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Weakly Stable Relations and Inductive Limits of C^*-algebras

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Abstract. We show that if $\mathcal{A}$ is a class of C^*-algebras for which the set of formal relations $\mathcal{R}$ is weakly stable, then $\mathcal{R}$ is weakly stable for the class $\mathcal{B}$ that contains $\mathcal{A}$ and all the inductive limits that can be constructed with the C^*-algebras in $\mathcal{A}$.

A set of formal relations $\mathcal{R}$ is said to be *weakly stable* for a class $\mathcal{C}$ of C^*-algebras if, in any C^*-algebra $A \in \mathcal{C}$, close to an approximate representation of the set $\mathcal{R}$ in A there is an exact representation of $\mathcal{R}$ in A.

1 Introduction

It seems that the first appearance of a universal C^*-algebra was when the functional calculus of normal operators was developed. After that, the use of matrix units in von Neumann algebras implicitly associated a copy of M_n with a set of matrix units. In a more recent language, we could say that M_n is isomorphic to the C^*-algebra generated by a set of matrix units:

$$M_n(\mathbb{C}) \cong C^*\langle x_{ij} : 1 \leq i, j \leq n \mid x_{ij} = x_{ji}^*, x_{ij}x_{kl} = \delta_{jk}x_{il} : 1 \leq i, j, k, l \leq n \rangle.$$

After Gelfand and Naimark defined C^*-algebras more abstractly, other examples came. The Cuntz algebras, and Brown's non-commutative unitary groups and Grassmanians are some of them, from the early 80's. Of course there is also the Toeplitz algebra, an important example dated from 1967. The existence of such C^*-algebras was proven one by one.

The first to show some concern about knowing whether a set of generators and relations defines a C^*-algebra was Blackadar, in his *Shape Theory for C^*-algebras* paper [1]. He introduced the concept of *admissible* sets of generators and relations and defined the universal C^*-algebra of an admissible pair $(\mathcal{G}, \mathcal{R})$. In that paper, Blackadar considered relations of the form $\|p(x_{\alpha_1}, \dots, x_{\alpha_n}, x_{\alpha_1}^*, \dots, x_{\alpha_n}^*)\| \leq k$, where p is a polynomial in $2n$ non-commuting variables, $x_{\alpha_1}, \dots, x_{\alpha_n} \in \mathcal{G}$, and $k \geq 0$.

The class of all relations that define universal C^*-algebras was first studied by Hadwin [8], Phillips [14], and later by Loring [10, 11]. To prove our main result we will use the notions of representation of relations and approximate representation of relations. We have found that the formalism regarding these notions as described in Loring [11] and also in the paper [6] by Eilers, Loring and Pedersen is the one that suits our needs better.

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In Section 2 we present the definitions of formal relations, natural relations, representations and approximate representations with more details than in the original paper ([6]).

In Section 3 we investigate how sets of relations and inductive limits of C^*-algebras behaved with respect to weak stability. We prove a closure theorem that establishes that if a set of bounded relations is weakly stable for a class $\mathcal{A}$ of C^*-algebras then it is weakly stable for the class $\mathcal{B}$ that contains $\mathcal{A}$ and all the possible inductive limits that can be obtained from the C^*-algebras in $\mathcal{A}$.

2 Generators and Relations

In this section we address certain issues regarding generators and relations which will be needed later. The notions that we will use were introduced mostly in [6], and also in [11].

For each real number $\alpha > 0$, we denote by

$$\mathcal{F}_m^{(\alpha)} = C^*\langle g_1, \dots, g_m \mid \|g_j\| \leq \alpha \rangle,$$

the universal C^*-algebra generated by a finite set $\mathcal{G} = \{g_1, g_2, \dots, g_m\}$ of non-commuting indeterminates, also called *generators*.

If A is a C^*-algebra and $a_1, \dots, a_m \in A$, we will denote $\mathbf{a} = (a_1, \dots, a_m) \in A^m$, and use the norm $\|\mathbf{a}\| = \max\{\|a_1\|, \dots, \|a_m\|\}$ on A^m.

An important property of $\mathcal{F}_m^{(\alpha)}$ that we shall use along the way is the following. A proof can be found in [11], Section 3.1.

If A is a C^*-algebra that contains elements $a_1, \dots, a_m$ such that $\|a_j\| \leq \alpha$, then there exists a unique $*$-homomorphism $\Phi_{\mathbf{a}} \colon \mathcal{F}_m^{(\alpha)} \to A$ such that $\Phi_{\mathbf{a}}(g_j) = a_j$, for $1 \leq j \leq m$.

Experience showed that establishing a general definition of relation is very difficult. Next we will see a clear way of doing it, which appeared in [6].

Definition 2.1 A *formal relation* in m-indeterminates is any element of $\mathcal{F}_m^{(1)}$.

