frac{2\pi}{\alpha},\frac{2\pi\tau}{\alpha}) < \wp(1;\frac{2\pi}{\alpha},\frac{2\pi\tau}{\alpha})$ (cf., e.g., [L89, Chapter 6.11]). Hence, for $0 < x < \frac{\pi\tau}{\mathrm{i}\alpha}$ the value of $\frac{1}{\mathrm{E}^2(x)}$ decreases monotonically from $+\infty$ to $\frac{1}{[2]^2}\Big(\frac{2\vartheta_1(\frac{\alpha}{2};p)}{\alpha\vartheta_1'(0;p)}\Big)^2\Big(\wp(1;\frac{2\pi}{\alpha},\frac{2\pi\tau}{\alpha}) - \wp(\frac{\pi\tau}{\alpha};\frac{2\pi}{\alpha},\frac{2\pi\tau}{\alpha})\Big) > 0$. □

One learns from Proposition 2.2 that for any value of the spectral parameter $-2(\frac{\pi}{\alpha}-1)\le\xi\le 2(\frac{\pi}{\alpha}-1)$ there exists a unique $-\frac{\pi\tau}{\mathrm{i}\alpha}\le x(\xi)\le\frac{\pi\tau}{\mathrm{i}\alpha}$ such that

$$\frac{2}{\mathrm{i}\alpha}\mathrm{Log}\left(\frac{[1+\mathrm{i}x(\xi)]}{[1-\mathrm{i}x(\xi)]}\right) = \xi, \tag{2.13a}$$

that is,

$$\boxed{x(\xi) := \mathrm{F}^{-1}(\xi) \text{ for } -2(\tfrac{\pi}{\alpha}-1)\le\xi\le 2(\tfrac{\pi}{\alpha}-1).} \tag{2.13b}$$

When combining with Proposition 2.1, this entails the following family of solutions to the difference Lamé equation at $g=2$.

Corollary 2.3 (Parametrized wave functions). *For $-2(\frac{\pi}{\alpha}-1)<\xi<2(\frac{\pi}{\alpha}-1)$, the single-gap wave function $\psi\big(z;\xi,x(\xi)\big)$ provides a meromorphic solution to the difference Lamé equation (2.6) with $g=2$ and eigenvalue $-\mathrm{E}(\frac{\pi\tau}{\mathrm{i}\alpha}) < \mathrm{E}(x(\xi)) < \mathrm{E}(\frac{\pi\tau}{\mathrm{i}\alpha})$.*

Notice that in Corollary 2.3 the dependence of $x(\xi)$ (and thus of $\psi\big(z;\xi,x(\xi)\big)$) on the spectral parameter ξ is smooth in view of the inverse function theorem.

3. Finite-dimensional reduction

3.1. Smooth periodic wave functions

The wave function in Corollary 2.3 has simple poles in $\mathbb{C}$ congruent to $z=0$, $z=1$ and $z=-1$ (modulo $\frac{2\pi}{\alpha}(\mathbb{Z}+\tau\mathbb{Z})$). To get rid of the singularities at $z=0$ and $z=\pm 1$, we extract the even part of the wave function:

$$\begin{aligned}\phi(z;\xi,x) &:= \psi(z;\xi,x)+\psi(-z;\xi,x) \qquad (3.1)\\ &= \frac{[\mathrm{i}x-z]\exp\big(\frac{\mathrm{i}z}{2}(\frac{\alpha\xi}{2}-\pi)\big) - [\mathrm{i}x+z]\exp\big(-\frac{\mathrm{i}z}{2}(\frac{\alpha\xi}{2}-\pi)\big)}{[z+1][z][z-1]}.\end{aligned}$$

Notice that for z on the real axis $\phi(z;\xi,x) = 2\mathrm{Re}\big(\psi(z;\xi,x)\big)$. At discrete values of the spectral variable of the form

$$\boxed{\xi_k := 2(\tfrac{\pi}{\alpha}-1-k)\quad (k\in\mathbb{Z}),} \tag{3.2}$$

the even wave function $\phi\big(z;\xi,x(\xi)\big)$ becomes periodic or antiperiodic in z with period $\frac{2\pi}{\alpha}$ (depending on the parity of k), which gets rid of all singularities on the real axis.

Proposition 3.1 (Smooth periodic wave functions). (i) *For $-2(\frac{\pi}{\alpha}-1)<\xi<2(\frac{\pi}{\alpha}-1)$, the wave function $\phi\big(z;\xi,x(\xi)\big)$ solves the difference Lamé equation (2.6) with $g=2$ and eigenvalue $-\mathrm{E}(\frac{\pi\tau}{\mathrm{i}\alpha}) < \mathrm{E}\big(x(\xi)\big) < \mathrm{E}(\frac{\pi\tau}{\mathrm{i}\alpha})$.*

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(ii) *For any integer* $0 < k < 2(\frac{\pi}{\alpha} - 1)$*, the wave function* $\phi\big(z; \xi_k, x(\xi_k)\big)$ *is (anti)periodic in* z *with period* $\frac{2\pi}{\alpha}$*:*

$$\phi\big(z + \tfrac{2\pi}{\alpha}; \xi_k, x(\xi_k)\big) = (-1)^{k-1}\phi\big(z; \xi_k, x(\xi_k)\big). \tag{3.3}$$

(iii) *When restricting to the real axis, the (anti)periodic meromorphic wave function* $\phi\big(z; \xi_k, x(\xi_k)\big)$ *in part (ii) extends continuously to a smooth function of* $z \in \mathbb{R}$ *(which subsequently will be denoted by* $\phi\big(z; \xi_k, x(\xi_k)\big)$ *as well).*

Proof. (i) In the situation of Corollary 2.3, it is plain that for z on the real axis $\phi\big(z; \xi, x(\xi)\big) = 2\mathrm{Re}\big(\psi\big(z; \xi, x(\xi)\big)\big)$ solves the difference Lamé equation with $g = 2$ and eigenvalue $-\mathrm{E}(\frac{\pi\tau}{\mathrm{i}\alpha}) < \mathrm{E}(x(\xi)) < \mathrm{E}(\frac{\pi\tau}{\mathrm{i}\alpha})$ (because the imaginary parts of the coefficients of the difference equation vanish on the real axis). This real meromorphic solution $\phi\big(z; \xi, x(\xi)\big)$ extends in turn from the real axis to the complex plane by analyticity in z.

(ii) To ensure that $\phi\big(z; \xi, x(\xi)\big)$ is (anti)periodic in z with period $\frac{2\pi}{\alpha}$ it suffices to choose the spectral parameter ξ such that the exponential factor $\exp\big(\frac{\mathrm{i}z}{2}(\frac{\alpha\xi}{2} - \pi)\big)$ is (anti-)periodic. This is achieved for $\xi \in \frac{2\pi}{\alpha} + 2\mathbb{Z}$. The requirement in Corollary 2.3 that $-2(\frac{\pi}{\alpha} - 1) < \xi < 2(\frac{\pi}{\alpha} - 1)$ narrows this down to the spectral values ξ_k (3.2) with $0 < k < 2(\frac{\pi}{\alpha} - 1)$. The value of the sign follows from the observation that $\exp\big(\frac{\mathrm{i}z}{2}(\frac{\alpha\xi_k}{2} - \pi)\big) = (-1)^{k-1}$ at $z = \frac{2\pi}{\alpha}$.

(iii) The wave function $\phi\big(z; \xi, x(\xi)\big)$ has simple poles on the real axis arising from the denominator at $z = 0 \bmod \frac{2\pi}{\alpha}\mathbb{Z}$ and $z = \pm 1 \bmod \frac{2\pi}{\alpha}\mathbb{Z}$. Since $\phi\big(z; \xi, x(\xi)\big)$ is even in z, it is clear that its residue at $z = 0$ vanishes. The vanishing of the residues of $\phi\big(z; \xi, x(\xi)\big)$ at $z = \pm 1$ follows in turn via relation (2.7c). By picking the spectral parameter ξ such that $\phi\big(z; \xi, x(\xi)\big)$ is (anti-)periodic in z with period $\frac{2\pi}{\alpha}$, one guarantees that the residues of all poles on the real axis vanish. In other words, at these spectral values the wave function extends to a smooth function of $z \in \mathbb{R}$. □

3.2. *Truncated discretization*

Given $g > 0$ and $m \in \mathbb{Z}_{>1}$, we will from now on scale the periods by putting

$$\boxed{\alpha = \frac{2\pi}{2g + m - 1}.} \tag{3.4}$$

It was shown in [DG21] that then the difference Lamé equation can be truncated onto the space of complex functions supported on $\{g, g+1, \ldots, g+m-1\}$. Indeed, by substituting $z = g + l - 1$ in the difference Lamé equation (2.6) and writing $\boxed{\Psi_l := \psi(g + l - 1),}$ one arrives at a finitely truncated discrete Lamé equation of the form

$$\frac{[m - l]}{[g + m - l]}\Psi_{l+1} + \frac{[l - 1]}{[g + l - 1]}\Psi_{l-1} = \mathrm{E}\Psi_l, \quad l = 1, \ldots, m, \tag{3.5}$$

where the coefficient of Ψ_{l+1} was simplified by means of the reflection relation $[\frac{2\pi}{\alpha} - z] = [z]$ (cf. Equation (2.4)). Notice in this connection that the coefficients of Ψ_{l+1} and of Ψ_{l-1} vanish when $l = m$ and when $l = 1$, respectively.

