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ON THE WEAK GROTHENDIECK GROUP OF A BEZOUT RING

O. S. SOROKIN
Department of Algebra and Logic, Faculty of Mechanics and Mathematics,
Ivan Franko National University of L'viv, 1 Universytetska str., 79000, Lviv, Ukraine
e-mail: neverhalluet@gmail.com

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Abstract. The K-theoretical aspect of the commutative Bezout rings is established using the arithmetical properties of the Bezout rings in order to obtain a ring of all Smith normal forms of matrices over the Bezout ring. The internal structure and basic properties of such rings are discussed as well as their presentations by the Witt vectors. In a case of a commutative von Neumann regular rings the famous Grothendieck group $K_0(R)$ obtains the alternative description.

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1. Introduction. In [**12**], it is proved that for any element a of a commutative morphic ring R, there is an element $b \in R$ such that the ideals aR, bR coincides with the annihilators $\mathrm{Ann}(b)$, $\mathrm{Ann}(a)$. Therefore, one can group principal ideals into the annihilator pairs, allowing that some ideal may be its own annihilator. So, the structure of principal ideals determines all the properties of the morphic rings.

We will construct an analogue of the Grothendieck group $K_0(R)$ of a Bezout ring R using the cyclically presented modules instead of the finitely generated projective R-modules. Such abelian group, that is denoted as $K_0'(R)$ and is called a weak Grothendieck group of a ring R, becomes a ring if we define a product of two elements of this group using the tensor product of cyclically presented modules. Moreover, the elements of such ring can be interpreted as classes of equivalence of the Smith normal forms of the matrices over a ring R.

In order to obtain a convenient way for the multiplication and the addition for the elements of $K_0'(R)$, we will establish the connection with the subring $W'(G(R))$ of the ring of Witt vectors over some ring. As a direct consequence, it will be proved that the functors K_0' and $W'G$ are naturally equivalent.

As any morphic ring is a Bezout one, we also describe $K_0'(R)$ when R is a morphic ring. In a case of a commutative von Neumann regular ring, the structure of $K_0'(R)$ becomes simpler and, as a consequence, $K_0'(R)$ and the usual Grothendieck group $K_0(R)$ coincide.

The main motivation of these investigations is that in [**13**] it is proved that a commutative Bezout domain is an elementary divisor ring if and only if any quotient ring R/aR is so, where a is an arbitrary nonzero element of R. Since any quotient ring R/aR of a commutative Bezout domain R is a morphic ring [**19**], then the studies of the ring $K_0'(R/aR)$ become related to the famous elementary divisor ring problem [**7,8,10,18**].

2. Preliminaries. All the rings considered in the paper are supposed to be commutative with the nonzero identity element. Let $U(R)$ be a set of all invertible elements of a ring R. By the *Jacobson radical* $J(R)$ of a ring R, we mean the set $J(R) = \{x \in R | \forall a \in R : 1 - ax \in U(R)\}$, and the *nilradical* $Nil(R)$ is defined as an ideal of all nilpotent elements of a ring R. A ring R is called a *reduced* ring if $Nil(R) = \{0\}$.

Suppose that A is a subset in a ring R. A set $\mathrm{Ann}(A) = \{x | Ax = 0\}$ is called an *annihilator* of a set A. If an element $a \in R$ is a divisor of an element $b \in R$, then we will write $a|b$.

We start with recalling of some definitions and facts that we will need below in our proofs.

DEFINITION 1.

(1) If any finitely generated ideal of a ring R is principal, then a ring R is said to be a *Bezout ring*.
(2) We say that a rectangular matrix A over a ring R *admits canonical diagonal reduction* if there are two invertible matrices P, Q of the appropriate sizes such that the matrix $PAQ = D = (d_i)$ is a diagonal matrix with an additional condition: for all indices we have the inclusion of the ideals $d_{i+1}R \subseteq d_iR$.
(3) If every row matrix (a, b) (column matrix $(a, b)^T$) admits canonical diagonal reduction, then we say that R is a *right (left) Hermite ring*.
(4) If every matrix over a ring R admits canonical diagonal reduction, then R is said to be an *elementary divisor ring*.
By the cyclically presented module over a ring R, we mean the R-module R/aR, where $a \in R$. Using Bourbaki [**2,3**], one can show that a finitely presented cyclic module over a Bezout ring is cyclically presented.

DEFINITION 2. We say that a ring R has *the stable range 1* if for any elements $a, b \in R$ the equality $aR + bR = R$ implies that there is some $x \in R$ such that $(a + bx)R = R$ [**1**, **15**].

DEFINITION 3. A ring R is called a *morphic ring* if for any $a \in R$ there is an R-module isomorphism $R/aR \cong \mathrm{Ann}(a)$ [**12**].

Here is a Nicholson's criterion for a morphic ring.

THEOREM 1 ([**12**]). *The following statements are equivalent for a ring R:*
(a) *R is a morphic ring;*
(b) *For any $a \in R$, one can find $b \in R$ such that* $\mathrm{Ann}(a) = bR$, $\mathrm{Ann}(b) = aR$;
(c) *For any $a \in R$, one can find $b \in R$ such that* $\mathrm{Ann}(a) = bR$, $\mathrm{Ann}(b) \cong aR$.

DEFINITION 4. An element a of a ring R is said to be a *von Neumann regular element* if there is some $b \in R$ such that $aba = a$. If all elements of a ring R are von Neumann regular, then R is called a *von Neumann regular ring* [**5**].

Also in [**12**], it is proved every the commutative von Neumann regular ring is a morphic one. Moreover, there are morphic rings, that are not regular. For example, a ring $H = \mathbb{Z} + x\mathbb{Q}[[x]]$ is a commutative Bezout domain, but by [**19**] H/xH is a morphic ring of stable range 2 (take $a = 5, b = 7$), so it is not von Neumann regular, as they are the rings of stable range 1. In addition, it is useful to mention that a pair (a, b) of elements of a ring R in the previous theorem is called a **morphic pair** and this fact will be denoted as $aR \sim_M bR$, since any morphic pair is determined by the pair of some principal ideals, but not elements.

3. The weak Grothendieck group. Let R be a commutative Bezout ring. We will try to construct an analogue $K_0'(R)$ of the Grothendieck group $K_0(R)$ considering the isomorphism classes of the finite direct sums of cyclically presented modules over R as the basic objects and using some ideas from [**11**].