Definition 2.2 A *representation* of a set of formal relations $\mathcal{R} \subset \mathcal{F}_m^{(1)}$ in a C^*-algebra A is a m-tuple $\mathbf{a} = (a_1, \dots, a_m) \in A^m$ such that $\|\mathbf{a}\| \leq 1$ and $\Phi_{\mathbf{a}}(r) = 0, \forall r \in \mathcal{R}$.

Definition 2.3 A C^*-algebra U is said to be *universal for the set of formal relations* $\mathcal{R} \subset \mathcal{F}_m^{(1)}$ if U contains elements $u_1, \dots, u_m$ such that $\mathbf{u} = (u_1, \dots, u_m)$ is a representation of $\mathcal{R}$ and, for every C^*-algebra A that contains a representation $\mathbf{a} = (a_1, \dots, a_m)$ of $\mathcal{R}$ in A, there exists a unique $*$-homomorphism $\varphi \colon U \to A$ such that $\varphi(u_j) = a_j$.

It is easy to see that if $\mathcal{R} \subset \mathcal{F}_m^{(1)}$ is a countable set of formal relations, and $I_{\mathcal{R}}$ is the ideal generated by $\mathcal{R}$, then $\mathcal{F}_m^{(1)}/I_{\mathcal{R}}$ is universal for $\mathcal{R}$.

The C^*-algebra universal for the set $\mathcal{R} \subset \mathcal{F}_m^{(1)}$ will be denoted by

$$C^*\langle g_1, \dots, g_m \mid \mathcal{R} \rangle,$$

or, even by $C^*\langle \mathcal{G} \mid \mathcal{R} \rangle$.

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As we see, the construction above assures the existence of all kinds of C^*-algebras generated by a finite number of bounded generators and formal relations.

Let A and B be C^*-algebras. If $\varphi\colon A \to B$ is a $*$-homomorphism, we will still denote by φ the map $\varphi(a_1, \dots, a_m) = \big(\varphi(a_1), \dots, \varphi(a_m)\big)$, from A^m to B^m.

Definition 2.4 A *natural relation* in m-indeterminates is any property $\mathcal{P}$ of m-tuples in a C^*-algebra A, such that if $\mathbf{a} = (a_1, \dots, a_m)$ satisfies $\mathcal{P}$ then $\varphi(\mathbf{a})$ satisfies $\mathcal{P}$, for every $*$-homomorphism $\varphi\colon A \to B$.

In an informal language, a natural relation is any property that exists in some C^*-algebra A and that is preserved via $*$-homomorphisms. To a formal relation r we associate the following natural relation: $\|\mathbf{a}\| \le 1$ and $\Phi_{\mathbf{a}}(r) = 0$.

Definition 2.5 If $a_1, \dots, a_m$ belong to a C^*-algebra A and satisfy the natural relations described by the finite set $\mathcal{P}$, we say that $\mathbf{a} = (a_1, \dots, a_m)$ is a *representation of the natural relations $\mathcal{P}$ in A*.

Definition 2.6 Let $\mathcal{G}$ be a finite set of generators and $\mathcal{P}$ be a set of natural relations. A C^*-algebra A, together with a representation $\pi\colon \mathcal{G} \to A$ of $\mathcal{P}$ is *universal for* $\mathcal{P}$ if, for all C^*-algebra B, the map $\varphi \mapsto \pi \circ \varphi$ from the set of all $*$-homomorphisms from A to B to the set of all representations of $\mathcal{P}$ in B is a bijection.

It is clear from this definition that the universal C^*-algebra determined by $\mathcal{G}$ and $\mathcal{P}$ is unique, up to isomorphisms.

The usual in the literature is to describe a universal C^*-algebra by a set of *natural* relations but, unfortunately, if $\mathcal{P}$ is a set of *natural* relations, the C^*-algebra $C^*\langle \mathcal{G} \mid \mathcal{P}\rangle$ may not exist. The existence of such C^*-algebras will be assured if the natural relations correspond to a set of formal relations contained in $\mathcal{F}_m^{(1)}$. The correspondence is in Lemma 2.8, which appeared in [6] as Lemma 2.2.1 (albeit with an error, pointed out to me by Terry Loring.) Below we will include a proof of a corrected statement.

Definition 2.7 We say that a natural relation $\mathcal{P}$ is *closed under products* if, for any family of C^*-algebras A_j and any family of representations $\mathbf{a}_j$ of $\mathcal{P}$ in A_j, then $(\mathbf{a}_j)_j$ is a representation of $\mathcal{P}$ in the product $\prod A_j$.