Specifically, for $g = 2$ (so $\alpha = \frac{2\pi}{m+3}$ and $\Psi_l = \psi(l + 1)$) the truncated discrete Lamé equation amounts to the following m-dimensional tridiagonal eigenvalue problem:

$$\boldsymbol{L}^{(m)}\Psi^{(m)} = \mathrm{E}\Psi^{(m)}, \tag{3.6a}$$

with

$$\boldsymbol{L}^{(m)} = \begin{bmatrix} 0 & \frac{[m-1]}{[m+1]} & 0 & \cdots & 0 \\ \frac{[1]}{[3]} & 0 & \ddots & & \vdots \\ 0 & \frac{[2]}{[4]} & \ddots & \frac{[2]}{[4]} & 0 \\ \vdots & & \ddots & 0 & \frac{[1]}{[3]} \\ 0 & \cdots & 0 & \frac{[m-1]}{[m+1]} & 0 \end{bmatrix} \quad \text{and} \quad \Psi^{(m)} = \begin{bmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \vdots \\ \vdots \\ \Psi_m \end{bmatrix}. \tag{3.6b}$$

We will now show that by restricting the smooth (anti)periodic solutions $\phi(z; \xi_k, x(\xi_k))$ of the difference Lamé equation from Proposition 3.1 to the lattice $\{2, 3, \ldots, m+1\}$, one arrives at the spectrum and a corresponding eigenbasis for the matrix $\boldsymbol{L}^{(m)}$ (3.6b). To this end, let us write for $1 \leq l, k \leq m (= 2(\frac{\pi}{\alpha} - 1) - 1)$:

$$\Phi^{(m)}(\xi_k) := \begin{bmatrix} \Phi_1(\xi_k) \\ \Phi_2(\xi_k) \\ \vdots \\ \Phi_m(\xi_k) \end{bmatrix} \quad \text{with} \quad \Phi_l(\xi_k) := \phi(l+1; \xi_k, x(\xi_k)), \tag{3.7}$$

where $\xi_k \overset{(3.2),(3.4)}{=} m + 1 - 2k$.

Theorem 3.2 (Diagonalization of $\boldsymbol{L}^{(m)}$). (i) *The vectors $\Phi^{(m)}(\xi_1), \ldots, \Phi^{(m)}(\xi_m)$ (3.7) constitute an eigenbasis for $\boldsymbol{L}^{(m)}$ (3.6b) such that*

$$\boldsymbol{L}^{(m)} \Phi^{(m)}(\xi_k) = \mathrm{E}(x(\xi_k)) \Phi^{(m)}(\xi_k) \quad \textit{for } k = 1, \ldots, m. \tag{3.8a}$$

(ii) *The corresponding eigenvalues are evenly distributed around the origin and numbered in decreasing order: $\mathrm{E}(x(\xi_{m+1-k})) = -\mathrm{E}(x(\xi_k))$ and*

$$\mathrm{E}\big(\tfrac{\pi\tau}{\mathrm{i}\alpha}\big) > \mathrm{E}(x(\xi_1)) > \mathrm{E}(x(\xi_2)) > \cdots > \mathrm{E}(x(\xi_m)) > -\mathrm{E}\big(\tfrac{\pi\tau}{\mathrm{i}\alpha}\big) \tag{3.8b}$$

(where the bound on the spectrum is of the form $\mathrm{E}\big(\frac{\pi\tau}{\mathrm{i}\alpha}\big) = \mathrm{E}(x(m+1)) = \mathrm{E}(x(\xi_0))$).

(iii) *The matrix of eigenvectors $\boldsymbol{\Phi}^{(m)} = [\Phi_l(\xi_k)]_{1 \leq l, k \leq m}$ enjoys the following palindromic (anti)symmetries along the rows and columns:*

$$\Phi_l(\xi_{m+1-k}) = (-1)^{l-1} \Phi_l(\xi_k) \quad \textit{and} \quad \Phi_{m+1-l}(\xi_k) = (-1)^{k-1} \Phi_l(\xi_k). \tag{3.8c}$$

Proof. (i) Since the matrix $\boldsymbol{L}^{(m)}$ and the vectors $\Phi^{(m)}(\xi_k)$ were obtained by restricting the difference Lamé equation and its smooth solutions on the real axis taken from Proposition 3.1, the eigenvalue equations (3.8a) are satisfied manifestly. Furthermore, Proposition 2.2 ensures that the mapping $\xi \to \mathrm{E}(x(\xi))$, $-m-1 \leq \xi \leq m+1 = 2(\frac{\pi}{\alpha} - 1)$ is injective, so the eigenvalues $\mathrm{E}(\xi_k)$ are all distinct. To confirm that the vectors in question indeed constitute a complete eigenbasis it remains to infer that none of them is equal to the zero vector. To this end, we will check that $\Phi_1(\xi_k) = \phi(2; \xi_k, x(\xi_k)) \neq 0$ for $k = 1, \ldots, m$. Indeed, if $\phi(2; \xi, x) = 0$ for $\xi, x \in \mathbb{C}$ subject to the constraint (2.7c), then we see from (3.1) that

$$\frac{[\mathrm{i}x + 2]}{[\mathrm{i}x - 2]} = e^{\mathrm{i}\alpha\xi} \overset{(2.7c)}{\Longrightarrow} \frac{[\mathrm{i}x + 2]}{[\mathrm{i}x - 2]} \frac{[\mathrm{i}x - 1]^2}{[\mathrm{i}x + 1]^2} = 1. \tag{3.9}$$

The left-hand side of the latter equation is an elliptic function of $\mathrm{i}x$ of order 3, so (when counting with multiplicity) this equation has three solutions modulo the period lattice $\frac{2\pi}{\alpha}(\mathbb{Z} + \tau\mathbb{Z})$. It is readily seen that the identity in fact holds at the three half-periods, so the three solutions are $\mathrm{i}x = \frac{\pi}{\alpha}$, $\mathrm{i}x = \frac{\pi}{\alpha}\tau$ and

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$\mathrm{i}x = \frac{\pi}{\alpha}(1+\tau)$. In other words, for $-(m+1) < \xi < m+1 = 2(\frac{\pi}{\alpha}-1)$ the function $\phi(2;\xi,x(\xi))$ does not vanish (while in the limit $\xi \to \pm 2(\frac{\pi}{\alpha}-1) \overset{\text{Proposition. 2.2}}{\Longrightarrow} x(\xi) \to \pm\frac{\pi\tau}{\mathrm{i}\alpha}$ and thus $\phi(2;\xi,x(\xi)) \to 0$). In particular, one has that $\phi\big(2;\xi_k,x(\xi_k)\big) \neq 0$ for $k=1,\ldots,m$ (while $\lim_{\xi\to\xi_k}\phi\big(2;\xi,x(\xi)\big)=0$ if $k=0$ or $k=m+1$).

(ii) First, since

$$2(\tfrac{\pi}{\alpha}-1) = m+1 = \xi_0 > \xi_1 > \xi_2 > \cdots > \xi_m > \xi_{m+1} = -\xi_0 = -2(\tfrac{\pi}{\alpha}-1),$$

the ordering of the eigenvalues in (3.8b) is immediate from Proposition 2.2. Secondly, the mapping $\xi \to \mathrm{E}(x(\xi))$ is odd in ξ, so the eigenvalues inherit the manifest antisymmetry $\xi_{m+1-k} = -\xi_k$.

(iii) Since $\phi\big(z;\xi_k,x(\xi_k)\big)$ is even in z, the palindromic (anti)symmetry along the columns follows from the (anti)periodicity:

$$\begin{aligned}\Phi_{m+1-l}(\xi_k) &\overset{(3.7)}{=} \phi\big(m+2-l;\xi_k,x(\xi_k)\big) \overset{(3.3)}{=} (-1)^{k-1}\phi\big(-l-1;\xi_k,x(\xi_k)\big)\\ &= (-1)^{k-1}\phi\big(l+1;\xi_k,x(\xi_k)\big) = (-1)^{k-1}\Phi_l(\xi_k).\end{aligned}$$

The palindromic (anti)symmetry along the rows is verified similarly:

$$\begin{aligned}\Phi_l(\xi_{m+1-k}) = \Phi_l(-\xi_k) &\overset{(3.7)}{=} \phi\big(l+1;-\xi_k,-x(\xi_k)\big)\\ &\overset{(3.1)}{=} (-1)^{l-1}\phi\big(l+1;\xi_k,x(\xi_k)\big) = (-1)^{l-1}\Phi_l(\xi_k).\end{aligned}$$ □

3.3. *Orthogonality*

In [DG21], the solutions of the finitely truncated discrete Lamé equation (3.5) were constructed for $g>0$ in terms of polynomials on the spectrum. In this context, the Christoffel–Darboux formula gives rise to an orthogonality relation for the eigenfunctions in question. Theorem 3.2 provides for $g=2$: *(*i) an alternative compact representation for the eigenfunctions in terms of elliptic numbers (i.e., theta functions) and *(ii)* formulas parametrizing the corresponding eigenvalues explicitly (barring the inversion of $\mathrm{F}(x)$ (2.8)). By applying [DG21, Proposition 7] in the case $g=2$, one establishes the following orthogonality relation for the eigenbasis $\Phi^{(m)}(\xi_k)$, $k=1,\ldots,m$.