Let $\Delta(R) = \{R/a_1R \oplus \cdots \oplus R/a_nR | a_1, \ldots, a_n \in R\}$ be a set of all finite direct sums of the cyclically presented modules over R. Then, we consider a relation "$\sim$" on the set $\Delta(R)$ defined as

$$g_1 \sim g_2 \ \Leftrightarrow g_1 \cong g_2$$

for $g_1, g_2 \in \Delta(R)$. Then, let $F(R)$ be a free abelian group generated by the set $\Delta(R)/\sim$. Since every element of $\Delta(R)/\sim$ is in the one-to-one correspondence with the set of all finite diagonal matrices of R and by [**10**] every finitely presented module named by the diagonal matrix D have canonical form, and D is equivalent to its Smith normal form (shortly SNF), so in any class of the equivalent elements in $\Delta(R)$ we can choose some SNF that represents this class in $\Delta(R)/\sim$. In fact, one can consider $F(R)$ as a free abelian group generated by the classes of equivalence of all SNF of the matrices over R. The elements of the set $\Delta(R)/\sim$ we will denote as $\mathrm{SNF}(g)$, where $g \in \Delta(R)$.

Definition 5. The quotient group $K_0'(R)$ of a free abelian group $F(R)$ by the subgroup generated by all expressions of the form $\mathrm{SNF}(g) + \mathrm{SNF}(g') - \mathrm{SNF}(g \oplus g')$ we will call a *weak Grothendieck group* of a morphic ring R. The elements of $K_0'(R)$ will be denoted as $[g]$.

In other words, $K_0'(R)$ is an abelian group of all classes of isomorphic finite direct sums of principal ideals of a morphic ring R with the following property:

$$[g] + [g'] = [g \oplus g'],$$

for any $[g], [g'] \in K_0'(R)$.

Remark, that $\mathrm{SNF}(R/a_1R \oplus \cdots \oplus R/a_nR) = \mathrm{SNF}(R/b_1R \oplus \cdots \oplus R/b_mR)$ in $\Delta(R)/\sim$ if and only if

$$\bigoplus_{i=1}^{n} R/a_iR \cong \bigoplus_{j=1}^{m} R/b_jR$$

and

$$\sum_{j=1}^{m} \mathrm{SNF}(g_j) = \sum_{k=1}^{l} \mathrm{SNF}(g_k')$$

in $F(R)$ if and only if $m = l$ and there is a permutation $\pi \in S_m$ such that $\forall j : \ g_j \cong g'_{\pi(j)}$.

Lemma 1. *Two elements* $[g], [g'] \in K_0'(R)$ *are equal if and only if*

$$g \oplus X \cong g' \oplus X,$$

for some $X = R/a_1R \oplus \cdots \oplus R/a_nR$*, where* $a_1, \ldots, a_n \in R$.

Proof. Suppose that $[g] = [g']$. Then

$$\mathrm{SNF}(g) - \mathrm{SNF}(g') = \sum_i (\mathrm{SNF}(x_i) + \mathrm{SNF}(y_i) - \mathrm{SNF}(x_i \oplus y_i))$$
$$- \sum_j (\mathrm{SNF}(x'_j) + \mathrm{SNF}(y'_j) - \mathrm{SNF}(x'_j \oplus y'_j)),$$

for some $\mathrm{SNF}(x_i),\ \mathrm{SNF}(y_i),\ \mathrm{SNF}(x'_j),\ \mathrm{SNF}(y'_j) \in F(R)$.

After placing the summands with negative signs to the another part of the equality and using the previous remark, we obtain that

$$g \oplus X \cong g' \oplus X,$$

where $X = (\bigoplus_i (x_i \oplus y_i)) \oplus (\bigoplus_j (x'_j \oplus y'_j)) \in \Delta(R)$.

Conversely, if $g \oplus X \cong g' \oplus X$ then $\mathrm{SNF}(g \oplus X) = \mathrm{SNF}(g' \oplus X)$ in $\Delta(R)/\sim$. Hence, $[g \oplus X] = [g' \oplus X]$ in $K'_0(R)$ implies that $[g] + [X] = [g'] + [X]$ and $[g] = [g']$. The lemma is proved. □

Now we need to formulate one well-known property of the tensor product of modules over the commutative ring.

PROPOSITION 1 **[9]**. *Let M be a R-module and I, J are some ideals of a commutative ring R. Then*

(i) $M \otimes_R R/I \cong M/IM$;
(ii) $R/I \otimes_R R/J \cong R/(I + J)$.

In the following lemma, we apply this Proposition in order to obtain one surprising property of the principal ideals of a morphic ring, and one rather obvous for the Bezout ring.

LEMMA 2. *For every pair of elements $a, b \in R$ of a morphic ring R: $aR \otimes_R bR \cong aR \cap bR$. If R is a Bezout ring, then $R/aR \otimes_R R/bR \cong R/(aR + bR)$.*

Proof. By the definition of a morphic ring and the above-mentioned result, we have that $aR \otimes_R bR \cong R/\mathrm{Ann}(a) \otimes_R R/\mathrm{Ann}(b) \cong R/(\mathrm{Ann}(a) + \mathrm{Ann}(b)) \cong \mathrm{Ann}(\mathrm{Ann}(a) + \mathrm{Ann}(b)) \cong aR \cap bR$ as was desired. The other statement follows from Proposition 1. □

It is useful to remak, that from previous lemma we obtain that tensor product of two principal ideals in the morphic ring is isomorphic to principal ideal as the intersection of principal ideals in any Bezout (and hence morphic) ring is again principal one **[10]**.

In the classical K-theoretical investigations, the Grothendieck's group $K_0(R)$ can be considered as a ring if we assume that R is a commutative ring and the product is defined as

$$[P] \cdot [Q] = [P \otimes_R Q]$$

for any finitely generated projective R-modules P and Q over a commutative ring R. In the similar manner, we obtain

THEOREM 2. *Let R be a commutative morphic ring. Then, an additive abelian group $K_0'(R)$ becomes a commutative ring with 1, if we define a product*

$$[R/aR][R/bR] = [R/aR \otimes_R R/bR],$$

for any $a, b \in R$, and extend it on the arbitrary elements of $K_0'(R)$ by the linearity.

REMARK. The previous lemma shows that the multiplication in $K_0'(R)$ is defined correctly: for Bezout rings $aR + bR$ is principal ideal, and $aR \cap bR$ is principal too for the case of morphic rings. Remark that any element any element of $K_0'(R)$ can be written as $[A] - [B]$, where $A, B \in \Delta(R)$ are reduced to the SNF and there is no pair of terms R/a_iR, R/b_jR such that it can be cancellated in the expression $[A] - [B]$. In the case of a morphic ring

$$\Delta(R) = \{a_1R \oplus \cdots \oplus a_nR | a_1, \ldots, a_n \in R\},$$

so in fact we deal with principal ideals and $[aR][bR] = [aR \cap bR]$.

Now we try to understand how $K_0'(R)$ behaves under the base ring R change and how one can describe its structure in the simplest case.