Lemma 2.8 *If $\mathcal{P}$ is a natural relation, closed under products, then there exists a formal relation r such that*

$$\|\mathbf{a}\| \le 1 \quad \textit{and} \quad \mathcal{P} \Leftrightarrow \|\mathbf{a}\| \le 1 \quad \textit{and} \quad \Phi_{\mathbf{a}}(r) = 0.$$

Proof Let $\mathcal{P}$ be a natural relation, and define B as the universal C^*-algebra generated by $a_1, \dots, a_m$, satisfying $\|\mathbf{a}\| \le 1$ and $\mathcal{P}$. By Theorem 3.1.1 in [11], we know that B exists. Take the map $\Phi_{\mathbf{a}}\colon \mathcal{F}_m^{(1)} \to B$ that sends g_j to a_j, $1 \le j \le m$. Clearly $\Phi_{\mathbf{a}}$ is surjective. Let I be the kernel of $\Phi_{\mathbf{a}}$. There exists a positive element $h \in I$ such that $I = \overline{h\mathcal{F}_m^{(1)}h}$.

Let $H(h)$ be the smallest hereditary C^*-subalgebra of $\mathcal{F}_m^{(1)}$ containing $\{h\}$. By Lemma 1.2.3 in Loring [11], we conclude that $H(h) = I$, and that I is a C^*-subalgebra generated by a single element: h. Since $\Phi_{\mathbf{a}}(h) = 0$, and the universal C^*-algebra generated by $\{g_1, \dots, g_m\}$ subjected to the formal relation h is $\mathcal{F}_m^{(1)}/I \cong B$, h is the formal relation we need. ∎

Remark 2.9 If the natural relation is not closed under products, the result may be false. An example for this could be the relation $\mathcal{P}$: x is nilpotent. It is easy to see that $\mathcal{P}$ is not closed under products and that $C^*\langle x \mid \|x\| \leq 1 \text{ and } x^n = 0, \text{ for some } n \geq 1\rangle$ does not exist.

Remark 2.10 For a given natural relation, the set of formal relations that corresponds to it is not necessarily unique. For instance, the C^*-algebra $A = C^*\langle a, b, c \mid \mathcal{R}\rangle$, where $\mathcal{R} = \{a-b, b-c\} \subset \mathcal{F}_3^{(1)}$ is isomorphic to $\mathcal{F}_1^{(1)}$. In order to see that, define $\varphi\colon \mathcal{F}_3^{(1)} \to \mathcal{F}_1^{(1)}$ by $\varphi(a) = \varphi(b) = \varphi(c) = x$, and notice that $\ker\varphi = I_{\{a-b,b-c\}}$. The quotient map $\bar{\varphi}\colon A \to \mathcal{F}_1^{(1)}$ sends $\bar{a} = \bar{b} = \bar{c}$ to x. On the other hand, there exists a unique $*$-homomorphism $\phi\colon \mathcal{F}_1^{(1)} \to A$ that sends x to $\bar{a}$. Since $\bar{\varphi} \circ \phi = 1_{\mathcal{F}_1^{(1)}}$ and $\phi \circ \bar{\varphi} = 1_A$, we have $A \cong \mathcal{F}_1^{(1)}$.

A non-trivial illustration of the non-uniqueness under consideration is the following.

Example 2.11 It is well known that the C^*-algebra $C^*\langle h \mid 0 \leq h \leq 1\rangle$ is isomorphic to $\mathcal{C}_0(]0,1])$. Let's see two different sets of generators and formal relations that define this C^*-algebra. Since an element h of a C^*-algebra A satisfies $0 \leq h \leq 1$ if, and only if, $\|h\| \leq 1$ and $|h| = h$, the set $\mathcal{R}_1 = \{x - |x|\} \subset \mathcal{F}_1^{(1)}$ is a set of formal relations associated to $0 \leq h \leq 1$. Hence $\mathcal{F}_1^{(1)}/I_{\{x-|x|\}} \cong C^*\langle h \mid 0 \leq h \leq 1\rangle$.

Another equivalent way of saying that an element h of a C^*-algebra A satisfies $0 \leq h \leq 1$ is: $h \geq 0$ and $\|h\| \leq 1$. Since $h \geq 0 \Leftrightarrow \exists a \in A \mid a = a^*$, and $h = a^2$, and noticing that $\|a\|^2 = \|a^*a\| = \|a^2\| = \|h\| \leq 1$, we have:

$$\begin{aligned} 0 \leq h \leq 1 &\iff \|h\| \leq 1, \text{ and } \exists a \in A, \quad \|a\| \leq 1,\ a = a^*,\ h = a^2 \\ &\iff \|h\| \leq 1, \text{ and } \exists a \in A, \quad \|a\| \leq 1,\ a - a^* = 0,\ h - a^2 = 0. \end{aligned}$$

Therefore, the set $\mathcal{R} = \{x - x^*, y - x^2\} \subset \mathcal{F}_2^{(1)}$ also is a formal set of formal relations associated to the natural relation $0 \leq h \leq 1$.