Proposition 3.3 (Orthogonality relation). *The eigenbasis $\Phi^{(m)}(\xi_1),\ldots,\Phi^{(m)}(\xi_m)$ (3.7) satisfies the following orthogonality relation:*

$$\sum_{l=1}^{m}\Phi_l(\xi_k)\Phi_l(\xi_{\tilde{k}})\Delta_l = \begin{cases}\hat{\Delta}_k^{-1}\,\Phi_1^2(\xi_k) & \text{if } \tilde{k}=k,\\ 0 & \text{if } \tilde{k}\neq k,\end{cases} \tag{3.10a}$$

where the orthogonality measure is of the following palindromic form

$$\Delta_l := \frac{[l+1]}{[2]}\begin{bmatrix} m-1\\ l-1\end{bmatrix} \quad \text{with} \quad \begin{bmatrix} m-1\\ l-1\end{bmatrix} := \frac{\prod_{j=1}^{m-1}[j]}{\prod_{j=1}^{l-1}[j]\,\prod_{j=1}^{m-l}[j]} \tag{3.10b}$$

(so $\Delta_{m+1-l}=\Delta_l$), and the quadratic norms factorize in terms of

$$\hat{\Delta}_k := \frac{1}{[3]}\prod_{\substack{i=1\\ i\neq k}}^{m}\big|\mathrm{E}\big(x(\xi_k)\big)-\mathrm{E}\big(x(\xi_i)\big)\big|^{-1} \tag{3.10c}$$

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and

$$\Phi_1(\xi_k) = \frac{2}{[3][2]}\mathrm{Re}\Big([2 - \mathrm{i}x(\xi_k)]e^{\frac{\mathrm{i}\alpha}{2}\xi_k}\Big). \tag{3.10d}$$

Proof. Proposition 7 of [DG21] furnishes an orthogonality relation for the eigenvectors of the matrix in [DG21, Eqs. (2.8a), (2.8b)] with $g > 0$, where it is assumed that the eigenvectors are normalized such that their first component is equal to 1. The asserted orthogonality relation for $\Phi^{(m)}(\xi_1), \ldots, \Phi^{(m)}(\xi_m)$ readily follows from this proposition upon substituting $g = 2$, $\textsc{m} = m - 1$, and accommodating for the current normalization stemming from Equation (3.1). □

4. Isotropic XY chain

4.1. Hamiltonian

Upon assuming that α is of the form in Equation (3.4) with $g = 2$ (so $\alpha = \frac{2\pi}{m+3}$), we consider the Hamiltonian of an open quantum spin chain on the finite lattice $\{1, 2, \ldots, m\}$ with positive coupling constants expressed in terms of elliptic integers

$$\boldsymbol{H}^{(m)} := \frac{1}{2}\sum_{l=1}^{m-1} \jmath_l\Big(\sigma_l^{\mathrm{x}}\sigma_{l+1}^{\mathrm{x}} + \sigma_l^{\mathrm{y}}\sigma_{l+1}^{\mathrm{y}}\Big) \quad \text{with} \quad \jmath_l = \sqrt{\tfrac{[l][m-l]}{[l+2][m+2-l]}}. \tag{4.1}$$

This inhomogeneous quantum spin Hamiltonian acts in a 2^m-dimensional state space

$$\mathcal{F}^{(m)} = \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{m \text{ times}} \tag{4.2a}$$

that is endowed with the standard sesquilinear inner product determined by

$$\langle u_1 \otimes \cdots \otimes u_m | v_1 \otimes \cdots \otimes v_m\rangle = \prod_{l=1}^{m} \langle u_l | v_l\rangle, \tag{4.2b}$$

with $u_l = \begin{bmatrix} (u_l)_1 \\ (u_l)_2 \end{bmatrix} \in \mathbb{C}^2$, $v_l = \begin{bmatrix} (v_l)_1 \\ (v_l)_2 \end{bmatrix} \in \mathbb{C}^2$and $\langle u_l|v_l\rangle := \overline{(u_l)_1}(v_l)_1 + \overline{(u_l)_2}(v_l)_2$. The local spin operators at site l act by means of Pauli matrices:

$$\sigma_l^{\mathrm{w}} := \underbrace{I \otimes \cdots \otimes I}_{l-1 \text{ times}} \otimes \sigma^{\mathrm{w}} \otimes \underbrace{I \otimes \cdots \otimes I}_{m-l \text{ times}} \qquad (\mathrm{w} \in \{\mathrm{x}, \mathrm{y}, \mathrm{z}\}), \tag{4.3a}$$

with

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \sigma^{\mathrm{x}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma^{\mathrm{y}} = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix} \quad \sigma^{\mathrm{z}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{4.3b}$$

Following [LSM61], the quantum spin Hamiltonian (4.1) can be rewritten in terms of fermionic creation– and annihilation operators (cf., e.g., [W17, Chapter 27]):

$$\boldsymbol{H}^{(m)} = \sum_{l=1}^{m-1} \jmath_l\big(c_l^* c_{l+1} + c_{l+1}^* c_l\big), \tag{4.4a}$$

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where

$$c_l^* := (-1)^{l-1} \underbrace{\sigma^z \otimes \cdots \otimes \sigma^z}_{l-1 \text{ times}} \otimes \sigma^+ \otimes \underbrace{I \otimes \cdots \otimes I}_{m-l \text{ times}}, \tag{4.4b}$$

$$c_l := (-1)^{l-1} \underbrace{\sigma^z \otimes \cdots \otimes \sigma^z}_{l-1 \text{ times}} \otimes \sigma^- \otimes \underbrace{I \otimes \cdots \otimes I}_{m-l \text{ times}}, \tag{4.4c}$$

with

$$\sigma^+ = \frac{1}{2}(\sigma^x + i\sigma^y) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \sigma^- = \frac{1}{2}(\sigma^x - i\sigma^y) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{4.4d}$$

The fermionic creation – and annihilation operators satisfy the canonical anticommutation relations

$$\{c_l^*, c_{\tilde{l}}^*\} = 0 \quad \{c_l, c_{\tilde{l}}\} = 0 \quad \{c_l^*, c_{\tilde{l}}\} = \delta_{l,\tilde{l}}\,\mathrm{Id}, \tag{4.5}$$

where $\{a, b\} := ab + ba$ and $\delta_{l,\tilde{l}}$ represents the Kronecker delta. Moreover, since the coupling constants $\jmath_l$ are positive and the operators c_l^* and c_l are adjoints in $\mathcal{F}^{(m)}$, that is,

$$\forall u, v \in \mathcal{F}^{(m)}: \quad \langle u | c_l^* v \rangle = \overline{\langle v | c_l u \rangle}, \tag{4.6a}$$

it is clear that the Hamiltonian $\boldsymbol{H}^{(m)}$ (4.4a)–(4.4d) is self-adjoint:

$$\forall u, v \in \mathcal{F}^{(m)}: \quad \langle u | \boldsymbol{H}^{(m)} v \rangle = \overline{\langle v | \boldsymbol{H}^{(m)} u \rangle}. \tag{4.6b}$$

4.2. n-particle Hamiltonian

Starting from a normalized vacuum vector that is annihilated by c_l ($l = 1, \ldots, m$)

$$|\emptyset\rangle := \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{m \text{ times}}, \tag{4.7a}$$

the standard orthonormal basis for $\mathcal{F}^{(m)}$ (4.2a), (4.2b) can be generated by acting with fermionic creation operators associated with strict partitions $\lambda = (\lambda_1, \ldots, \lambda_n)$ that have bounded row- and column sizes $\leq m$:

$$|\lambda\rangle := c_{\lambda_n}^* \cdots c_{\lambda_2}^* c_{\lambda_1}^* |\emptyset\rangle. \tag{4.7b}$$

Here, the parts of the strict partition $(\lambda_1, \ldots, \lambda_n)$ labeling the basis vector $|\lambda\rangle$ represent the coordinates of $n \leq m$ places on the lattice $\{1, 2, \ldots, m\}$, where in the vaccum vector $|\emptyset\rangle$ the local state $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ has been flipped to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. This gives rise to the following orthogonal decomposition of the state space $\mathcal{F}^{(m)}$ in n-particle subspaces:

$$\mathcal{F}^{(m)} = \bigoplus_{n=0}^{m} \mathcal{F}^{(m,n)} \quad \text{with} \quad \mathcal{F}^{(m,n)} := \mathrm{Span}_{\mathbb{C}}\{|\lambda\rangle \mid \lambda \in \Lambda^{(m,n)}\}, \tag{4.8a}$$

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where

$$\Lambda^{(m,n)} := \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \mid m \geq \lambda_1 > \lambda_2 > \cdots > \lambda_n \geq 1\} \tag{4.8b}$$

with the convention that $\Lambda^{(m,0)} := \{\emptyset\}$. Notice that the $\binom{m}{n}$-dimensional n-particle subspace $\mathcal{F}^{(m,n)}$ is stable with respect to the action of $\boldsymbol{H}^{(m)}$ (4.4a)–(4.4d).