PROPOSITION 2. *K_0' is a functor from the category **BezoutRings** of the Bezout rings to the **Rings** category.*

Proof. Let $f : R \to R'$ be a homomorphism of the morphic rings. For any element

$$M = R/m_1R \oplus \cdots \oplus R/m_nR \in \Delta(R)$$

one can define an element $M' = f_\sharp(M)$ in the following manner:

$$f_\sharp(M) = R' \otimes_R M \cong \bigoplus_{i=1}^{n}(R' \otimes_R R/m_iR) \cong \bigoplus_{i=1}^{n}(R'/f(m_i)R') \in \Delta(R').$$

Thus, a ring's map $f : R \to R'$ rises a correspondence

$$f_* : \begin{cases} K_0'(R) & \to K_0'(R') \\ [A] - [B] & \mapsto [f_\sharp(A)] - [f_\sharp(B)] \end{cases}$$

of the abelian groups $K_0'(R)$ and $K_0'(R')$. Moreover, if $[R/aR], [R/bR] \in K_0'(R)$ then

$$f_*([R/aR \otimes_R R/bR]) = f_*([R/(aR + bR)]) = [R' \otimes_R R/(aR + bR)] =$$

$$= [R'/(f(a)R' + f(b)R')] = [R'/f(a)R' \otimes_R R'/f(b)R'] = [R'/f(a)R'] \cdot [R'/f(b)R']$$

So f_* becomes a ring's homomorphism. Therefore,

$$K_0' : \textbf{BezoutRings} \rightsquigarrow \textbf{Rings}$$

is a map from the category **BezoutRings** of all Bezout rings and their homomorphisms to the category **Rings** defined by the rule

$$K_0' : \begin{cases} R & \mapsto K_0'(R) \\ R \xrightarrow{f} R' & \mapsto K_0'(R) \xrightarrow{f_*} K_0'(R') \end{cases}.$$

Then, we need to verify: is it a functor or not? Indeed, if $f = 1_R : R \to R$ then for any $[A] - [B] \in K_0'(R)$ we obtain

$$f_*([A] - [B]) = [f_\sharp(A)] - [f_\sharp(B)] = [R \otimes_R A] - [R \otimes_R B] = [A] - [B]$$

and hence $K_0'(1_R) = 1_{K_0'(R)}$. If $R \xrightarrow{f} R' \xrightarrow{g} R''$ are two homomorphisms of the morphic rings, then we need to prove that

$$(g \circ f)_* = g_* \circ f_*.$$

Without loss of the generality, take any $[R/aR] \in K_0'(R)$. Then

$$(g \circ f)_*([R/aR]) = [R'' \otimes_R R/aR] = [R''/g(f(a))R'']$$

and

$$\begin{aligned} g_*(f_*([R/aR])) &= g_*([R' \otimes_R R/aR]) = g_*(R'/f(a)R') \\ &= [R'' \otimes_{R'} R'/f(a)R'] = [R''/g(f(a))R''] \end{aligned}$$

as was desired. So, K_0' is a functor. The proposition is proved. □

The similar arguments can prove that K_0' : **MorphicRings** $\rightsquigarrow$ **Rings** is a functor too. As a consequence, it can be shown that K_0' preserves direct products of the Bezout rings as well as morphic ones:

$$K_0'\Big(\prod_i R_i\Big) \cong \prod_i K_0'(R_i).$$

THEOREM 3. *Let R be a Bezout ring. Then, $K_0'(R)$ has a direct summand isomorphic to the ring of integers $\mathbb{Z}$.*

Proof. Considering any maximal ideal M of R, we define a natural homomorphism

$$f : R \to F = R/M$$

of a ring R onto a field F. Then, K_0' induces a homomorphism

$$f_* : K_0'(R) \to \mathbb{Z} \cong K_0'(F).$$

A map

$$i_* : \begin{cases} \mathbb{Z} & \to K_0'(R) \\ n & \mapsto n[R] \end{cases}$$

is a monomorphism such that $f_* i_* = 1_{\mathbb{Z}}$ and the following short exact sequence

$$0 \to \ker f_* \to K_0'(R) \xrightarrow{f_*} \mathbb{Z} \to 0$$

splits

$$K_0'(R) \cong \mathbb{Z} \oplus \ker f_*.$$

The latter isomorphism proves the theorem. □

THEOREM 4. *If R is a Bezout ring, then $K_0'(R) \cong \mathbb{Z}$ if and only if R is a field.*

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Proof. By the previous theorem, we know that $K_0'(R) \cong \mathbb{Z}$ if and only if for any maximal ideal M of a ring R

$$\ker f_* = 0$$

where $f : R \to F = R/M$ is a natural homomorphism.

Suppose that $\ker f_* = \{[A] - [B] \mid f_*([A]) = f_*([B])\} = 0$. This means that whether $f_*([A]) = f_*([B])$ then $[A] = [B]$. Since R is a morphic ring an element $A \in \Delta(R)$ can be considered in the form

$$A = R/a_1R \oplus \cdots \oplus R/a_nR$$

where $a_1R \subseteq \cdots \subseteq a_nR$. Then

$$f_\sharp(R/a_iR) = R/M \otimes_R R/a_iR \cong R/(M + a_iR) = \begin{cases} F, & a_i \in M \\ 0, & a_i \notin M \end{cases}.$$

Hence, $f_*([A]) = k$, where $k \in \{1, \dots, n\}$ is such that $a_1, a_2, \dots, a_k \in M$, $a_{k+1}, \dots, a_n \notin M$. Thus, the equality $f_*([A]) = f_*([B])$ means that the chains of the ideals $a_1R \subseteq \cdots \subseteq a_nR$ and $b_1R \subseteq \cdots \subseteq b_mR$ have the same number of ideals inside M. Therefore, the condition $\ker f_* = 0$ means that there is at most one chain of the ideals $a_1R \subseteq \cdots \subseteq a_kR$ of the length k inside M for any $k \geq 0$. But if we take a unique chain of the ideals of the length $k \geq 2$ inside M, then any term of this chain is itself a chain of the length 1 inside M. But all chains of the length 1 are equal, so there is at most one principal ideal aR inside M. If $M \neq aR$ then there is $b \in M \setminus aR$ such that $aR \neq bR$. But this is impossible since aR is a unique chain of the length 1, so $M = aR$.

Moreover, if we take $[A] = [R \oplus R \oplus R/aR]$ and $[B] = [R \oplus R/aR]$, then $f_*([A]) = f_*([B]) = 1$ and hence $[A] = [B]$. Then

$$[R \oplus R \oplus R/aR] = [R \oplus R/aR]$$

implies that $[R] = [0]$ that is a contradiction, and such maximal ideal M cannot exist. After repeating the similar procedure to the other maximal ideals, we obtain that there are no maximal ideals in R and R have to be a field. The theorem is proved. □

4. The connection of K_0' and the Witt vectors. In the current section, we will try to find a convenient way for the addition and multiplication of the elements of K_0'.