We claim that $B = \mathcal{F}_2^{(1)}/I_{\{x-x^*,y-x^2\}}$ is isomorphic to $C^*\langle h \mid 0 \leq h \leq 1\rangle$. In fact, notice that $\bar{y} = y + I_{\{x-x^*,y-x^2\}}$ is a positive element of B, since $\bar{y} = (\bar{x})^2$ and $\bar{x} = (\bar{x})^*$. So $\bar{y}$ is a representation of $\mathcal{R}_1$ in B. The universality of $C^*\langle h \mid 0 \leq h \leq 1\rangle$ implies that there exists a unique $\phi\colon C^*\langle h \mid 0 \leq h \leq 1\rangle \to B$ such that $\phi(h) = \bar{y}$. On the other hand, we know that there exists a $*$-homomorphism

$$\varphi_h\colon \mathcal{F}_2^{(1)} \to C^*\langle h \mid 0 \leq h \leq 1\rangle$$

such that $\varphi_h(x) = \sqrt{h}$ and $\varphi_h(y) = h$. Clearly, $\varphi_h(x - x^*) = \sqrt{h} - (\sqrt{h})^* = 0$ and $\varphi_h(y - x^2) = h - (\sqrt{h})^2 = 0$. Then the map

$$\bar{\varphi}_h\colon \mathcal{F}_2^{(1)}/\ker\varphi = B \to C^*\langle h \mid 0 \leq h \leq 1\rangle$$

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is well defined, and we have:

1. $\overline{\varphi_h} \circ \phi(h) = \overline{\varphi_h}(\bar{y}) = h$, which means that $\overline{\varphi_h} \circ \phi = 1_{C^*\langle h|0 \leq h \leq 1\rangle}$;
2. $\phi \circ \overline{\varphi_h}(\bar{x}) = \phi(\sqrt{h}) = \sqrt{\bar{y}} = \bar{x}$, $\phi \circ \overline{\varphi_h}(\bar{y}) = \phi(h) = \bar{y}$, and hence $\phi \circ \overline{\varphi_h} = 1_B$.

Example 2.12 For a fixed positive real number α, the C^*-algebra B_α was defined in [3] as the universal C^*-algebra generated by two unitaries u_α, v_α and a self-adjoint h_α, $\|h_\alpha\| \leq \alpha$ such that $u_\alpha v_\alpha u_\alpha^* v_\alpha^* = e^{ih_\alpha}$. Let's see how B_α could be defined as the C^*-algebra generated by a finite set of generators and a finite set of formal relations.

Consider the C^*-algebra $A_\alpha = C^*\langle x, y, z, 1 \mid \mathcal{R}_\alpha\rangle$, where $\mathcal{R}_\alpha \subset \mathcal{F}_4^{(1)}$ is the set

$$\mathcal{R}_\alpha = \{1^* - 1, 1 - 1^2, 1x - x1, 1x - x, 1y - y1, 1y - y, 1z - z1, 1z - z,$$
$$xx^* - 1, x^*x - 1, yy^* - 1, y^*y - 1, z - z^*, xyx^*y^* - e^{i\alpha z}\}.$$

There exists the $*$-homomorphism $\phi\colon \mathcal{F}_4^{(1)} \to B_\alpha$ that sends x to u_α, y to v_α 1 to 1, and z to $\alpha^{-1}h_\alpha$. The ideal $I_{\mathcal{R}_\alpha}$ generated by the set $\mathcal{R}_\alpha$ coincide with the kernel of ϕ. So we have determined the $*$-homomorphism $\bar{\phi}\colon C^*\langle x, y, z, 1 \mid \mathcal{R}_\alpha\rangle \cong \mathcal{F}_4^{(1)}/I_{\mathcal{R}_\alpha} \to B_\alpha$.

Also, the universality of B_α implies that there exists $\psi\colon B_\alpha \to C^*\langle x, y, z, 1 \mid \mathcal{R}_\alpha\rangle$ that sends u_α to $\bar{x}$, v_α to $\bar{y}$, 1 to $\bar{1}$, h_α to $\alpha\bar{z}$. The maps $\bar{\phi} \circ \psi$ and 1_{B_α} coincide on the generators of B_α, as well as the maps $\psi \circ \bar{\phi}$ and 1_{A_α} coincide on the generators x, y, z, 1. Therefore $\bar{\phi} \circ \psi = 1_{B_\alpha}$, $\psi \circ \bar{\phi} = 1_{A_\alpha}$, and then $C^*\langle x, y, z, 1 \mid \mathcal{R}_\alpha\rangle \cong B_\alpha$.

2.1 Approximate Representations

For an element a in a C^*-algebra A, define

$$\lfloor a \rfloor = ak(|a|),$$

where $k(t) = \min\{1, t^{-1}\}$, $t > 0$.