Proposition 4.1 (Matrix elements of the Hamiltonian). *The matrix elements of the quantum spin Hamiltonian $\boldsymbol{H}^{(m)}$ (4.1) with respect to the standard basis $|\lambda\rangle$ (4.7a), (4.7b) read for $\lambda \in \Lambda^{(m,n)}$ with $0 < n < m$:*

$$\boldsymbol{H}^{(m)}|\lambda\rangle = \sum_{\substack{1\leq j\leq n \\ \lambda+e_j\in\Lambda^{(m,n)}}} \jmath_{\lambda_j}|\lambda + e_j\rangle + \sum_{\substack{1\leq j\leq n \\ \lambda-e_j\in\Lambda^{(m,n)}}} \jmath_{\lambda_j-1}|\lambda - e_j\rangle \tag{4.9}$$

(while $\boldsymbol{H}^{(m)}|\lambda\rangle = 0$ for $\lambda \in \Lambda^{(m,n)}$ if $n = 0$ or $n = m$). Here, the vectors $e_1, \ldots e_n$ refer to the standard unit basis of $\mathbb{Z}^n$.

Proof. Let us act with $\boldsymbol{H}^{(m)}$ (4.4a)–(4.4d) on $|\lambda\rangle$ with $\lambda \in \Lambda^{(m,n)}$. For $0 < n < m$ and $1 \leq l < m$, it is seen from the anticommutation relations (4.5) that the term

$$\jmath_l c_l^* c_{l+1}|\lambda\rangle = \jmath_l c_l^* c_{l+1} c^*_{\lambda_n} \cdots c^*_{\lambda_{j+1}} c^*_{\lambda_j} \cdots c^*_{\lambda_1}|\emptyset\rangle$$

vanishes unless $l \notin \{\lambda_1, \ldots, \lambda_n\}$ and $l+1 \in \{\lambda_1, \ldots, \lambda_n\}$, that is, unless $\exists 1 \leq j \leq n$ such that $l = \lambda_j - 1$ and $\lambda_{j+1} < l$ (with the convention that $\lambda_{n+1} := 0$), or equivalently: Unless $\exists 1 \leq j \leq n$ such that $l = \lambda_j - 1$ and $\lambda - e_j \in \Lambda^{(m,n)}$. In this case, one has that

$$\begin{aligned}
\jmath_l c_l^* c_{l+1}|\lambda\rangle &= \jmath_l c^*_{\lambda_n} \cdots c^*_{\lambda_{j+1}} c_l^* c_{l+1} c^*_{\lambda_j} c^*_{\lambda_{j-1}} \cdots c^*_{\lambda_1}|\emptyset\rangle \\
&= \jmath_{\lambda_j-1} c^*_{\lambda_n} \cdots c^*_{\lambda_{j+1}} c^*_{\lambda_j-1} c_{\lambda_j} c^*_{\lambda_j} c^*_{\lambda_{j-1}} \cdots c^*_{\lambda_1}|\emptyset\rangle \\
&= \jmath_{\lambda_j-1} c^*_{\lambda_n} \cdots c^*_{\lambda_{j+1}} c^*_{\lambda_j-1} c^*_{\lambda_{j-1}} \cdots c^*_{\lambda_1}|\emptyset\rangle = \jmath_{\lambda_j-1}|\lambda - e_j\rangle.
\end{aligned}$$

In the same manner, one deduces that for $0 < n < m$ and $1 \leq l < m$ the term $\jmath_l c^*_{l+1} c_l|\lambda\rangle$ vanishes unless $l \in \{\lambda_1, \ldots, \lambda_n\}$ and $l+1 \notin \{\lambda_1, \ldots, \lambda_n\}$, that is, unless $\exists 1 \leq j \leq n$ such that $l = \lambda_j$ and $\lambda_{j-1} > l+1$ (with the convention that $\lambda_0 := m+1$), or equivalently: Unless $\exists 1 \leq j \leq n$ such that $l = \lambda_j$ and $\lambda + e_j \in \Lambda^{(m,n)}$. In this case, one has that

$$\begin{aligned}
\jmath_l c_l c^*_{l+1}|\lambda\rangle &= \jmath_l c^*_{\lambda_n} \cdots c^*_{\lambda_{j+1}} c^*_{l+1} c_l c^*_{\lambda_j} c^*_{\lambda_{j-1}} \cdots c^*_{\lambda_1}|\emptyset\rangle \\
&= \jmath_{\lambda_j} c^*_{\lambda_n} \cdots c^*_{\lambda_{j+1}} c^*_{\lambda_j+1} c_{\lambda_j} c^*_{\lambda_j} c^*_{\lambda_{j-1}} \cdots c^*_{\lambda_1}|\emptyset\rangle \\
&= \jmath_{\lambda_j} c^*_{\lambda_n} \cdots c^*_{\lambda_{j+1}} c^*_{\lambda_j+1} c^*_{\lambda_{j-1}} \cdots c^*_{\lambda_1}|\emptyset\rangle = \jmath_{\lambda_j}|\lambda + e_j\rangle.
\end{aligned}$$

Notice that in both cases the relation between the pertinent values of $1 \leq l < m$ and $1 \leq j \leq n$ is one-to-one given λ (since our partitions are strict); hence, by summing over all terms from $1 \leq l < m$ the asserted formula for the matrix elements in Equation (4.9) follows. □

Remark 4.2. In the formulas for the Hamiltonian $\boldsymbol{H}^{(m)}$ (4.4a)–(4.4c) and for the standard basis $|\lambda\rangle$ (4.7a), (4.7b) (as well as in the proof of Proposition 4.1), the fermionic operators c_l^* and c_l can in principle be replaced simply by the spin raising and lowering operators $\sigma_l^+ = \sigma_l^{\mathrm{x}} + \mathrm{i}\sigma_l^{\mathrm{y}}$ and $\sigma_l^- = \sigma_l^{\mathrm{x}} - \mathrm{i}\sigma_l^{\mathrm{y}}$, respectively. Indeed, in view of the ordering of the parts of the strict partition one has in particular that for any $\lambda \in \Lambda^{(m,n)}$: $c^*_{\lambda_n} \cdots c^*_{\lambda_2} c^*_{\lambda_1}|\emptyset\rangle = \sigma^+_{\lambda_n} \cdots \sigma^+_{\lambda_2} \sigma^+_{\lambda_1}|\emptyset\rangle$. However, by employing fermionic operators instead it is automatic that all results below apply verbatim to the free-fermion description of the spin model under consideration.

Let $\ell^2(\Lambda^{(m,n)})$ denote the (n-magnon) Hilbert space of complex functions $\lambda \overset{\Psi}{\mapsto} \Psi_\lambda$, $\lambda \in \Lambda^{(m,n)}$ endowed with the inner product

$$\langle \tilde{\Psi}, \Psi \rangle := \sum_{\lambda \in \Lambda^{(m,n)}} \overline{\tilde{\Psi}}_\lambda \Psi_\lambda \qquad (\forall \Psi, \tilde{\Psi} \in \ell^2(\Lambda^{(m,n)})). \tag{4.10}$$

It is evident from the orthonormality of the standard basis $|\lambda\rangle$, $\lambda \in \Lambda^{(m,n)}$ that the injection $\mathcal{I}^{(m,n)} : \ell^2(\Lambda^{(m,n)}) \to \mathcal{F}^{(m,n)}$ given by

$$\mathcal{I}^{(m,n)}(\Psi) := \sum_{\lambda \in \Lambda^{(m,n)}} \Psi_\lambda |\lambda\rangle \qquad (\forall \Psi \in \ell^2(\Lambda^{(m,n)})) \tag{4.11}$$

defines an Hilbert-space isomorphism between $\ell^2(\Lambda^{(m,n)})$ and $\mathcal{F}^{(m,n)}$.

Definition 4.1 (n-particle Hamiltonian). Let us define the *n-particle Hamiltonian* $\boldsymbol{H}^{(m,n)} : \ell^2(\Lambda^{(m,n)}) \to \ell^2(\Lambda^{(m,n)})$ as the pull-back of the restriction of the quantum spin Hamiltonian $\boldsymbol{H}^{(m)}$ (4.1) to the n-particle subspace $\mathcal{F}^{(m,n)}$ (4.8a), (4.8b) with respect to the isomorphism $\mathcal{I}^{(m,n)} : \ell^2(\Lambda^{(m,n)}) \to \mathcal{F}^{(m,n)}$ (4.11):

$$\boldsymbol{H}^{(m,n)} = \left(\mathcal{I}^{(m,n)}\right)^{-1} \circ \boldsymbol{H}^{(m)}\Big|_{\mathcal{F}^{(m,n)}} \circ \mathcal{I}^{(m,n)}. \tag{4.12}$$

It is clear from this definition that $\boldsymbol{H}^{(m,n)}$ inherits the self-adjointness of $\boldsymbol{H}^{(m)}$. With the aid of Proposition 4.1, we arrive at the following explicit formula for the action of $\boldsymbol{H}^{(m,n)}$ in $\ell^2(\Lambda^{(m,n)})$.