DEFINITION 6. A *Witt ring* (or *Witt vectors*) for a commutative ring R is called a set

$$W(R) = 1 + tR[[t]] = \{1 + a_1t + a_2t^2 + \cdots | a_1, a_2, \dots \in R\}$$

that is an abelian group under the multiplication operation between the formal power series (this operation represents the additive operation of a ring $W(R)$) and the ring multiplication operation is defined by the convolution rule in the following way: any $f(t) \in W(R)$ can be written as

$$f(t) = \prod_{i=1}^{\infty}(1 + r_it)$$

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so for the arbitrary $r \in R$ we define

$$(1+rt) * f(t) = f(rt)$$

and extend this rule to the infinite products. An identity of a ring $W(R)$ is an element $1+t$ and 1 is a zero element.

Before applying the mentioned ring construction, we need to formulate the following definition.

DEFINITION 7. Let R be a commutative Bezout ring. A semiring

$$G(R) = \{aR \mid a \in R\}$$

of all its principal ideals under the addition and intersection of the ideals is called a *globalization of a ring R*.

Thus, for any be a commutative Bezout ring R, we can consider a subsemiring

$$W_0(G(R)) = \{1 + (a_1R)t + (a_2R)t^2 + \cdots + (a_nR)t^n | a_1R \supseteq \cdots \supseteq a_nR, n \geq 0\}$$

of a ring $W(G(R))$ considered as a semiring.

Any element $f(t) = 1 + (a_1R)t + (a_2R)t^2 + \cdots + (a_nR)t^n \in W_0(G(R))$ can be expressed in the form

$$f(t) = \prod_{i=1}^{n}(1 + a_iRt).$$

An identity element of $W_0(G(R))$ is $1 + Rt$ and 1 is its zero element. Furthermore, if

$$f(t) = 1 + \sum_{i=1}^{n}(a_iR)t^i, \; g(t) = 1 + \sum_{j=1}^{m}(b_jR)t^j$$

are any elements of $W_0(G(R))$ then their sum and product can be computed by the formulae

$$f(t) \cdot g(t) = 1 + \sum_{k=1}^{n+m}(\sum_{i+j=k}(a_iR \cap b_jR))t^k,$$
$$f(t) * g(t) = \prod_{j=1}^{m} f((b_jR)t) = \prod_{i=1}^{n} g((a_iR)t) = \prod_{i,j=1,1}^{n,m}(1 + (a_iR \cap b_jR)t).$$

After the direct computations, one can conclude that the above definitions of the sum and product also belongs to $W_0(G(R))$.

THEOREM 5. *If R is a Bezout ring, then*

$$K_0'(R) \cong W'(G(R)),$$

where $W'(R)$ is a ring completion of a semiring $W_0(G(R))$.

Proof. In the following consideration, the subtraction operation in the ring completion $W'(G(R))$ will be denoted by $\frac{f(t)}{g(t)}$, for $f(t), g(t)$. In fact, it is a formal

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polynomial's division. So, the ring $W'(R)$ can be described as

$$W'(R) = \left\{ \frac{1+(a_1R)t+\cdots+(a_nR)t^n}{1+(b_1R)t+\cdots+(b_mR)t^m} \,\middle|\, a_1R \supseteq \cdots \supseteq a_nR, \right.$$
$$\left. b_1R \supseteq \cdots \supseteq b_mR,\ n,m \geq 0 \right\}$$

with the addition

$$\frac{a(t)}{b(t)} \cdot \frac{c(t)}{d(t)} = \frac{a(t)c(t)}{b(t)d(t)}$$

and multiplication

$$\frac{a(t)}{b(t)} * \frac{c(t)}{d(t)} = \frac{(a(t)*c(t))(b(t)*d(t))}{(a(t)*d(t))(b(t)*c(t))}$$

for any $a(t), b(t), c(t), d(t) \in W_0(G(R))$. We define a map

$$F_R : \begin{cases} K_0'(R) & \to W'(G(R)) \\ [A]-[B] & \mapsto \frac{a(t)}{b(t)} = (1+\sum_{i=1}^n (a_iR)t^i)/(1+\sum_{j=1}^m (b_jR)t^j) \end{cases}$$

where $A = R/a_1R \oplus \cdots \oplus R/a_nR,\ B = R/b_1R \oplus \cdots \oplus R/b_mR$ are reduced to the SNF. The map F_R is a bijection since the SNF is defined uniquely and $a(t)$ and $b(t)$ have a uniquely determined decompositions

$$a(t) = \prod_{i=1}^{n}(1+(a_iR)t),\ \ b(t) = \prod_{j=1}^{m}(1+(b_jR)t).$$

Also, it is a homomorphism since

$$\begin{aligned} F_R(([A]-[B])+([C]-[D])) &= F_R([A\oplus C]-[B\oplus D]) = \frac{a(t)c(t)}{b(t)d(t)} \\ &= F_R([A]-[B]) \cdot F_R([C]-[D]) \end{aligned}$$

and

$$\begin{aligned} F_R(([A]-[B])([C]-[D])) &= F_R([A\otimes_R C]+[B\otimes_R D]-[B\otimes_R C]-[A\otimes_R D]) \\ &= \frac{\prod_{i,j}(1+(a_iR\cap c_jR)t)\prod_{k,l}(1+(b_kR\cap d_lR)t)}{\prod_{i,l}(1+(a_iR\cap d_lR)t)\prod_{k,j}(1+(b_kR\cap c_jR)t)} \\ &= \frac{(a(t)*c(t))(b(t)*d(t))}{(a(t)*d(t))(b(t)*c(t))} = \frac{a(t)}{b(t)} * \frac{c(t)}{d(t)} = F_R([A]-[B]) * F_R([C]-[D]). \end{aligned}$$

Thus, F_R is an isomorphism. The theorem is proved. □

PROPOSITION 3. *The maps*

$$\begin{aligned} &W' = W_0^{-1}W_0 : \textbf{\textit{Rings}} \rightsquigarrow \textbf{\textit{Rings}} \\ &W_0 : \textbf{\textit{Semirings}} \rightsquigarrow \textbf{\textit{Semirings}} \\ &G : \textbf{\textit{BezoutRings}} \rightsquigarrow \textbf{\textit{Semirings}} \end{aligned}$$

defined above are the functors.

