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Quasi-Integrals of the Plane Restricted Three-Body Problem

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ABSTRACT

The paper is concerned with studying the domain of possible motion and a field of the test body velocities in the plane restricted problem of three bodies. The study is based on existence of a quasi-integral of areas (similar to an integral of areas in the problem of two bodies) as well as on the Jacobi integral. The method of constructing the quasi-integrals is a standard one (see, for example, [1],[2].

QUASI-INTEGRALS

1.1 Preliminary material

Let us consider a plane dynamic system, consisting of the two point bodies of non-zero masses and a test body. We assume that point bodies move around their barycenter in circular orbits and a test body is moving around one of these bodies.

We denote by M_1 and M_2 masses of the point bodies, by R the distance between the point bodies, by 1 distance between the point body of the mass M_1 and barycenter, by H distance between the point body of the mass M_1 and the libration point L_1, by n the angular velocity of the point bodies, by μ ratio $M_1/(M_1+M_2)$. Assume, for definiteness, that a point body is moving around a body of the mass M_1. Further, we introduce the plane coordinate system xy (see Fig. 1) rotating with a constant angular velocity n, having origin in M_1, axis x, crossing M_2 and such unit of distance that for a test body

$$\max_t [x^2(t) + y^2(t)] = 1.$$

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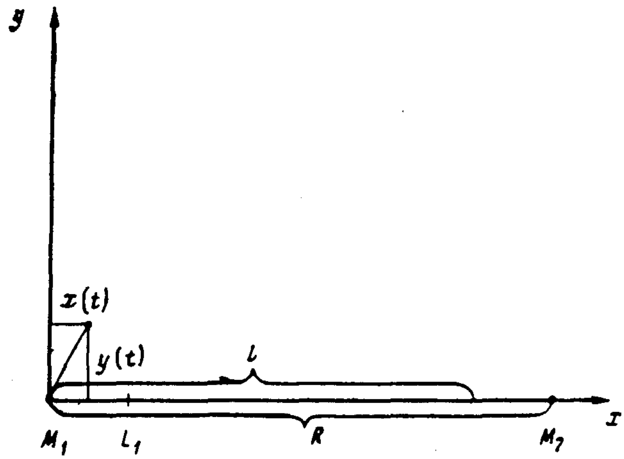

Fig. 1: Coordinate System

It may be easily proved that equations of motion for a test body in such restricted circular problem of three bodies under appropriate choice of the time unit have the form:

$$\begin{cases} \ddot{x}-2n\dot{y}-n^2(x-1) = \dfrac{\partial}{\partial x}\left[\dfrac{\mu}{\sqrt{x^2+y^2}} + \dfrac{1-\mu}{\sqrt{(R-x)^2+y^2}}\right], \\ \ddot{y}+2n\dot{x}-n^2 y = \dfrac{\partial}{\partial y}\left[\dfrac{\mu}{\sqrt{x^2+y^2}} + \dfrac{1-\mu}{\sqrt{(R-x)^2+y^2}}\right], \end{cases} \qquad (1)$$

where

$$\begin{cases} n = R^{-3/2} \\ 1 = R(1-\mu) \end{cases}$$

together with Bolzmann's equation, whose particular solutions are integrals of the system of equations (1) have the form:

$$\frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y} + \frac{\partial f}{\partial \dot{x}}\left\{\frac{2}{R^{3/2}}\dot{y} + \frac{x}{R^3} + \frac{1-\mu}{R^2} - \frac{\mu x}{(x^2+y^2)^{3/2}}\right.$$

$$+ \frac{(1-\mu)(R-x)}{((R-x)^2 + y^2)^{3/2}}$$

$$+ \frac{\partial f}{\partial \dot{y}} \{ \frac{2}{R^{3/2}} \dot{x} + \frac{y}{R^3} - \frac{\mu y}{(x^2+ y^2)^{3/2}} - \frac{(1-\mu)y}{((R-x)^2+ y^2)^{3/2}} \} = 0 \quad (2)$$

It is known that for the system of equations (1) there exists Jacobi integral. In addition to this classical integral we shall attempt constructing approximate integrals i.e. quasi-integrals for (1).