Proposition 4.3 (Action of the n-particle Hamiltonian). *For $0 < n < m$, one has that*

$$(\boldsymbol{H}^{(m,n)}\Psi)_\lambda = \sum_{\substack{1 \le j \le n \\ \lambda + e_j \in \Lambda^{(m,n)}}} \jmath_{\lambda_j} \Psi_{\lambda + e_j} + \sum_{\substack{1 \le j \le n \\ \lambda - e_j \in \Lambda^{(m,n)}}} \jmath_{\lambda_j - 1} \Psi_{\lambda - e_j} \tag{4.13}$$

$(\forall \Psi \in \ell^2(\Lambda^{(m,n)}))$.

Proof. Definition 4.1 implies that $\forall \Psi \in \ell^2(\Lambda^{(m,n)})$:

$$\mathcal{I}^{(m,n)} \boldsymbol{H}^{(m,n)} \Psi = \boldsymbol{H}^{(m)} \mathcal{I}^{(m,n)} \Psi,$$

or more explicitly (cf. Equation (4.11)):

$$\sum_{\lambda \in \Lambda^{(m,n)}} (\boldsymbol{H}^{(m,n)}\Psi)_\lambda |\lambda\rangle = \boldsymbol{H}^{(m)} \left(\sum_{\lambda \in \Lambda^{(m,n)}} \Psi_\lambda |\lambda\rangle \right). \tag{4.14}$$

With the aid of Proposition 4.1 the right-hand side of this intertwining relation is rewritten as

$$\sum_{\lambda \in \Lambda^{(m,n)}} \Psi_\lambda \left(\sum_{\substack{1 \le j \le n \\ \lambda + e_j \in \Lambda^{(m,n)}}} \jmath_{\lambda_j} |\lambda + e_j\rangle + \sum_{\substack{1 \le j \le n \\ \lambda - e_j \in \Lambda^{(m,n)}}} \jmath_{\lambda_j - 1} |\lambda - e_j\rangle \right),$$

which is equal to

$$\sum_{\substack{\lambda \in \Lambda^{(m,n)} \\ 1 \le j \le n}} \Psi_\lambda \big(\jmath_{\lambda_j} |\lambda + e_j\rangle + \jmath_{\lambda_j - 1} |\lambda - e_j\rangle \big)$$

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with the convention that $|\lambda \pm e_j\rangle = 0$ if $\lambda \pm e_j \notin \Lambda^{(m,n)}$. The translations $\lambda \to \lambda - e_j$ and $\lambda \to \lambda + e_j$ recast the respective terms in the form

$$\sum_{\substack{\lambda\in\Lambda^{(m,n)}\\ 1\le j\le n}} \bigl(\mathrm{J}_{\lambda_j-1}\Psi_{\lambda-e_j} + \mathrm{J}_{\lambda_j}\Psi_{\lambda+e_j}\bigr)|\lambda\rangle$$

with the convention that $\Psi_{\lambda\pm e_j} = 0$ if $\lambda \pm e_j \notin \Lambda^{(m,n)}$. When comparing with the left-hand side of Equation (4.14), the asserted action of $\boldsymbol{H}^{(m,n)}$ in $\ell^2(\Lambda^{(m,n)})$ given by Equation (4.13) follows. □

4.3. Spectrum and orthonormal eigenbasis

In the homogeneous case with $\mathrm{J}_1 = \mathrm{J}_2 = \cdots = \mathrm{J}_{m-1}$, a standard method for diagonalizing $\boldsymbol{H}^{(m)}$ (4.4a)–(4.4c) hinges on performing a (discrete) Fourier transform of the fermionic operators. In the inhomogeneous situation with site-dependent coupling constants, this discrete Fourier transform is replaced by the pertinent Bogoliubov transformation stemming from the eigenfunction transform diagonalizing the one-particle Hamiltonian (cf., e.g., [HSS12, Section 4]). Here, we will follow an alternative path, which is based on the observation that Theorem 3.2 and Proposition 3.3 allow for an explicit construction of an orthonormal eigenbasis diagonalizing the n-particle Hamiltonian $\boldsymbol{H}^{(m,n)}$ (4.13) in terms of Slater determinants.

To this end, some further notation is needed. For any $\lambda, \kappa \in \Lambda^{(m,n)}$, let us define

$$\xi_\kappa^{(m,n)} := (\xi_{\kappa_1}, \xi_{\kappa_2}, \ldots, \xi_{\kappa_n}), \tag{4.15a}$$

$$\Delta_\lambda^{(m,n)} := \prod_{1\le j\le n} \Delta_{\lambda_j} \quad\text{and}\quad \hat{\Delta}_\kappa^{(m,n)} := \prod_{1\le j\le n} \hat{\Delta}_{\kappa_j}, \tag{4.15b}$$

where (as before) $\xi_k = 2(\frac{\pi}{\alpha} - 1 - k)\big|_{\alpha=\frac{2\pi}{m+3}} = m + 1 - 2k$ while Δ_l and $\hat{\Delta}_k$ are of the form detailed in Proposition 3.3. The corresponding spectrum will be given by eigenvalues of the form

$$\boldsymbol{E}^{(m,n)}\bigl(\xi_\kappa^{(m,n)}\bigr) := \sum_{1\le j\le n} \mathrm{E}\bigl(x(\xi_{\kappa_j})\bigr) \tag{4.15c}$$

(with $\mathrm{E}\bigl(x(\xi_1)\bigr), \ldots, \mathrm{E}\bigl(x(\xi_m)\bigr)$ as in Theorem 3.2).

We define the n-particle wave function $\Psi^{(m,n)}\bigl(\xi_\kappa^{(m,n)}\bigr) \in \ell^2(\Lambda^{(m,n)})$ by means of its values on $\Lambda^{(m,n)}$ in terms of the following normalized Slater determinant:

$$\Psi_\lambda^{(m,n)}\bigl(\xi_\kappa^{(m,n)}\bigr) := \sqrt{\Delta_\lambda^{(m,n)}\hat{\Delta}_\kappa^{(m,n)}}\,\det\left[\frac{\Phi_{\lambda_j}(\xi_{\kappa_k})}{\Phi_1(\xi_{\kappa_k})}\right]_{1\le j,k\le n} \tag{4.16}$$

(with $\Phi_1(\xi_k), \ldots, \Phi_m(\xi_k)$ taken from Equation (3.7)). So here and below, when the superscripted dimensions (m,n) are omitted we are referring to the components of the eigenvectors, the eigenvalues and the weights of the orthogonality measures associated with the truncated difference Lamé matrix $\boldsymbol{L}^{(m)}$ (3.6b) to be imported from Section 3.

Theorem 4.4 (Diagonalization of $\boldsymbol{H}^{(m,n)}$). *For $0 < n < m$, the wave functions $\Psi^{(m,n)}\bigl(\xi_\kappa^{(m,n)}\bigr)$, $\kappa \in \Lambda^{(m,n)}$ constitute and orthogonal eigenbasis diagonalizing $\boldsymbol{H}^{(m,n)}$ (4.13) in $\ell^2(\Lambda^{(m,n)})$, that is, $\forall \kappa, \tilde{\kappa} \in \Lambda^{(m,n)}$:*

$$\boldsymbol{H}^{(m,n)}\Psi^{(m,n)}\bigl(\xi_\kappa^{(m,n)}\bigr) = \boldsymbol{E}^{(m,n)}\bigl(\xi_\kappa^{(m,n)}\bigr)\Psi^{(m,n)}\bigl(\xi_\kappa^{(m,n)}\bigr) \tag{4.17a}$$

and

$$\left\langle \Psi^{(m,n)}\left(\xi_\kappa^{(m,n)}\right), \Psi^{(m,n)}\left(\xi_{\tilde\kappa}^{(m,n)}\right)\right\rangle = \begin{cases} 1 & \text{if } \tilde\kappa = \kappa, \\ 0 & \text{if } \tilde\kappa \neq \kappa. \end{cases} \tag{4.17b}$$

Proof. Let us act with $\boldsymbol{H}^{(m,n)}$ (4.13) on $\Psi^{(m,n)}\left(\xi_\kappa^{(m,n)}\right)$ (4.16), and evaluate the result at $\lambda \in \Lambda^{(m,n)}$:

$$\sum_{\substack{1\leq i\leq n \\ \lambda+e_i\in\Lambda^{(m,n)}}} \mathrm{J}_{\lambda_i} \det\left[\Delta^{1/2}_{\lambda_j+\delta_{i,j}} \hat\Delta^{1/2}_{\kappa_k} \frac{\Phi_{\lambda_j+\delta_{i,j}}(\xi_{\kappa_k})}{\Phi_1(\xi_{\kappa_k})}\right]_{1\leq j,k\leq n}$$
$$+ \sum_{\substack{1\leq i\leq n \\ \lambda-e_i\in\Lambda^{(m,n)}}} \mathrm{J}_{\lambda_i-1} \det\left[\Delta^{1/2}_{\lambda_j-\delta_{i,j}} \hat\Delta^{1/2}_{\kappa_k} \frac{\Phi_{\lambda_j-\delta_{i,j}}(\xi_{\kappa_k})}{\Phi_1(\xi_{\kappa_k})}\right]_{1\leq j,k\leq n}.$$