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Proof. If we set

$$W'(f):\begin{cases} W'(R) & \to W'(R') \\ \frac{1+a_1t+\cdots+a_nt^n}{1+b_1t+\cdots+b_mt^m} & \mapsto \frac{1+f(a_1)t+\cdots+f(a_n)t^n}{1+f(b_1)t+\cdots+f(b_m)t^m} \end{cases}$$

$$W_0(f):\begin{cases} W_0(R) & \to W_0(R') \\ 1+a_1t+\cdots+a_nt^n & \mapsto 1+f(a_1)t+\cdots+f(a_n)t^n \end{cases}$$

$$G(f):\begin{cases} G(R) & \to G(R') \\ aR & \mapsto f(a)R' \end{cases}$$

for any homomorphism $f: R \to R'$ in the appropriate source category, then images of f such as $W'(f)$, $W_0(f)$, $G(f)$ are precisely the homomorphisms in the target categories of the given maps. The fact that W', W_0 and G preserves identity homomorphisms and the compositions can be shown by the routine calculations. So, W', W_0 and G are the functors. □

THEOREM 6. *There is a natural equivalence of functors*

$$K_0' \approx W'G.$$

Proof. By Theorem 5 $K_0'(R) \cong W'(G(R))$ for any Bezout ring R via the isomorphism F_R. So, if $f: R \to R'$ is any homomorphism of Bezout rings R and R' then

$$\begin{array}{ccc} K_0'(R) & \xrightarrow{F_R} & W'(G(R)) \\ \downarrow{\scriptstyle K_0'(f)} & & \downarrow{\scriptstyle W'(G(f))} \\ K_0'(R') & \xrightarrow{F_{R'}} & W'(G(R')) \end{array}$$

is a commutative diagram since

$$(W'(G(f)) \circ F_R)([A]-[B]) = W'(G(f))\left(\left(1+\sum_{i=1}^{n}(a_iR)t^i)/(1+\sum_{j=1}^{m}(b_jR)t^j\right)\right)$$

$$= \left(1+\sum_{i=1}^{n}(f(a_i)R')t^i\right) \Big/ \left(1+\sum_{j=1}^{m}(f(b_j)R')t^j\right) = \left(F_{R'} \circ K_0'(f)\right)([A]-[B]).$$

Thus, $K_0' \approx W'G$ as was desired. The theorem is proved. □

The latter result shows the way that one can compute the SNF of the block sum $A \oplus B$ and Kroneker's product $A \otimes B$ of two given matrices A and B that are already reduced to their SNF's. Since the multiplication in $K_0'(R)$ can be done after some number of the addition operations (this follows from the distributivity of the tensor product over the direct sums) then naturally arises a question: are there any other way to represent the elements of $K_0'(R)$ for more efficient evaluation of the sums of the given elements?

The answer is affirmative and below we give a solution. If $[A] = [R/a_1R \oplus \cdots \oplus R/a_nR]$ and $[B] = [R/b_1R \oplus \cdots \oplus R/b_mR]$ are the elements of $K_0'(R)$ that are reduced to their SNF, then we represent $[A]$ and $[B]$ in a form

$$[X] \mapsto \begin{pmatrix} x_n & x_{n-1} & \cdots & x_1 & 1 & 0 & 0 & \cdots 0 & 0 \\ 0 & x_n & \cdots & x_2 & x_1 & 1 & 0 & \cdots 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots x_n & x_{n-1} \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots 0 & x_n \end{pmatrix} \in M_{n+m}(G(R))$$

and multiply the respective matrices for $[A]$ and $[B]$ in $M_{n+m}(G(R))$, then the resulting matrix will represent the sum $[A \oplus B]$. In other words: if J is a Jordan matrix in $M_{n+m}(G(R))$ with the zero eigenvalue, then

$$[A] \leftrightarrow a_nE + a_{n-1}J + \cdots + a_1J^{n-1} + J^n$$

$$[B] \leftrightarrow b_nE + b_{m-1}J + \cdots + b_1J^{m-1} + J^m$$

$$[A \oplus B] \leftrightarrow (a_nE + \cdots + a_1J^{n-1} + J^n)(b_nE + \cdots + b_1J^{m-1} + J^m)$$

and the latter product will be the necessary result. On the other hand, it is not necessary to multiply pairwise every row and column in order to obtain the result. In fact, the situation can be solved even simpler – the product

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n+m} \end{pmatrix} = \begin{pmatrix} a_n & a_{n-1} & \cdots & a_1 & 1 & 0 & 0 & \cdots 0 & 0 & \\ 0 & a_n & \cdots & a_2 & a_1 & 1 & 0 & \cdots 0 & 0 & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots a_n & a_{n-1} & \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots 0 & a_n & \end{pmatrix} \times \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ b_1 \\ \vdots \\ b_{m-1} \\ b_m \end{pmatrix}$$

represents the sum $[C] = [A \oplus B]$ in the SNF.

5. The cancellation of cyclically presented modules over Bezout rings. In the case of Bezout and morphic rings, the computations that involves the elements of the group $K_0'(R)$ rise to the expressions with the principal ideals. So we need to know: how the preimages of the equal elements of $K_0'(R)$ can be described in the terms of R?

LEMMA 3 (Cancellation Lemma). *Let R be a Bezout ring and $A, B, R/xR \in \Delta(R)$ and A, B are reduced to the SNF. Then*

$$A \oplus R/xR \cong B \oplus R/xR \Leftrightarrow A = B.$$

Proof. Suppose that $A = R/a_1R \oplus \cdots \oplus R/a_nR$, $B = R/b_1R \oplus \cdots \oplus R/b_{n+m}R$, $a_1R \supseteq \cdots \supseteq a_nR$, $b_1R \supseteq \cdots \supseteq b_{n+m}R$. By **[8]** if A_1 and B_1 are the SNF of $A \oplus R/xR$

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and $B \oplus R/xR$, then $A_1 = B_1$. So we are going to compute explicitly SNF A_1 and B_1 using the Fitting invariants.

For the simplification of the notations in the proof below, we will write $a+b$ and ab for $aR+bR$ and $aR \cap bR$ respectively. The ordering "$\le$" corresponds to the natural inclusion of the sets.

After computing the Fitting invariants, the normal forms A_1 and B_1 are

$$\begin{aligned}A_1 &= (a_1+x)\oplus(a_2+a_1x)\oplus\cdots\oplus(a_n+a_{n-1}x)\oplus(a_nx)\\ B_1 &= (b_1+x)\oplus(b_2+b_1x)\oplus\cdots\oplus(b_n+b_{n-1}x)\oplus(b_{n+1}+b_nx)\\ &\qquad\oplus\cdots\oplus(b_{n+m-1}+b_{n+m}x)\oplus(b_{n+m}x)\end{aligned}$$

From the equality $A_1 = B_1$, we obtain the system of the principal ideal equations:

$$\begin{cases} b_1+x=a_1+x\\ b_2+b_1x=a_2+a_1x\\ \dots\\ b_n+b_{n-1}x=a_n+a_{n-1}x\\ b_{n+1}+b_nx=a_nx\end{cases},\qquad \begin{cases} b_{n+2}+b_{n+1}x=0\\ \dots\\ b_{n+m-1}+b_{n+m}x=0\\ b_{n+m}x=0\end{cases}$$

From the equation $b_{n+2}+b_{n+1}x=0$, we obtain that $b_{n+2}=0$ and so $b_{n+2}=\cdots=b_{n+m}=0$. As $b_{n+1}+b_nx=a_nx$, we conclude that $b_{n+1}\le x$ and $b_nx=a_nx$.