In this paper we assume that one accomplishes the inequalities

$$\begin{cases} 1 < H, \\ 1 << R \end{cases}$$

We introduce a small parameter $\gamma = \sqrt{1/R}$; decompose the coefficients by $\frac{\partial f}{\partial \dot{x}}$ and $\frac{\partial f}{\partial \dot{y}}$ from (2) into series in powers of γ; then the euqation (2) may be rewritten in the form

$$\frac{\partial f}{\partial x} \dot{x} + \frac{\partial f}{\partial y} \dot{y} + \frac{\partial f}{\partial \dot{x}} \{ - \frac{\mu x}{(x^2+ y^2)^{3/2}} + \gamma^3 2\dot{y} + \gamma^6 (3-2\mu)x + \ldots \} +$$

$$+ \frac{\partial f}{\partial \dot{y}} \{ - \frac{\mu y}{(x^2+y^2)^{3/2}} \gamma^3 2\dot{x} + \gamma^6 \mu y + \ldots \} = 0 \quad (3)$$

Let a single-valued particular solution of the equation (3), having the form

$$\sum_{i=0}^{\infty} f_i(x,y,\dot{x},\dot{y})\gamma^i = \text{const.}, \quad (4)$$

be called a quasi-integral of the order q of a test body of the circular restricted three-body problem.

1.2 Jacobi's integral and quasi-integral.

A simple verification shows that the relation

$$0.5(\dot{x}^2 + \dot{y}^2) - 0.5(x^2+y^2)n^2 + n^2 1x - \frac{\mu}{(x^2+y^2)^{1/2}} -$$

$$- \frac{1-\mu}{((R-x)^2+ y^2)^{1/2}} = h \quad (5)$$

where h = const. is an integral. It is the Jacobi integral of restricted circular problem of three bodies. Let us rewrite (5) leaving the terms to the sixth order inclusive:

$$0.5(\dot{x}^2+\dot{y}^2)-\frac{\mu}{(x^2+y^2)^{1/2}}-\gamma^2(1-\mu)-\gamma^6((1.5-\mu)x^2+\frac{\mu y^2}{2})=h \tag{6}$$

It is easy to see that (6) is a quasi-integral of the order 6. Let us name this quasi-integral as the Jacobi quasi-integral.

1.3 Quasi-integral of the areas.

The equation

$$(x\dot{y}-y\dot{x})+\gamma^3\{(x^2+y^2)+\frac{15}{16}(1-\mu)[\frac{-\mu x/(x^2+y^2)^{1/2}+\dot{y}(x\dot{y}-y\dot{x})}{+(\dot{x}^2+\dot{y}^2)/2-\mu/(x^2+y^2)^{1/2}}]\}+$$

$$\gamma^6\{3(1-\mu)\int xy\,dt+15\frac{1-\mu}{4}$$

$$\times\int\frac{(y\dot{y}^2-y\dot{x}^2+2x\dot{x}\dot{y})(-\mu x/(x^2+y^2)^{1/2}+\dot{y}(x\dot{y}-y\dot{x}))}{((\dot{x}^2+\dot{y}^2)/2-\mu/(x^2+y^2)^{1/2})^2}dt\}=C \tag{7}$$

where C = const. shows that it is a quasi-integral of the order 6. We call it quasi-integral of the areas of a test body of the restricted circular problem of three-bodies (if $\gamma \Rightarrow 0$, then (7) becomes the integral of areas $x\dot{y}-y\dot{x}=C$).

Procedures of constructing the quasi-integral (7) consists in a direct substitution of (4) into (3) and by selecting the terms at equal powers γ and searching the single-valued particular solutions of the equations obtained. Thus, function f_o should be a particular solution of the equation.

$$\frac{\partial f_o}{\partial x}\dot{x}+\frac{\partial f_o}{\partial y}\dot{y}-\frac{\partial f_o}{\partial \dot{x}}\frac{\mu x}{(x^2+y^2)^{3/2}}-\frac{\partial f_o}{\partial \dot{y}}\frac{\mu y}{(x^2+y^2)^{3/2}}=0. \tag{8}$$