By means of the relations $\mathrm{J}_{\lambda_i}\Delta^{1/2}_{\lambda_i+1} = \Delta^{1/2}_{\lambda_i}\frac{[m-\lambda_i]}{[m+2-\lambda_i]}$ for $1 \leq \lambda_i < m$ and $\mathrm{J}_{\lambda_i-1}\Delta^{1/2}_{\lambda_i-1} = \Delta^{1/2}_{\lambda_i}\frac{[\lambda_i-1]}{[\lambda_i+1]}$ for $1 < \lambda_i \leq m$, the expression in question is rewritten as

$$= \sum_{1\leq i\leq n} \det\left[\Delta^{1/2}_{\lambda_j}\hat\Delta^{1/2}_{\kappa_k} \frac{\left(\frac{[m-\lambda_i]}{[m+2-\lambda_i]}\right)^{\delta_{i,j}} \Phi_{\lambda_j+\delta_{i,j}}(\xi_{\kappa_k})}{\Phi_1(\xi_{\kappa_k})}\right]_{1\leq j,k\leq n}$$
$$+ \sum_{1\leq i\leq n} \det\left[\Delta^{1/2}_{\lambda_j}\hat\Delta^{1/2}_{\kappa_k} \frac{\left(\frac{[\lambda_i-1]}{[\lambda_i+1]}\right)^{\delta_{i,j}} \Phi_{\lambda_j-\delta_{i,j}}(\xi_{\kappa_k})}{\Phi_1(\xi_{\kappa_k})}\right]_{1\leq j,k\leq n},$$

where the restrictions on the summations in i were suppressed in the end since the resulting determinants vanish manifestly for $\lambda \in \Lambda^{(n,m)}$ with $\lambda + e_i \notin \Lambda^{(n,m)}$ or with $\lambda - e_i \notin \Lambda^{(n,m)}$, respectively (either because of a vanishing ith row or because of rows i and $i-1$ or i and $i+1$ being linearly dependent). Upon exploiting the linearity in the ith row, the determinants can be merged pairwise

$$= \sum_{1\leq i\leq n} \det\left[\frac{\Delta^{1/2}_{\lambda_j}\hat\Delta^{1/2}_{\kappa_k}}{\Phi_1(\xi_{\kappa_k})2^{1-\delta_{i,j}}}\left(\left(\frac{[m-\lambda_i]}{[m+2-\lambda_i]}\right)^{\delta_{i,j}}\Phi_{\lambda_j+\delta_{i,j}}(\xi_{\kappa_k}) + \left(\frac{[\lambda_i-1]}{[\lambda_i+1]}\right)^{\delta_{i,j}}\Phi_{\lambda_j-\delta_{i,j}}(\xi_{\kappa_k})\right)\right]_{1\leq j,k\leq n},$$

so as to enable invoking of the eigenvalue equations from Theorem 3.2 on each element of the ith row

$$= \sum_{1\leq i\leq n} \det\left[\Delta^{1/2}_{\lambda_j}\hat\Delta^{1/2}_{\kappa_k}\left(\mathrm{E}\left(x(\xi_{\kappa_k})\right)\right)^{\delta_{i,j}} \frac{\Phi_{\lambda_j}(\xi_{\kappa_k})}{\Phi_1(\xi_{\kappa_k})}\right]_{1\leq j,k\leq n}. \tag{4.18}$$

By first expanding the latter determinants along the ith row and then interchanging the order of the two summations, a comparison of the result with the expansion of the determinant $\det\left[\Delta^{1/2}_{\lambda_j}\hat\Delta^{1/2}_{\kappa_k}\frac{\Phi_{\lambda_j}(\xi_{\kappa_k})}{\Phi_1(\xi_{\kappa_k})}\right]_{1\leq j,k\leq n}$ along the kth column readily reveals that Equation (4.18) can be rewritten as

$$= \left(\sum_{1\leq k\leq n} \mathrm{E}\left(x(\xi_{\kappa_k})\right)\right) \det\left[\Delta^{1/2}_{\lambda_j}\hat\Delta^{1/2}_{\kappa_k}\frac{\Phi_{\lambda_j}(\xi_{\kappa_k})}{\Phi_1(\xi_{\kappa_k})}\right]_{1\leq j,k\leq n},$$

which settles the proof of the eigenvalue equation (4.17a).

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The proof of the orthogonality relation (4.17b) hinges in turn on Proposition 3.3 and the Cauchy–Binet formula. Indeed, $\forall\kappa, \kappa \in \Lambda^{(m,n)}$:

$$
\begin{aligned}
&\left\langle \Psi^{(m,n)}\big(\xi_\kappa^{(m,n)}\big), \Psi^{(m,n)}\big(\xi_{\tilde\kappa}^{(m,n)}\big)\right\rangle = \sum_{\lambda\in\Lambda^{(m,n)}} \overline{\Psi_\lambda^{(m,n)}\big(\xi_\kappa^{(m,n)}\big)}\Psi_\lambda^{(m,n)}\big(\xi_{\tilde\kappa}^{(m,n)}\big)\\
&\overset{(i)}{=} \sum_{m\geq\lambda_1>\cdots>\lambda_n\geq 1} \det\left[\hat\Delta_{\kappa_j}^{1/2}\frac{\Phi_{\lambda_k}(\xi_{\kappa_j})}{\Phi_1(\xi_{\kappa_j})}\Delta_{\lambda_k}^{1/2}\right]_{1\leq j,k\leq n} \det\left[\Delta_{\lambda_j}^{1/2}\frac{\Phi_{\lambda_j}(\xi_{\tilde\kappa_k})}{\Phi_1(\xi_{\tilde\kappa_k})}\hat\Delta_{\tilde\kappa_k}^{1/2}\right]_{1\leq j,k\leq n}\\
&\overset{(ii)}{=} \det\left(\left[\hat\Delta_{\kappa_j}^{1/2}\frac{\Phi_l(\xi_{\kappa_j})}{\Phi_1(\xi_{\kappa_j})}\Delta_l^{1/2}\right]_{\substack{1\leq j\leq n\\ 1\leq l\leq m}}\left[\Delta_l^{1/2}\frac{\Phi_l(\xi_{\tilde\kappa_k})}{\Phi_1(\xi_{\tilde\kappa_k})}\hat\Delta_{\tilde\kappa_k}^{1/2}\right]_{\substack{1\leq l\leq m\\ 1\leq k\leq n}}\right)\\
&\overset{(iii)}{=} \det\left[\frac{\hat\Delta_{\kappa_j}^{1/2}\hat\Delta_{\tilde\kappa_k}^{1/2}}{\Phi_1(\xi_{\kappa_j})\Phi_1(\xi_{\tilde\kappa_k})}\sum_{1\leq l\leq m}\Phi_l(\xi_{\kappa_j})\Phi_l(\xi_{\tilde\kappa_k})\Delta_l\right]_{1\leq j,k\leq n}\\
&\overset{(iv)}{=} \det\big[\delta_{\kappa_j,\tilde\kappa_k}\big]_{1\leq j,k\leq n} \overset{(v)}{=} \begin{cases} 1 & \text{if } \kappa=\tilde\kappa,\\ 0 & \text{if } \kappa\neq\tilde\kappa.\end{cases}
\end{aligned}
$$

The algorithm to justify the above chain of equalities reads as follows. *(i)* Use the definitions (4.15b) and (4.16), where the first matrix was replaced by the transposed and the complex conjugation was omitted because all functions are real-valued. *(ii)* Apply the Cauchy–Binet formula. *(iii)* Perform the matrix multiplication. *(iv)* Apply the orthogonality relation of Proposition 3.3. *(v)* Observe that if $\kappa_1 \neq \tilde\kappa_1$, then either all elements in the first row (if $\kappa_1 > \tilde\kappa_1$) or all elements in the first column (if $\kappa_1 < \tilde\kappa_1$) of the matrix in question are equal to zero; next, if $\kappa_1 = \tilde\kappa_1$, proceed inductively in the dimension. □

The diagonalization of the quantum spin hamiltonian $\boldsymbol{H}^{(m)}$ is immediate from Theorem 4.4 via the isomorphism (4.11).