Multiplying (in fact intersecting!) by a_n the equation $b_n+b_{n-1}x=a_n+a_{n-1}x$, we obtain

$$a_n+a_nx=a_nb_n+a_nb_{n-1}x=a_nb_n+b_nb_{n-1}x=a_nb_n+b_nx\le b_n.$$

But $a_n=a_n+a_nx$ and so $a_n\le b_n$. Analogously, multiplying the same equation by b_n we will have that $b_n\le a_n$. So $a_n=b_n$.

Again, from the equation $b_{n+2}+b_{n+1}x=0$ we conclude that $b_{n+1}x=0$. Therefore, multiplying the equation $b_{n+1}+b_nx=a_nx$ by the ideal b_{n+1} we have

$$b_{n+1}=b_{n+1}+b_nb_{n+1}x=a_nb_{n+1}x=b_nb_{n+1}x=0.$$

So, the given system of the principal ideal equations simplifies and we have

$$\begin{cases} b_1+x=a_1+x\\ b_2+b_1x=a_2+a_1x\\ \dots\\ b_{n-1}+b_{n-2}x=a_{n-1}+a_{n-2}x\\ b_n+b_{n-1}x=a_n+a_{n-1}x\\ b_nx=a_nx\end{cases}$$

Again, we multiply the equation $b_{n-1}+b_{n-2}x=a_{n-1}+a_{n-2}x$ by a_{n-1} and hence obtain

$$a_{n-1}b_{n-1}+a_{n-1}b_{n-2}x=a_{n-1}$$

If we multiply by x the equation $b_n+b_{n-1}x=a_n+a_{n-1}x$, we will obtain

$$b_nx+b_{n-1}x=a_nx+a_{n-1}x$$

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and hence $b_{n-1}x = a_{n-1}x$. Thus, the equation

$$a_{n-1} = a_{n-1}b_{n-1} + a_{n-1}b_{n-2}x$$

implies that $a_{n-1} \leq b_{n-1}$. Similarly, $b_{n-1} \leq a_{n-1}$. Therefore, $a_{n-1} = b_{n-1}$. After the finite number of steps using the prescribed procedure, we will have the following finally reduced system:

$$\begin{cases} b_1 + x = a_1 + x \\ b_1 x = a_1 x \end{cases}$$

But, multiplying the first equation by a_1, we obtain

$$a_1 = a_1 + a_1 x = a_1 b_1 + a_1 x = a_1 b_1 + b_1 x \leq b_1.$$

Again, by the similar consideration, we can conclude that $a_1 = b_1$.

So, having SNFs of A and B such that $A \oplus R/xR \cong B \oplus R/xR$, we have obtained that the summand R/xR can be cancelled and $A = B$ as was desired. The lemma is proved. □

In the case of morphic ring $\Delta(R) = \{a_1 R \oplus \cdots \oplus a_n R | a_1, \ldots, a_n \in R\}$ and hence we can cancel not only cyclically presented modules, but principal ideals too.

As a corollary, we obtain the following result.

THEOREM 7. *Let R be a Bezout ring and $[A], [B] \in K_0'(R)$. Then $[A] = [B]$ if and only if $A = B$ in case when A, B are reduced to the SNF, and $A \cong B$ in the other case.*

Proof. Suppose that $[A] = [B]$. Then $A \oplus X \cong B \oplus X$ for some $X = R/x_1 R \oplus \cdots \oplus R/x_n R$ by Lemma 3. Let A' and B' be the SNFs of

$$A \oplus (R/x_1 R \oplus \cdots \oplus R/x_{n-1}R), \;\; B \oplus (R/x_1 R \oplus \cdots \oplus R/x_{n-1}R)$$

respectively. Then by the previous lemma $A' \oplus R/x_n R \cong B' \oplus R/x_n R$ implies that $A' = B'$ and hence

$$A \oplus (R/x_1 R \oplus \cdots \oplus R/x_{n-1}R) \cong B \oplus (R/x_1 R \oplus \cdots \oplus R/x_{n-1}R).$$

Continuing this process, we will finally obtain that $A \cong B$. If A and B are reduced to the SNF, then $A = B$. The theorem is proved. □

REMARK. In the Cancellation lemma, the existence of canonical forms of direct sums of cyclically presented modules is equivalent to the fact that R is a Bezout ring [**10**]. Moreover, as it is shown in [**4**], if R is a ring of stable range 1, then the isomorphism of cokernels of matrices implies the equivalence of matrices, so $\{a_1 R \oplus \cdots \oplus a_n R | a_1, \cdots, a_n \in R\} \cong \{R/a_1 R \oplus \cdots \oplus R/a_n R | a_1, \ldots, a_n \in R\}$. The same situation we saw in the case of morphic rings.

In general, when $\Delta(R)$ is a set of finitely generated modules that have canonical forms over the ring R, where R is a ring in some subcategory $\mathbf{C}$ of **Rings** we need the following properties in order to construct the weak Grothendieck group functor $K_0' : \mathbf{C} \rightsquigarrow \mathbf{Rings}$:

- the direct sum and tensor product of any two elements of $\Delta(R)$ are isomorphic to elements of $\Delta(R)$;

- the image of any element of $\Delta(R)$ under the base ring R change to R' is isomorphic to an element of $\Delta(R')$;
- the finitely generated direct summands of elements of $\Delta(R)$ are isomorphic to elements of $\Delta(R)$.

Note that, the mentioned properties provide all the above statements about K_0' to be true even in such general case. Before we continue the investigation of K_0' for the case of Bezout rings, it is good to provide a few examples when the mentioned construction works:

(1) elementary divisor rings and finitely presented modules over them [**10**];
(2) principal ideal rings and finitely generated modules over them;
(3) CF-rings and direct sums of cyclic modules [**14**].

In the K-theoretical literature, there is also some notion that is between the classical $K_0(R)$ and $K_0'(R)$. It is known as $G_0(R)$ and defined on the all isomorphism classes of finitely generated R-modules modulo the relations generated by the short exact sequences [**17**]. If one consider a morphic ring R and a short exact sequence $0 \to aR \to R \to \mathrm{Ann}(a) \to 0$, then in $G_0(R)$ we have $[R] = [aR] + [\mathrm{Ann}(a)]$. The same relation is obtained in $K_0'(R)$ when $R \cong aR \oplus \mathrm{Ann}(a)$, that means that a is a von Neumann regular element. Not all morphic rings are von Neumann regular and hence the arithmetical properties of $G_0(R)$ and $K_0'(R)$ differ.