Obviously

$$f_o=x\dot{y}-y\dot{x} \tag{9}$$

is the particular solution of (8). Similarly, one finds

$$f_1\equiv 0,\quad f_2\equiv 0. \tag{10}$$

For function f_3 one obtains, by taking into account (9) and (10), the equation

$$\frac{\partial f_3}{\partial x}\dot{x} + \frac{\partial f_3}{\partial y}\dot{y} - \frac{\partial f_3}{\partial \dot{x}}\frac{\mu x}{(x^2+y^2)^{3/2}} - \frac{\partial f_3}{\partial \dot{y}}\frac{\mu y}{(x^2+y^2)^{3/2}}$$

$$- 2(x\dot{x} + y\dot{y}) = 0; \qquad (11)$$

for its particular solution

$$f_3 = x^2 + y^2. \qquad (12)$$

Similarly as for functions f_1 and f_2 one finds

$$f_4 \equiv 0, \quad f_5 \equiv 0. \qquad (13)$$

For f_6 the equations (if one considers (9), (10), (12) and (13)) has the form

$$\frac{\partial f_6}{\partial x}\dot{x} + \frac{\partial f_6}{\partial y}\dot{y} - \frac{\partial f_6}{\partial \dot{x}}\frac{\mu x}{(x^2+y^2)^{3/2}} - \frac{\partial f_6}{\partial \dot{y}}\frac{\mu y}{(x^2+y^2)^{3/2}}$$

$$- 3(1-\mu)xy = 0. \qquad (14)$$

$$f_6 = 3(1-\mu)\int xy\, dt \qquad (15)$$

is (see Supplementary notes) the solution of (14). However, if in (15) one passes to osculating elements (see, for example, [3]) then after integrating we obtain

$$f_6 = 3(1-\mu)(a^7/m)^{1/2}\{(1-e^2)^{1/2}(\cos^2\omega-\sin^2\omega)[e\cos E$$

$$+ \frac{1}{3}e\cos^3 E - \frac{1}{2}(1-e^2)\cos^2 E] + \cos\omega\ \sin\omega[\frac{5}{2}e^2 E$$

$$- e(3+e^2)\sin E + \frac{1}{3}e(2-e^2)\sin^3 E + \frac{1}{4}(2+e^2)\sin 2E]\}$$

Since

$$E = (-1)^k \arcsin\frac{x\dot{x} + y\dot{y}}{\left[\frac{\mu^2}{2\mu/(x^2+y^2)^{1/2} - (\dot{x}^2+\dot{y}^2)} - (x\dot{y}-y\dot{x})\right]^{1/2}} + k\pi$$

(where k = 0, ±1, ±2, ...), - the function thus obtained f_6 is ambiguous. (In it one has a summand the term)

$$\frac{15}{2}(1-\mu)(a^7/\mu)^{1/2}\cos\omega\ \sin\omega\ e^2E).$$

In order to find the desired quasi-integral of areas of order 6, instead of (12) one chooses the following single-valued particular solution of equation (11) (in substance, one introduce restrictions on some kinds of symmetry):

$$f_3 = x^2+y^2+\frac{15}{16}(1-\mu)\left[\frac{-\frac{\mu x}{(x^2+y^2)^{1/2}}+\dot{y}(x\dot{y}-y\dot{x})}{0.5(\dot{x}^2+\dot{y}^2)-\mu/(x^2+y^2)^{1/2}}\right]^2 \tag{16}$$

and instead of (15)

$$f_6=3(1-\mu)\int xy\,dt+\frac{15}{4}(1-\mu)\int(y\dot{y}^2-y\dot{x}^2+2x\dot{x}\dot{y})$$

$$\times\frac{-\frac{\mu x}{(x^2+y^2)^{1/2}}+\dot{y}(x\dot{y}-y\dot{x})}{(0.5(\dot{x}^2+\dot{y}^2)-\mu/(x^2+y^2)^{1/2})^2}\,dt. \tag{17}$$

1.4 Addition

By the method presented in section 1.3, one can show that there is no quasi-integral in the system (1), which at $\gamma \Rightarrow 0$ is turned into Laplace's integral.

THE VELOCITY FIELD

Theorem 1. Let C and h be the constants of the quasi-integral of areas and of the Jacobi-quasi-integral of a test body in the restricted circular three-body problem. If $C\neq 0$, $h<0$ and $\mu^2+2hC^2 \geq 0$, then in the vicinity of the mass M_1 the velocity field is double-valued.