Notice that even when A in non-unital, $\lfloor a \rfloor$ is in A. In fact, if $k(|a|)$ does not belong to A, but to $\tilde{A}$, the product $ak(|a|)$ must be in A, since A is an ideal of $\tilde{A}$.

Proposition 2.13 *Let A and B be any C^*-algebras. For any $a \in A$ the following are true:*

1. $\|\lfloor a \rfloor\| \leq 1$;
2. *if* $\|a\| \leq 1$, *then* $\lfloor a \rfloor = a$;
3. *if* $\varphi\colon A \to B$ *is a* $*$-*homomorphism then* $\varphi(\lfloor a \rfloor) = \lfloor \varphi(a) \rfloor$.

Proof Using functional calculus, we have:

1. $\|\lfloor a \rfloor\|^2 = \|\lfloor a \rfloor^* \lfloor a \rfloor\| = \left\| a^*a\big(k(|a|)\big)^2 \right\| = \||a|k(|a|)\|^2 = \|g(t)\|^2 \leq 1$, where $g(t) = tk(t)$, for $t > 0$;
2. if $\|a\| \leq 1$ then $k(|a|) = 1$, and consequently $\lfloor a \rfloor = a$;
3. $\varphi(\lfloor a \rfloor) = \varphi\big(ak(|a|)\big) = \varphi(a) \cdot \varphi\big(k(|a|)\big) = \varphi(a) \cdot k\big(|\varphi(a)|\big) = \lfloor \varphi(a) \rfloor$. ■

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If $\mathbf{a} = (a_1, \dots, a_m)$, denote $\lfloor \mathbf{a} \rfloor = (\lfloor a_1 \rfloor, \dots, \lfloor a_m \rfloor)$.

Definition 2.14 Suppose $0 \le \delta \le 1$ and let $\mathcal{R} \subset \mathcal{F}_m^{(1)}$ be finite. A m-tuple $\mathbf{a} = (a_1, \dots, a_m)$ in A^m is a *δ-representation of $\mathcal{R}$ in the C^*-algebra A* if:

(i) $\|\mathbf{a}\| \le 1 + \delta$;
(ii) $\|\Phi_{\lfloor \mathbf{a} \rfloor}(r)\| \le \delta, \forall r \in \mathcal{R}$.

3 Weakly Stable Relations and Inductive Limits

3.1 Inductive Limits

For an inductive system $A_1 \xrightarrow{\theta_1} A_2 \xrightarrow{\theta_2} A_3 \xrightarrow{\theta_3} \cdots$ we denote its inductive limit by $(A_\infty, \theta_{n,\infty})$, where the $*$-homomorphisms $\theta_{n,\infty} \colon A_n \to A_\infty$ have the property $\theta_{n+1,\infty} \circ \theta_n = \theta_{n,\infty}$. We also denote $\theta_{n,r} \colon A_n \to A_r$ the multiple compositions: $\theta_{n,r} = \theta_{r-1} \circ \cdots \circ \theta_{n+1} \circ \theta_n$, for $n + 1 < r < \infty$.

Next proposition contains some properties that will be used several times. The reader may consult Loring [11], Section 13.1, for more details.

Proposition 3.1 *Let $(A_\infty, \theta_{n,\infty}) = \varinjlim(A_n, \theta_n)$, and $\theta_{n,r}$ be the same as before. Then:*

1. $A_\infty = \overline{\bigcup_n \theta_{n,\infty}(A_n)}$;
2. *for a_n in A_n, $\|\theta_{n,\infty}(a_n)\| = \lim_{k\to\infty} \|\theta_{n,k}(a_n)\|$;*
3. *given $a \in A_n$, $b \in A_r$, such that $\theta_{n,\infty}(a) = \theta_{r,\infty}(b)$ and given $\epsilon > 0$, then there exists $k_0 \ge n$, r such that $\|\theta_{n,k}(a) - \theta_{r,k}(b)\| < \epsilon$, $\forall k \ge k_0$.*

Let $(A_n, \theta_n)_{n\in\mathbb{N}}$ be an inductive system with limit $(A_\infty, \theta_{n,\infty})$. Let $n_0 \in \mathbb{N}$, $\mathbf{a}_{(n_0)} = (a^1_{(n_0)}, \dots, a^m_{(n_0)}) \in (A_{n_0})^m$, and define, for $k \ge n_0$,

$$\mathbf{a}_{(k)} = \theta_{n_0,k}(\mathbf{a}_{(n_0)}), \quad \text{and} \quad \mathbf{a} = \theta_{n_0,\infty}(\mathbf{a}_{(n_0)}).$$

Consider, for $k \ge n_0$, the $*$-homomorphisms $\Phi_{\lfloor \mathbf{a}_{(k)} \rfloor} \colon \mathcal{F}_m^{(1)} \to A_k$ and $\Phi_{\lfloor \mathbf{a} \rfloor} \colon \mathcal{F}_m^{(1)} \to A_\infty$ given by $\Phi_{\lfloor \mathbf{a}_{(k)} \rfloor}(x_j) = \lfloor a^j_{(k)} \rfloor$, and $\Phi_{\lfloor \mathbf{a} \rfloor}(x_j) = \lfloor a^j \rfloor$, for $1 \le j \le m$.