6. The internal properties of K_0'.

For this section, we assume that R is a morphic ring or a Bezout ring of stable range 1. However, the same can be proved for case of an arbitrary Bezout ring, as we use only the Cancellation lemma, Fitting invariants, distributivity of intersection over the addition of principal ideals and uniqueness of canonical forms.

LEMMA 4. *Every idempotent e of $K_0'(R)$ can be written in the following form:*

$$e = [aR]([R] - [bR]),$$

for some $a, b \in R$.

Proof. Let $[A] - [B]$ be some idempotent of $K_0'(R)$ that is written in the reduced form (i.e. A, B are reduced to the SNF and there is no pair of a_iR, b_jR that can be cancellated). Then, $([A] - [B])^2 = [A] - [B]$ and thus

$$[A]^2 + [B]^2 + [B] = [A] + 2[A][B].$$

Suppose that $A' = a_2R \oplus \cdots \oplus a_nR \neq 0$. Hence, $[A]^2 = [a_1R \oplus A'][A] = [A] + [A'][A]$ and so

$$[A]^2 + [B]^2 + [B] = [A] + 2[A][B].$$

Since the highest terms (with respect to the inclusion) of the LHS and RHS have to be equal then $a_1R \cap a_2R + b_1R = a_1 \cap b_1$ and hence $a_2R \subseteq b_1R$, $a_2R = a_1R \cap b_1R$. Moreover, if $A'' = a_3R \oplus \cdots \oplus a_nR$, then

$$\begin{aligned}
[A'][A] &= [a_2R \oplus A''][a_1R \oplus a_2R \oplus A''] = [a_2R \oplus a_2R \oplus 3A''] + [A'']^2 = \cdots \\
&= [2a_2R \oplus 4a_3R \oplus \cdots \oplus (2n-2)a_nR], \\
[B]^2 + [B] &= [2b_1R \oplus 4b_2R \oplus \cdots \oplus (2m)(b_mR)].
\end{aligned}$$

Therefore,

$$\sum_{k=1}^{m} 2k[b_kR] + \sum_{i=2}^{n}(2i-2)[a_iR] = 2[a_1R \cap b_1R]$$
$$+ 2\sum_{i=2}^{n}[a_iR \cap b_1R] + 2\sum_{k=2}^{m}[a_1R \cap b_kR] + 2\sum_{i,k=2}^{n,m}[a_iR \cap b_kR].$$

But $a_iR \cap b_1R = a_iR$, for $2 \leq i \leq n$ and $a_2R = a_1R \cap b_1R$, so cancellating both parts we have

$$\sum_{k=1}^{m} 2k[b_kR] + \sum_{i=3}^{n}(2i-4)[a_iR] = 2\sum_{i=2}^{n}[a_iR \cap b_2R] + 2\sum_{k=2}^{m}[a_1R \cap b_kR]$$
$$+ 2\sum_{i=2,k=3}^{n,m}[a_iR \cap b_kR].$$

Again, using the equality of the highest terms we have that $b_1R + a_3R = a_1R \cap b_2R$ and hence $b_1R \subseteq a_1R,\ \ b_1R = b_2R$. Using the fact that $a_1R \cap B_kR = b_kR$, for $1 \leq k \leq m$, we can obtain a simplification

$$4[b_2R] + \sum_{k=3}^{m}(2k-2)[b_kR] + \sum_{i=4}^{n}(2i-6)[a_iR] = 2[a_2R] + 2\sum_{i=2,k=3}^{n,m}[a_iR \cap b_kR].$$

The equality of the highest terms implies that $a_2R = b_1R = b_2R$ and hence the combination $[A] - [B]$ is not written in the reduced form. The obtained contradiction implies that $A' = 0$ and $A = aR$, for some $a \in R$. Then, $([aR] - [B])^2 = [aR] - [B]$ implies that

$$[aR] + [B]^2 + [B] = [aR] + 2[aR][B]$$

and hence

$$\sum_{k=1}^{m}(2k)[b_kR] = 2\sum_{k=1}^{m}[aR \cap b_kR].$$

Since the both parts of the latter equality are already written in the SNF, then $b_1R = aR \cap b_1R$. Therefore, $b_1R \subseteq aR$ and hence

$$\sum_{k=1}^{m}(2k)[b_kR] = 2\sum_{k=1}^{m}[b_kR].$$

The number of terms in the LHS is $m^2 + m$ and in the RHS is $2m$. Since we have assumed that $[B]$ is in the reduced form then the only possible case is $m^2 + m = 2m$ and hence $m = 1$.

So, any idempotent $[A]-[B]$ can be written as $[A]-[B]=[aR]-[aR\cap bR]$, where $aR\supseteq bR$ or equivalently

$$[A]-[B]=[aR]([R]-[bR]).$$

The lemma is proved. $\square$

LEMMA 5. *$[A]$ is an invertible element of $K_0'(R)$ if and only if $[A]=[R]$.*

Proof. Let $[A]([C]-[D])=[R]$ and $[C]-[D]$ is written in the reduced form. Then, $[A][C]=[R]+[A][D]$ and hence the equality of the highest terms implies that $a_1R\cap c_1R=R$. The latter implies that $a_1R=c_1R=R$. Suppose that $A=R\oplus A',\ C=R\oplus C',\ C'\neq 0$. Then,

$$[A']+[C']+[A'][C']=[D]+[A'][D]$$

implies that $a_2R+c_2R=d_1R$. Let $A'=a_2R\oplus A'',\quad C'=c_2R\oplus C'',\quad D=(a_2R+c_2R)\oplus D'$.

After the substitution, we will obtain

$$\begin{aligned}&[a_2R]+[A'']+[c_2R]+[C'']+[a_2R\cap c_2R]+[a_2R][C'']+[c_2R][A'']\\&\quad+[A''][C'']=[a_2R+c_2R]+[D']+[a_2R+c_2R][A']+[a_2R+c_2R][D']\end{aligned}$$

and hence

$$\begin{aligned}&[a_2R+c_2R]+2[a_2R\cap c_2R]+[A'']+[C'']+[a_2R][C'']+[c_2R][A'']\\&\quad+[A''][C'']=[a_2R+c_2R]+2[D']+[a_2R]+[A''].\end{aligned}$$