Proof. First assume that $\dot{x},\dot{y}$ do not depend on γ, i.e.

$$\dot{x} = A_o(x,y),\qquad \dot{y} = B_o(x,y). \tag{18}$$

Then, it follows from (6) and (7), that possible are two pairs of functions:

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$$A_o = -\frac{Cy}{x^2+y^2} \pm \frac{x \text{ sign } x}{x^2+y^2}[-C^2 + 2(h + \frac{\mu}{(x^2+y^2)^{1/2}})(x^2+y^2)]^{1/2}$$

(19)

$$B_o = -\frac{Cx}{x^2+y^2} \pm \frac{y \text{ sign } x}{x^2+y^2}[-C^2+2(h + \frac{\mu}{(x^2+y^2)^{1/2}}(x^2+y^2)]^{1/2}$$

Introduce the following parameters

$$r = (x^2 + y^2)^{1/2}$$

$$\Delta = (-C^2 + 2(h+ /r)r^2)^{1/2} \qquad (20)$$

Now we assume that $\dot{x}, \dot{y}$ depends upon γ and upto order of γ^4, we take

$$\dot{x} = -Cy/r^2 + A_1\gamma + A_2\gamma^2 + A_3\gamma^3 + A_4\gamma^4$$

$$\pm \frac{x \text{ sign } x}{r^2}[\Delta^2 + B_1\gamma + B_2\gamma^2 + B_3\gamma^3 + B_4\gamma^4]^{1/2}$$

$$\dot{y} = -Cx/r^2 + E_1\gamma + E_2\gamma^2 + E_3\gamma^3 + E_4\gamma^4 + \qquad (21)$$

$$\pm \frac{y \text{ sign } x}{r^2}[\Delta^2 + F_1\gamma + F_2\gamma^2 + F_3\gamma^3 + F_4\gamma^4]^{1/2}$$

Substituting (21) into (6) and (7) and successively retaining terms for $\gamma, \gamma^2, \gamma^3$ and γ^4 , we find the functions

$$A_1 = B = E_1 = F_1 = 0; \qquad (22)$$

$$A_2 = E_2 = 0$$

$$B_2 = F_2 = 2(1-\mu)r^2; \qquad (23)$$

$$A_3 = W_2Cx^2y/2r^4\Delta^2 + W_1y/r^2,$$

$$E_3 = W_2Cy^2x/2r^4\Delta^2 + W_1x/r^2, \qquad (24)$$

$$B_3 = W_2y^2/r^2 + W_1 2C,$$

$$F_3 = W_2x^2/r^2 + W_1 2C,$$

where

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$$W_1 = r^2 + \frac{15}{16}\,\frac{1-\mu}{h^2}\,[\{-\frac{\mu x}{r} + \frac{C^2 x}{r^2}\}^2 + \frac{C^2 y^2 \Delta^2}{r^4}] \tag{25}$$

$$W_2 = \frac{15}{4}\,\frac{1-\mu}{h^2}\,\Delta^2 C[-\frac{\mu}{r} + \frac{C^2}{r^2}] \; ;$$

$$A_4 = B_4 = E_4 = F_4 = 0. \tag{26}$$

DOMAIN OF MOTION OF A TEST BODY

Denote by S_1 the domain of a possible motion of a test body in the vicinity of the point of mass M_1 while computing the boundaries of this domain we have taken into account the terms containing γ^1.

Theorem 2: Let C and h be constants of the quasi-integral of areas and those of the Jacobi quasi-integral in the restricted circular three-body problem. If $C \neq 0$ and $h < 0$, then:
with $\mu^2 + 2hC^2 = 0$ the domain S_3 is a circular ring

$$\frac{-\mu + (2C^2\gamma^2(1-\mu+C\gamma))^{1/2}}{2(h+1(1-\mu)\gamma^2 + C\gamma^3)} \leq r \leq \frac{-\mu-(2C^2\gamma^2(1-\mu + C\gamma))^{1/2}}{2(h+(1-\mu)\gamma^2 + C\gamma^3)}$$

with $\mu^2 + 2hC^2 > 0$ the domain S_3 is an elliptical ring (indices i and j are used to denote the inner and outer boundaries of the ring) having main axes

$$a_{i(e)} = r_{2i(e)} - r^2_{i(e)}\,\frac{C}{2hr_{i(e)} + \mu}[1 + \frac{15}{16}(1-\mu) \times \frac{(\mu r_{i(e)} - C^2)^2}{h^2 r^4_{i(e)}}]\gamma^3$$