Recalling that

$$\mathbf{a}_{(k+1)} = (a^1_{(k+1)}, \dots, a^m_{(k+1)}) = \big(\theta_k(a^1_{(k)}), \dots, \theta_k(a^m_{(k)})\big) = \theta_k(\mathbf{a}_{(k)}),$$

and using Proposition 2.13, we have, for $1 \le j \le m$, and for every $k \ge n_0$,

$$\Phi_{\lfloor \mathbf{a}_{(k+1)} \rfloor}(x_j) = \lfloor a^j_{(k+1)} \rfloor = \lfloor \theta_k(a^j_{(k)}) \rfloor = \theta_k\big(\lfloor (a^j_{(k)}) \rfloor\big) = \theta_k \circ \Phi_{\lfloor \mathbf{a}_{(k)} \rfloor}(x_j),$$

and

$$\begin{aligned}\Phi_{\lfloor \mathbf{a} \rfloor}(x_j) = \lfloor \theta_{n_0,\infty}(a^j_{(n_0)}) \rfloor = \theta_{n_0,\infty}(\lfloor a^j_{(n_0)} \rfloor) &= \theta_{n_0,\infty} \circ \Phi_{\lfloor \mathbf{a}_{(n_0)} \rfloor}(x_j) \\ &= \theta_{k,\infty} \circ \Phi_{\lfloor \mathbf{a}_{(k)} \rfloor}(x_j).\end{aligned}$$

We conclude that for $k \ge n_0$, $\mathbf{a}_{(k)} = \theta_{n_0,k}(\mathbf{a}_{(n_0)})$, and $\mathbf{a} = \theta_{n_0,\infty}(\mathbf{a}_{(n_0)})$, we have

$$\Phi_{\lfloor \mathbf{a}_{(k+1)} \rfloor} = \theta_k \circ \Phi_{\lfloor \mathbf{a}_{(k)} \rfloor}, \quad \text{and} \quad \Phi_{\lfloor \mathbf{a} \rfloor} = \theta_{k,\infty} \circ \Phi_{\lfloor \mathbf{a}_{(k)} \rfloor}. \tag{1}$$

3.2 A Closure Result

Definition 3.2 Suppose $\mathcal{C}$ is a class of C^*-algebras, and let $\mathcal{R} \subset \mathcal{F}_m^{(1)}$ be a finite set of formal relations. We say that $\mathcal{R}$ is *weakly stable with respect to* $\mathcal{C}$ if, for every $\epsilon > 0$ there exists $\delta > 0$ such that, for any m-tuple $\mathbf{a}$ in any C^*-algebra A of the class $\mathcal{C}$, if $\mathbf{a}$ is a δ-representation of $\mathcal{R}$ in A then there exists an m-tuple $\mathbf{b}$ in A that is a representation of $\mathcal{R}$ in A such that $\|\mathbf{a} - \mathbf{b}\| < \epsilon$.

Lemma 3.3 *Let $(A_n, \theta_n)_{n\in\mathbb{N}}$ be an inductive system with limit $(A_\infty, \theta_{n,\infty})$. Suppose $\mathcal{R} \subset \mathcal{F}_m^{(1)}$ is a finite set of formal relations, weakly stable for the class $\mathcal{A} = \{A_n, n \in \mathbb{N}\}$. Let $0 < \delta \leq 1$, and let $\mathbf{a} \in (A_\infty)^m$ be a δ-representation of $\mathcal{R}$ in A_∞. Then, for any $\epsilon > 0$, it is possible to find an index $k \in \mathbb{N}$ and $\mathbf{a}_{(k)} \in (A_k)^m$ such that $\|\mathbf{a} - \theta_{k,\infty}(\mathbf{a}_{(k)})\| < \epsilon$ and $\theta_{k,\infty}(\mathbf{a}_{(k)})$ is a 2δ-representation of $\mathcal{R}$ in A_∞.*

Proof Let $p \in \mathcal{R}$ and suppose $\mathbf{a} = (a^1, \dots, a^m)$ is a δ-representation of $\mathcal{R}$ in A_∞.