After the cancellation of the equal terms in the both sides, we will have

$$2[a_2R\cap c_2R]+[C'']+[a_2R][C'']+[c_2R][A'']+[A''][C'']=2[D']+[a_2R].$$

Then, $a_2R\cap c_2R=a_2R+d_2R$ and $d_2R\subseteq a_2R$. Moreover, $a_2R\cap c_2R=a_2R$ and $a_2R\subseteq c_2R$. Thus, we conclude that $d_1R=a_2R+c_2R=c_2R$ and so the combination $[C]-[D]$ is not written in the reduced form. The obtained contradiction implies that $C'=0$ and $c_2R=0$. Thus, $[A']=[A'][D]+[D]$ and hence $a_2R=d_1R$. Therefore, $[a_2R]+[A'']=([a_2R]+[A''])([a_2R]+[D'])+[a_2R]+[D']$ implies that $[a_2R]+2[D']+[A''][D']=0$ and hence $a_2R=0$. As a result, we have that $[A]=[R]$. $\square$

LEMMA 6. *If the highest terms of $[A],[B]\in K_0'(R)$ are coprime and have zero intersection, then $[A]-[B]\in U(K_0'(R))$ if and only if $[A]-[B]=[R]-2[xR]$, for some $[xR]\in K_0'(R)$.*

Proof. Suppose that $[A]-[B]\in U(K_0'(R))$. Then, $[A]^2+[B]^2=([A]-[B])^2\in U(K_0'(R))$ and by Lemma 5 $[A]^2+[B]^2=[R]$. The first and the second terms of $[A]^2+[B]^2$ are $a_1R+b_1R,\ a_2R+a_1Rb_1R+b_2R$ and by Theorem 6 we obtain that $a_1R+b_1R=R$ and $a_2R+b_2R=a_2R+a_1Rb_1R+b_2R=0$. Then, $a_2R=b_2R=0$ and $[A]-[B]=[aR]-[bR]$, for some coprime elements $a,b\in R$. So, $[A]-[B]=[aR\oplus bR]-2[bR]=[(aR+bR)\oplus(aR\cap bR)]-2[bR]=[R]-2[bR]$. The lemma is proved. $\square$

PROPOSITION 4. *A ring $K_0'(R)$ is a reduced, that is there are no nonzero nilpotent elements.*

Proof. Suppose that $[A]-[B] \in K_0'(R)$ is a nilpotent element such that

$$([A]-[B])^2 = [0].$$

Then, $[A]^2 + [B]^2 = 2[A \otimes_R B]$. By Theorem 6, this equality is equivalent to the fact that SNF of $(A \otimes_R A) \oplus (B \otimes_R B)$ and $(A \otimes_R B) \oplus (A \otimes_R B)$ are equal. But the highest term of the right-hand side is $a_1 R \cap b_1 R$ and $a_1 R + b_1 R$ of the left-hand side. Then

$$a_1 R, b_1 R \subseteq a_1 R + b_1 R = a_1 R \cap b_1 R \subseteq a_1 R, b_1 R.$$

Hence, $a_1 R = b_1 R$ and they can be cancellated in the expression $[A]-[B]$. Continuing this process, we obtain that $[A]-[B]$ is simply $[X] = [x_1 R \oplus \cdots \oplus x_k R]$ such that $[X]^2 = 0$.

But the highest term of $[X]^2$ is $x_1 R$ and it is zero, so the whole $[X]$ and $[A]-[B]$ are $[0]$. So, $K_0'(R)$ is a reduced ring. The proposition is proved. □

In a case when R is a commutative von Neumann regular ring, the structure of a ring $K_0'(R)$ becomes rather simple. Suppose that $a \in R$. Then, there is $x \in R$ such that $a^2 x = a$ and

$$aR \sim_M (1-ax)R.$$

Indeed, $(1-ax) \in \mathrm{Ann}(a)$, and $0 = \mathrm{Ann}(aR + (1-ax)R) = \mathrm{Ann}(a) \cap \mathrm{Ann}(1-ax)$, hence if $aR \sim_M bR$, $1-axR \sim_M cR$ then $bc = 0$ and $b \in \mathrm{Ann}(c) = (1-ax)R$. The las implies that $(1-ax)R = \mathrm{Ann}(a)$. Then

$$aR + \mathrm{Ann}(a) = R, \;\; aR \cap \mathrm{Ann}(a) = aR \cap (1-ax)R = aR \cdot (1-ax)R = 0.$$

Thus, if $[A]-[B] \in K_0'(R)$, then

$$[A]-[B] = \left[\bigoplus_{i=1}^{n} a_i R\right] + \left[\bigoplus_{j=1}^{m} \mathrm{Ann}(b_j)\right] - \sum_{j=1}^{m} [b_j R \oplus \mathrm{Ann}(b_j)] = [A'] - m[R]$$

for some $[A'] \in K_0'(R)$. In other words, we have obtained the following result.

Proposition 5. *If R is a commutative von Neumann regular ring, then*

$$K_0'(R) = \{[A] - m[R] | m \geq 0, A = a_1 R \oplus \cdots \oplus a_n R, \forall i: \; a_i^2 = a_i\}.$$

The latter result is rather important since it has a connection with the usual Grothendieck's group $K_0(R)$ of a von Neumann regular ring R.

Theorem 8. *If R is a commutative von Neumann regular ring, then*

$$K_0(R) = K_0'(R).$$

Proof. Since any von Neumann regular ring is an exchange one then by **[16]** any finitely generated projective module is a direct sum of principal idempotent ideals. Conversely, any principal idempotent ideal is a projective module, so are their direct sums. Therefore, $K_0(R) = K_0'(R)$ since their generators coincide. The theorem is proved. □

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The main advantage of the weak Grothendieck group is the fact that it always becomes a commutative ring. Looking forward, it is an open question how to construct a ring $K_0'(R)$ in a case of noncommutative ring R, even for unit-regular rings. These rings are two-sided morphic and every matrix over unit-regular ring is equivalent to some diagonal matrix. The construction of $K_0'(R)$ and its connection with $K_0(R)$ can help to solve an open problem of Goodearl [**6**]: is $K_0(R)$ of simple unit-regular ring necessarily strictly unperforated?

THEOREM 9. *If R is a commutative elementary divisor ring, then*

$$K_0(R) \subseteq K_0'(R).$$

Proof. Any finitely generated projective module over R is a finitely presented one and it decomposes into a direct sum of cyclically presented modules, so generators of $K_0(R)$ are among the generators of $K_0'(R)$. The theorem is proved. □

It is known that for any projective module P over a ring R there are some free R-module F and a submodule Q of F such that

$$P \oplus Q = F.$$

Thus, as any element of $K_0'(R)$ can be expressed in form $[P_1] - [P_2]$ then

$$[P_1] - [P_2] = [P_1 \oplus Q_2] - [P_2 \oplus Q_2] = [P_1 \oplus Q_2] - n[R],$$

where $P_2 \oplus Q_2 \cong R^N$ and we conclude that the studying of the structure of $K_0'(R)$ help to understand the Grothendieck's group $K_0(R)$.

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