$$b_{i(e)} = r^2_{2i(e)} - \frac{C}{2hr_{i(e)} + \mu}\,\gamma^3 , \tag{27}$$

where

$$r_{i(e)} = (-\mu_{(\pm)}(\mu^2 + 2hC^2)/2h, \tag{28}$$

$$r_{2i(e)} = (1 + \frac{1-\mu}{h}\gamma^2)\,\frac{-\mu_{(\pm)}(\mu^2 + 2hC^2)^{1/2}}{2h}$$

$$(\pm)\,\frac{(1-\mu)C^2}{2h(\mu^2 + 2hC^2)^{1/2}}\,\gamma^2 \tag{29}$$

The first part of the theorem follows from the formulae (20)-(25). Let us prove the second part of the theorem.

It follows from the formulae (20)-(22) that the boundary S_1 can be computed by formula (28); from formulae (20)-(23) it follows that the boundary S_2 can be computed by formula (29). Then, it follows from formulae (20)-(24) that the boundary S_3, which is described by the end of vector $r_{3i(e)}$ with the origin in M_1 and with the length

$$r_{3i(e)}(x_{i(e)}\ y_{i(e)}) = r_{i(e)} + \rho_{i(e)}(x_{i(e)}, y_{i(e)})\gamma^3, \qquad (30)$$

is followed by the equality

$$(2hr_{i(e)} + \mu)\rho_{i(e)} + C[r^2_{i(e)} + \frac{15}{16}(1-\mu)\frac{(\mu r_{i(e)} - C^2)x^2_{i(e)}}{h^2 r^4_{i(e)}}]=0. \qquad (31)$$

Equation (30) by virtue of (31) and (20) may be rewritten in the form

$$r_{3i(e)} = r_{2i(e)} - \{\frac{x^2_{i(e)}}{2hr_{i(e)} + \mu} C[1+\frac{15}{16}(1-\mu)\frac{(\mu r_{i(e)} - C^2)^2}{h^2 r^4_{i(e)}}] + \frac{y^2_{i(e)}\ C}{2hr_{i(e)} + \mu}\}\ \gamma^3 .$$

It remains to note that the end of the radius-vector $r_{3i(e)}$ sets an ellipse with main axes (27).

Consequence 1. S_1 and S_2 are circular rings.

Consequence 2. $S_2 \subset S_3|_{C>0}$, $S_2 \supset S_3|_{C<0}$, $S_3|_{C<0} \subset S_3|_{C>0}$ (see Fig. 2)

The first consequence follows from (28) and (29); the second one - from (27) and the obvious inequalities

$$2hr_e + \mu < 0,$$

$$2hr_1 + \mu > 0.$$

SUPPLEMENTARY NOTES

Let us pass in (14) from $x, y, \dot{x}, \dot{y}$ to the osculating elements a, e, ω, M i.e.

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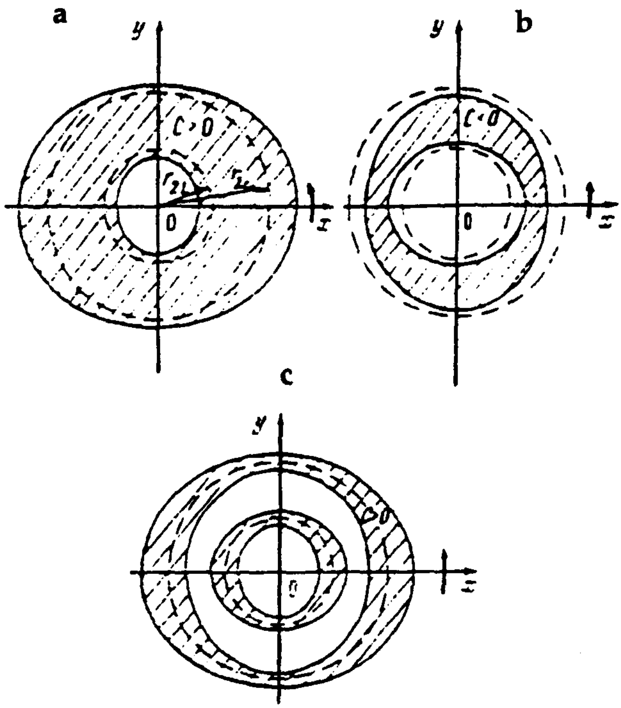