Corollary 4.5 (Diagonalization of $\boldsymbol{H}^{(m)}$). *In addition to the vacuum all-spins-down state $|\emptyset\rangle$ (4.7a) and the m-particle fully flipped all-spins-up state*

$$|(m, m-1, \ldots, 2, 1)\rangle = \underbrace{\begin{bmatrix}1\\0\end{bmatrix}\otimes\cdots\otimes\begin{bmatrix}1\\0\end{bmatrix}}_{m\ \textit{times}}, \tag{4.19a}$$

which both belong to the kernel of $\boldsymbol{H}^{(m)}$ (4.1), orthonormal eigenvectors and corresponding eigenvalues diagonalizing our quantum spin Hamiltonian in the state space $\mathcal{F}^{(m)}$ (4.2a), (4.2b) are given by

$$\sum_{\lambda\in\Lambda^{(m,n)}} \Psi_\lambda^{(m,n)}\big(\xi_\kappa^{(m,n)}\big)|\lambda\rangle \quad \textit{and} \quad \boldsymbol{E}^{(m,n)}\big(\xi_\kappa^{(m,n)}\big), \tag{4.19b}$$

with $\kappa \in \Lambda^{(m,n)}$ and $0 < n < m$.

The monotonicity of the eigenvalues in Theorem 3.2 makes it straightforward to read-off the highest – and lowest – eigenvalues of $\boldsymbol{H}^{(m,n)}$ and $\boldsymbol{H}^{(m)}$ from Theorem 4.4 and Corollary 4.5.

Corollary 4.6 (Highest – and lowest – eigenvalues). (i) *For $0 < n < m$ the highest and lowest eigenvalues of $\boldsymbol{H}^{(m,n)}$ (4.13) in $\ell^2(\Lambda^{(m,n)})$ are given by*

$$\boldsymbol{E}^{(m,n)}\big(\xi_{(n,n-1,\ldots,1)}^{(m,n)}\big) = \sum_{1\leq k\leq n} \varepsilon\big(x(\xi_k)\big) \tag{4.20a}$$

and

$$\boldsymbol{E}^{(m,n)}\left(\xi^{(m,n)}_{(m,m-1,\ldots,m+1-n)}\right) = -\sum_{1\le k\le n} \mathrm{E}\big(x(\xi_k)\big), \tag{4.20b}$$

respectively.

(ii) *The highest and lowest eigenvalues of* $\boldsymbol{H}^{(m)}$ *(4.1) in* $\mathcal{F}^{(m)}$ *are given for m odd by*

$$\boldsymbol{E}^{(m,\frac{m+1}{2})}\left(\xi^{(m,\frac{m+1}{2})}_{(\frac{m+1}{2},\ldots,2,1)}\right) = \boldsymbol{E}^{(m,\frac{m-1}{2})}\left(\xi^{(m,\frac{m-1}{2})}_{(\frac{m-1}{2},\ldots,2,1)}\right) = \sum_{1\le k\le\frac{m-1}{2}} \mathrm{E}\big(x(\xi_k)\big) \tag{4.21a}$$

and

$$\boldsymbol{E}^{(m,\frac{m+1}{2})}\left(\xi^{(m,\frac{m+1}{2})}_{(m,m-1,\ldots,\frac{m+1}{2})}\right) = \boldsymbol{E}^{(m,\frac{m-1}{2})}\left(\xi^{(m,\frac{m-1}{2})}_{(m,m-1,\ldots,\frac{m+3}{2})}\right) = -\sum_{1\le k\le\frac{m-1}{2}} \mathrm{E}\big(x(\xi_k)\big), \tag{4.21b}$$

respectively, and for m even by

$$\boldsymbol{E}^{(m,\frac{m}{2})}\left(\xi^{(m,\frac{m}{2})}_{(\frac{m}{2},\ldots,2,1)}\right) = \sum_{1\le k\le\frac{m}{2}} \mathrm{E}\big(x(\xi_k)\big) \tag{4.22a}$$

and

$$\boldsymbol{E}^{(m,\frac{m}{2})}\left(\xi^{(m,\frac{m}{2})}_{(m,m-1,\ldots,\frac{m}{2}+1)}\right) = -\sum_{1\le k\le\frac{m}{2}} \mathrm{E}\big(x(\xi_k)\big), \tag{4.22b}$$

respectively.

Remark 4.7. The all spins up/down states $|(m, m-1, \ldots, 2, 1)\rangle$ and $|\emptyset\rangle$ are not the only spin states in the kernel of the spin Hamiltonian $\boldsymbol{H}^{(m)}$. Indeed, since the eigenvalues of the truncated difference Lamé matrix $\boldsymbol{L}^{(m)}$ (3.6b) are distributed symmetrically around the origin (cf. Theorem 3.2), that is,

$$\mathrm{E}\big(x(\xi_k)\big) + \mathrm{E}\big(x(\xi_{m+1-k})\big) = 0 \quad \text{for } 1 \le k \le \tfrac{m+1}{2},$$

it is clear that $\boldsymbol{E}^{(m,n)}\left(\xi^{(m,n)}_{\kappa}\right) = 0$, $\forall \kappa \in \Lambda^{(m,n)}$ such that $\kappa_j + \kappa_{n+1-j} = m+1$ for $1 \le j \le \frac{n+1}{2}$.

4.4. On the trigonometric and rational degenerations

The spectral problem for the trigonometric $p \to 0$ degeneration of the finite discrete Lamé equation (3.5) can be conveniently solved *for any* $g \in (0,\infty)$ in terms of Rogers' q-ultraspherical polynomials [DG21, Section 4.2] (cf. also [R90, Section 3C2]). As such, the inhomogeneous isotropic XY chain associated with the pertinent Jacobi matrix works out an example of the exactly solvable quantum spin models stemming from the Askey scheme of (basic) hypergeometric polynomials, cf. [CNV19, FG20, GVZ16, GVZ13, JV10, SV11, VZ12]. Indeed, the Hamiltonian of the corresponding trigonometric spin chain is of the form in $\boldsymbol{H}^{(m)}$ (4.1) with (cf. Equation (2.5), (3.4))

$$\mathrm{J}_l \to \sqrt{\frac{[l]_q[m-l]_q}{[l+g]_q[m-l+g]_q}} \quad \text{where} \quad [z]_q := \frac{q^{\frac{z}{2}} - q^{-\frac{z}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \text{ and } q = e^{\frac{2\pi \mathrm{i}}{2g+m-1}} \tag{4.23}$$

(for any $g > 0$). Theorem 4.4 and Corollaries 4.5, 4.6 apply verbatim to this case, provided we replace the eigenvalues and eigenvectors of $\boldsymbol{L}^{(m)}$ (3.6b) stemming from Theorem 3.2 by those drawn from

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[DG21, Section 4.2] for the tridiagonal matrix

$$\boldsymbol{L}^{(m)} \to \begin{bmatrix} 0 & \frac{[m-1]_q}{[m-1+g]_q} & 0 & \cdots & 0 \\ \frac{[1]_q}{[1+g]_q} & 0 & \ddots & & \vdots \\ 0 & \frac{[2]_q}{[2+g]_q} & \ddots & \frac{[2]_q}{[2+g]_q} & 0 \\ \vdots & & \ddots & 0 & \frac{[1]_q}{[1+g]_q} \\ 0 & \cdots & 0 & \frac{[m-1]_q}{[m-1+g]_q} & 0 \end{bmatrix}. \tag{4.24}$$

Concretely, this boils down to performing the following substitutions in Equations (4.15a)–(4.15c) and Equation (4.16) so as to adapt the input data for Theorem 4.4 and Corollaries 4.5, 4.6 accordingly:

$$\xi_k \to g + k - 1, \quad \mathrm{E}\big(x(\xi)\big) \to q^{\frac{\xi}{2}} + q^{-\frac{\xi}{2}}, \tag{4.25a}$$

$$\Delta_l \to \frac{[g+l-1]_q}{[g]_q} \begin{bmatrix} m-1 \\ l-1 \end{bmatrix}_q \quad \text{with} \quad \begin{bmatrix} m-1 \\ l-1 \end{bmatrix}_q := \frac{\prod_{j=1}^{m-1} [j]_q}{\prod_{j=1}^{l-1} [j]_q \prod_{j=1}^{m-l} [j]_q}, \tag{4.25b}$$

$$\hat{\Delta}_k \to \frac{\Delta_k}{\mathcal{N}} \quad \text{with} \quad \mathcal{N} := \sum_{l=1}^{m} \Delta_l = 2 \prod_{1<k<m} \left| q^{\frac{\xi_k}{2}} - q^{-\frac{\xi_k}{2}} \right|, \tag{4.25c}$$

and

$$\Phi_l(\xi) \to q^{(l-1)g/2} \tfrac{(q;q)_{l-1}}{(q^{1-m};q)_{l-1}} C_{l-1}\Big(\tfrac{1}{2}(q^{\frac{\xi}{2}} + q^{-\frac{\xi}{2}}); q^g | q\Big), \tag{4.25d}$$

where $C_{l-1}(x; q^g|q)$ refers to Rogers' q-ultraspherical polynomial of degree $l-1$ in $x = \frac{1}{2}(q^{\frac{\xi}{2}} + q^{-\frac{\xi}{2}})$ [KLS10, Chapter 14.10.1] (and the normalization factors are expressed in terms of standard q-Pochhammer symbols).Since the pertinent q-ultraspherical polynomials form a one-parameter subfamily of the q-Racah polynomials [DV98, Section 5.4], in principle the corresponding spin chain is a special instance of the inhomogeneous isotropic XY chains proposed in [JV10, Section 4.4] (at least formally, because here we have that $|q| = 1$ while in [JV10, Section 4.4] the authors rather pick $0 < q < 1$). A reminiscent though not identical reduction of the isotropic XY chain associated with the q-Racah polynomials (also in the regime $0 < q < 1$) can be found in [VZ12, Section V].