(i) Suppose p is a $*$-polynomial in $x_1, \dots, x_m$. Then $\Phi_{\lfloor \mathbf{a} \rfloor}(p)$ is a $*$-polynomial in $\lfloor a^1 \rfloor, \lfloor a^2 \rfloor, \dots, \lfloor a^m \rfloor$. Continuity implies that there exists $\alpha_p > 0$ such that if $\mathbf{c} \in (A_\infty)^m$, and $\|\mathbf{a} - \mathbf{c}\| < \alpha_p$, then $|\|\Phi_{\lfloor \mathbf{a} \rfloor}(p)\| - \|\Phi_{\lfloor \mathbf{c} \rfloor}(p)\|| < \delta$. Consequently

$$\|\Phi_{\lfloor \mathbf{a} \rfloor}(p)\| - \delta < \|\Phi_{\lfloor \mathbf{c} \rfloor}(p)\| < \|\Phi_{\lfloor \mathbf{a} \rfloor}(p)\| + \delta \leq 2\delta.$$

(ii) If $p \in \mathcal{R}$ is any element of $\mathcal{F}_m^{(1)}$, take a $*$-polynomial q such that $\|p - q\|_u < \frac{\delta}{4}$. There is $\alpha_p > 0$ such that $\|\mathbf{a} - \mathbf{c}\| < \alpha_p \Rightarrow \big| \|\Phi_{\lfloor \mathbf{a} \rfloor}(q)\| - \|\Phi_{\lfloor \mathbf{c} \rfloor}(q)\| \big| < \frac{\delta}{4}$. On the other hand,

$$\begin{aligned}\|\Phi_{\lfloor \mathbf{a} \rfloor}(q)\| &\leq \|\Phi_{\lfloor \mathbf{a} \rfloor}(q) - \Phi_{\lfloor \mathbf{a} \rfloor}(p)\| + \|\Phi_{\lfloor \mathbf{a} \rfloor}(p)\| \leq \\ &\leq \|p - q\|_u + \|\Phi_{\lfloor \mathbf{a} \rfloor}(p)\| < \frac{\delta}{4} + \delta = \frac{5\delta}{4}.\end{aligned}$$

Then

$$\|\Phi_{\lfloor \mathbf{c} \rfloor}(q)\| < \|\Phi_{\lfloor \mathbf{a} \rfloor}(q)\| + \frac{\delta}{4} < \frac{5\delta}{4} + \frac{\delta}{4} = \frac{3\delta}{2},$$

and

$$\begin{aligned}\|\Phi_{\lfloor \mathbf{c} \rfloor}(p)\| &\leq \|\Phi_{\lfloor \mathbf{c} \rfloor}(p) - \Phi_{\lfloor \mathbf{c} \rfloor}(q)\| + \|\Phi_{\lfloor \mathbf{c} \rfloor}(q)\| \leq \\ &\leq \|p - q\|_u + \|\Phi_{\lfloor \mathbf{c} \rfloor}(q)\| < \frac{\delta}{2} + \frac{3\delta}{2} = 2\delta.\end{aligned}$$

Since $A_\infty = \overline{\bigcup \theta_{n,\infty}(A_n)}$, choose $\mathbf{c} = \theta_{k,\infty}(\mathbf{a}_{(k)})$ for k large enough such that

$$\|\mathbf{a} - \mathbf{c}\| < \min\{\epsilon\,;\, \alpha_p, p \in \mathcal{R}\}.$$

The minimum is a strictly positive number because $\mathcal{R}$ is finite. ∎

Theorem 3.4 *Let $(A_n, \theta_n)_{n\in\mathbb{N}}$ be an inductive system with limit $(A_\infty, \theta_{n,\infty})$. Suppose $\mathcal{R} \subset \mathcal{F}_m^{(1)}$ is a finite set of formal relations. If $\mathcal{R}$ is weakly stable for the class $\mathcal{A} = \{A_n, n \in \mathbb{N}\}$, then $\mathcal{R}$ is weakly stable for the class $\mathcal{B} = \{A_n, n \in \mathbb{N}\} \cup \{A_\infty\}$.*

Proof Let $\epsilon > 0$. Use the hypothesis to find a δ, $0 < \delta < 1$, with the property that for any $n \in \mathbb{N}$, if $\mathbf{a}_{(n)}$ is a δ-representation of $\mathcal{R}$ in A_n then there exists a representation $\mathbf{b}_{(n)}$ of $\mathcal{R}$ in A_n such that $\|\mathbf{a}_{(n)} - \mathbf{b}_{(n)}\| < \frac{\epsilon}{2}$. For this δ, take any $\frac{\delta}{4}$-representation $\mathbf{a}$ of $\mathcal{R}$ in A_∞.