Fig. 2: The domain $S_3|_{C>0}$ (a); $S_3|_{C<0}$(b), and $S_3|_{C>0}$ / $S_3|_{C<0}$ (c) if $|C|$ = const., h = const.

$$\frac{\partial f_6}{\partial x}\dot{x} + \frac{\partial f_6}{\partial y}\dot{y} - \frac{\partial f_6}{\partial \dot{x}}\frac{\mu x}{(x^2+y^2)^{3/2}} - \frac{\partial f_6}{\partial \dot{y}}\frac{\mu y}{(x^2+y^2)^{3/2}}$$

$$= \frac{\partial f_6}{\partial a}\left[\frac{\partial a}{\partial x}\dot{x} + \frac{\partial a}{\partial y}\dot{y} - \frac{\partial a}{\partial \dot{x}} - \frac{\partial a}{\partial \dot{x}}\frac{\mu x}{(x^2+y^2)^{3/2}} - \frac{\partial a}{\partial \dot{y}}\frac{\mu y}{(x^2+y^2)^{3/2}}\right]$$

$$+ \frac{\partial f_6}{\partial e}\left[\frac{\partial e}{\partial x}\dot{x} + \frac{\partial e}{\partial y}\dot{y} - \frac{\partial e}{\partial \dot{x}}\frac{\mu x}{(x^2+y^2)^{3/2}} - \frac{\partial e}{\partial \dot{y}}\frac{\mu y}{(x^2+y^2)^{3/2}}\right]$$

$$+ \frac{\partial f_6}{\partial \omega}\left[\frac{\partial \omega}{\partial x}\dot{x} + \frac{\partial \omega}{\partial y}\dot{y} - \frac{\partial \omega}{\partial \dot{x}}\frac{\mu x}{(x^2+y^2)^{3/2}} - \frac{\partial \omega}{\partial \dot{y}}\frac{\mu y}{(x^2+y^2)^{3/2}}\right]$$

$$+ \frac{\partial f_6}{\partial M}\left[\frac{\partial M}{\partial x}\dot{x} + \frac{\partial M}{\partial y}\dot{y} - \frac{\partial M}{\partial \dot{x}}\frac{\mu x}{(x^2+y^2)^{3/2}} - \frac{\partial M}{\partial \dot{y}}\frac{\mu y}{(x^2+y^2)^{3/2}}\right] \quad (32)$$

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By using well known formulae

$$(\mu a(1-e)^2)^{1/2} = x\dot{y} - y\dot{x}$$

$$\frac{2\mu}{a(1-e\cos E)} - \frac{\mu}{a} = \dot{x}^2 + \dot{y}^2,$$

$$a(\cos E - e) = x\cos\omega + y\sin\omega\ ,$$

$$a(1-e^2)\sin E = -x\sin\omega + y\cos\omega,$$

$$M = E - e\cos E,$$

one can show that in (32).

$$\frac{\partial a}{\partial x}\dot{x} + \frac{\partial a}{\partial y}\dot{y} - \frac{\partial a}{\partial \dot{x}}\frac{\mu x}{(x^2+y^2)^{3/2}} - \frac{\partial a}{\partial \dot{y}}\frac{\mu y}{(x^2+y^2)^{3/2}} = 0,$$

$$\frac{\partial e}{\partial x}\dot{x} + \frac{\partial e}{\partial y}\dot{y} - \frac{\partial e}{\partial \dot{x}}\frac{\mu x}{(x^2+y^2)^{3/2}} - \frac{\partial e}{\partial \dot{y}}\frac{\mu y}{(x^2+y^2)^{3/2}} = 0,$$

$$\frac{\partial \omega}{\partial x}\dot{x} + \frac{\partial \omega}{\partial y}\dot{y} - \frac{\partial \omega}{\partial \dot{x}}\frac{\mu x}{(x^2+y^2)^{3/2}} - \frac{\partial \omega}{\partial \dot{y}}\frac{\mu y}{(x^2+y^2)^{3/2}} = 0,$$

$$\frac{\partial M}{\partial x}\dot{x} + \frac{\partial M}{\partial y}\dot{y} - \frac{\partial M}{\partial \dot{x}}\frac{\mu x}{(x^2+y^2)^{3/2}} - \frac{\partial M}{\partial \dot{y}}\frac{\mu y}{(x^2+y^2)^{3/2}} = 0$$

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