Finally, the rational limit $\alpha \to 0$ corresponds in our picture to the limit $g \to \infty$ because of the relation (3.4) between the periods and the coupling parameter. After normalizing properly, the finite discrete Lamé equation (3.5) degenerates in this limit to the eigenvalue problem for the celebrated Kac–Sylvester tridiagonal matrix [DG21, Section 4.1]:

$$\boldsymbol{L}^{(m)} \to \begin{bmatrix} 0 & m-1 & 0 & \cdots & 0 \\ 1 & 0 & \ddots & & \vdots \\ 0 & 2 & \ddots & 2 & 0 \\ \vdots & & \ddots & 0 & 1 \\ 0 & \cdots & 0 & m-1 & 0 \end{bmatrix}. \tag{4.26}$$

The associated inhomogeneous isotropic XY chain has a Hamiltonian of the form $\boldsymbol{H}^{(m)}$ (4.1) with

$$\mathrm{J}_l \to \sqrt{l(m-l)}. \tag{4.27}$$

Hence, this recovers the Krawtchouk chain, a particular inhomogeneous open spin chain that is ubiquitous in the literature as a toy model for the transfer of qubit states through quantum wires, cf., for example, [A-E04, B07, BV17, CV10, GS18, K10, NPL03, VZ12] and references therein. The corresponding substitutions adapting the formulas of Theorem 4.4 and Corollaries 4.5, 4.6 to the present case read (cf. [DG21, Section 4.1]):

$$\xi_k \to m + 1 - 2k, \quad \mathrm{E}\big(x(\xi)\big) \to \xi, \quad \Phi_l(\xi_k) \to K_{l-1}(k-1; \tfrac{1}{2}, m-1), \tag{4.28a}$$

and

$$\Delta_l \to \binom{m-1}{l-1}, \qquad \hat{\Delta}_k \to \frac{1}{2^{m-1}}\binom{m-1}{k-1}, \tag{4.28b}$$

where $K_{l-1}(x; \frac{1}{2}, m-1)$ denotes the Krawtchouk polynomial of degree $l-1$ in x at the parameter value $p = \frac{1}{2}$ [KLS10, Chapter 9.11].

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References

[A-E04] C. Albanese, M. Christandl, N. Datta and A. Ekert, 'Mirror inversion of quantum states in linear registers', *Phys. Rev. Lett.* **93**(23) (2004), 230502.

[B07] S. Bose, 'Quantum communication through spin chain dynamics: An introductory overview', *Contemp. Phys.* **48**(1) (2007), 13–30.

[BV17] E.-O. Bossé and L. Vinet, 'Coherent transport in photonic lattices: A survey of recent analytic results', *SIGMA Symmetry Integrability Geom. Methods Appl.* **13** (2017), 074.

[CV10] R. Chakrabarti and J. Van der Jeugt, 'Quantum communication through a spin chain with interaction determined by a Jacobi matrix', *J. Phys. A: Math. Theor.* **43**(8) (2010), 085302.

[CNV19] N. Crampé, R. I. Nepomechie and L. Vinet, 'Free-fermion entanglement and orthogonal polynomials', *J. Stat. Mech.: Theory Exp.* 2019(9) (**2019**), 093101.

[DG21] J.F. van Diejen and T. Görbe, 'Elliptic Kac–Sylvester matrix from difference Lamé equation', *Ann. Henri Poincaré* **23**(1) (2022), 49–65.

[DV98] J. F. van Diejen and L. Vinet, 'The quantum dynamics of the compactified trigonometric Ruijsenaars–Schneider model', *Comm. Math. Phys.* **197**(1) (1998), 33–74.

[D-F23] F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl and M. A. McClain (eds.) *NIST Digital Library of Mathematical Functions*, Release 1.1.11 of 2023-09-15, https://dlmf.nist.gov/.

[FV96] G. Felder and A. Varchenko, 'Algebraic Bethe ansatz for the elliptic quantum group $E_{\tau,\eta}(\mathrm{sl}_2)$', *Nuclear Phys. B* **480**(2) (1996), 485–503.

[FG20] F. Finkel and A. González-López, 'Inhomogeneous XX spin chains and quasi-exactly solvable models', *J. Stat. Mech.: Theory Exp.* 2020(9) (**2020**), 093105.

[GR04] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, second edn., Encyclopedia of Mathematics and its Applications, vol. 96 (Cambridge University Press, Cambridge, 2004).

[GVZ16] V. X. Genest, L. Vinet and A. Zhedanov, 'Quantum spin chains with fractional revival', *Ann. Physics* **371** (2016), 348–367.

[GS18] K. Groenland and K. Schoutens, 'Many-body strategies for multiqubit gates: Quantum control through Krawtchouk-chain dynamics', *Phys. Rev. A* **97**(4) (2018), 042321.

[GVZ13] F. A. Grünbaum, L. Vinet and A. Zhedanov, 'Birth and death processes and quantum spin chains', *J. Math. Phys.* **54**(6) (2013), 062101.

[HSS12] E. Hamza, R. Sims and G. Stolz, 'Dynamical localization in disordered quantum spin systems', *Comm. Math. Phys.* **315**(1) (2012), 215–239.

[I23] V. Inozemtsev, *Integrable Many-Particle Systems* (World Scientific, Hackensack, NJ, 2023).

[JV10] E. I. Jafarov and J. Van der Jeugt, 'Quantum state transfer in spin chains with q-deformed interaction terms', *J. Phys. A: Math. Theor.* **43**(40) (2010), 405301.

[K10] A. Kay, 'A review of perfect, efficient, state transfer and its application as a constructive tool', *International Journal of Quantum Information* **8**(4) (2010), 641–676.

[KL22] R. Klabbers and J. Lamers, 'How coordinate Bethe ansatz works for Inozemtsev model', *Comm. Math. Phys.* **390**(2) (2022), 827–905.

[KLS10] R. Koekoek, P. A. Lesky and R. Swarttouw, *Hypergeometric Orthogonal Polynomials and Their q-Analogues*, Springer Monographs in Mathematics (Springer-Verlag, Berlin, 2010).

[KZ95] I.M. Krichever and A. Zabrodin, 'Spin generalization of the Ruijsenaars–Schneider model, the non-abelian $2D$ Toda chain, and representations of the Sklyanin algebra', *Russian Math. Surveys* **50**(6) (1995), 1101–1150.

[L89] D. Lawden, *Elliptic Functions and Applications*, Applied Mathematical Sciences, vol. 80 (Springer-Verlag, New York, 1989).

[LSM61] E. Lieb, T. Schultz and D. Mattis, 'Two soluble models of an antiferromagnetic chain', *Ann. Physics* **16**(3) (1961), 407–466.

[NPL03] G.M. Nikolopoulos, D. Petrosyan and P. Lambropoulos, 'Coherent electron wavepacket propagation and entanglement in array of coupled quantum dots', *Europhys. Lett.* **65**(3) (2004), 297–303.

[R04] H. Rosengren, 'Sklyanin invariant integration', *Int. Math. Res. Not. IMRN* **2004**(60) (2004), 3207–3232.

[R90] S. N. M. Ruijsenaars, 'Finite-dimensional soliton systems', In *Integrable and Superintegrable Systems* (World Scientific, Singapore, 1990), 165–206.

[R99a] S. N. M. Ruijsenaars, 'Systems of Calogero–Moser type', In *Particles and Fields*, CRM Series in Mathematical Physics (Springer, New York, 1999), 251–352.

[R99b] S. N. M. Ruijsenaars, 'Generalized Lamé functions. I. The elliptic case', *J. Math. Phys.* **40**(3) (1999), 1595–1626.

[R99c] S. N. M. Ruijsenaars, 'Relativistic Lamé functions: the special case $g = 2$', *J. Phys. A: Math. Gen.* **32**(9) (1999), 1737–1772.

[S83] E. K. Sklyanin, 'Some algebraic structures connected with the Yang–Baxter equation. Representations of quantum algebras', *Functional Anal. Appl.* **17**(4) (1983), 273–284.

[SV11] N. I. Stoilova and J. Van der Jeugt, 'An exactly solvable spin chain related to Hahn polynomials', *SIGMA Symmetry Integrability Geom. Methods Appl.* **7** (2011), 033.

[VZ12] L. Vinet and A. Zhedanov, 'How to construct spin chains with perfect state transfer', *Phys. Rev. A* **85**(1) (2012), 012323.

[W78] H. S. Wilf, *Mathematics for the Physical Sciences* (Dover Publications, Inc., New York, 1978).

[W17] P. Woit, *Quantum Theory, Groups and Representations. An Introduction*, (Springer-Verlag, Cham, 2017).

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