Case 1 Suppose that there exist $n_0 \in \mathbb{N}$ and $\mathbf{a}_{(n_0)} \in (A_{n_0})^m$ such that $\mathbf{a} = \theta_{n_0,\infty}(\mathbf{a}_{(n_0)})$. Being a $\frac{\delta}{4}$- representation of $\mathcal{R}$ in A_∞, $\mathbf{a}$ clearly can be considered a $\frac{\delta}{2}$-representation of $\mathcal{R}$ in A_∞. Possibly having to increase n_0, since $\|\mathbf{a}\| \leq 1 + \frac{\delta}{2}$ and $\|\mathbf{a}\|$ is the limit of the non-increasing sequence $\left(\|\theta_{n_0,k}(\mathbf{a}_{(n_0)})\|\right)_k$, we can assume that $\|\mathbf{a}_{(n_0)}\| \leq 1 + \delta$.

For $k \geq n_0$, define $\mathbf{a}_{(k)} = \theta_{n_0,k}(\mathbf{a}_{(n_0)})$. Since $\|\Phi_{\lfloor \mathbf{a} \rfloor}(p)\| \leq \frac{\delta}{2}$ for all $p \in \mathcal{R}$, we can choose $k_0 \geq n_0$ such that $\mathbf{a}_{(k_0)}$ is a δ-representation of $\mathcal{R}$ in A_{k_0}. Then, using the hypothesis, there exists a representation $\mathbf{b}_{(k_0)}$ of $\mathcal{R}$ in A_{k_0}, such that $\|\mathbf{a}_{(k_0)} - \mathbf{b}_{(k_0)}\| < \frac{\epsilon}{2}$. Recall that this means that $\|\mathbf{b}_{(k_0)}\| \leq 1$ and $\Phi_{\mathbf{b}_{(k_0)}}(p) = 0$, for all $p \in \mathcal{R}$.

Define $\mathbf{b} \in (A_\infty)^m$ as $\mathbf{b} = \theta_{k_0,\infty}(\mathbf{b}_{(k_0)})$. It is clear that

(i) $\|\mathbf{b}\| = \|\theta_{k_0,\infty}(\mathbf{b}_{(k_0)})\| = \lim_k \|\theta_{k_0,k}(\mathbf{b}_{(k_0)})\| \leq \lim_k \|\mathbf{b}_{(k)}\| \leq \|\mathbf{b}_{(k_0)}\| \leq 1$;
(ii) $\Phi_{\mathbf{b}}(p) = \theta_{k_0,\infty} \circ \Phi_{\mathbf{b}_{(k_0)}}(p) = 0$, for all $p \in \mathcal{R}$,

which imply that $\mathbf{b}$ is a representation of $\mathcal{R}$ in A_∞.

We have:

$$\begin{aligned}\|\mathbf{b} - \theta_{k_0,\infty}(\mathbf{a}_{(k_0)})\| &= \|\theta_{k_0,\infty}(\mathbf{b}_{(k_0)}) - \theta_{k_0,\infty}(\mathbf{a}_{(k_0)})\| = \|\theta_{k_0,\infty}(\mathbf{b}_{(k_0)} - \mathbf{a}_{(k_0)})\| \\ &= \lim_k \|\theta_{k_0,k}(\mathbf{b}_{(k_0)} - \mathbf{a}_{(k_0)})\| \leq \|\mathbf{b}_{(k_0)} - \mathbf{a}_{(k_0)}\| < \frac{\epsilon}{2}.\end{aligned}$$

General Case Let $\mathbf{a}$ be any $\frac{\delta}{4}$-representation of $\mathcal{R}$ in A_∞. Use Lemma 3.3 to choose $k_0 \in \mathbb{N}$ such that $\|\mathbf{a} - \theta_{k_0,\infty}(\mathbf{a}_{(k_0)})\| < \frac{\epsilon}{2}$, and $\theta_{k_0,\infty}(\mathbf{a}_{(k_0)})$ is a $\frac{\delta}{2}$-representation of $\mathcal{R}$ in A_∞. Using Case 1, there exists a representation $\mathbf{b}$ of $\mathcal{R}$ in A_∞ such that $\|\mathbf{b} - \theta_{k_0,\infty}(\mathbf{a}_{(k_0)})\| < \frac{\epsilon}{2}$. We also have:

$$\|\mathbf{b} - \mathbf{a}\| \leq \|\mathbf{b} - \theta_{k_0,\infty}(\mathbf{a}_{(k_0)})\| + \|\theta_{k_0,\infty}(\mathbf{a}_{(k_0)}) - \mathbf{a}\| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \qquad \blacksquare$$

In [9], Huaxin Lin proved that pairs of almost commuting selfadjoint contractive matrices are uniformly close to commuting pairs of selfadjoint contractive matrices. In other words, Lin proved that the set of formal relations $\mathcal{R} = \{x - x^*, y - y^*, xy - yx\} \subset \mathcal{F}_2^{(1)}$ is weakly stable for the class $\mathcal{A}$ of all algebras $M_n(\mathbf{C})$. A corollary of Theorem 3.4 is that the relations $\mathcal{R}$ above are weakly stable for the class of all AF-algebras. The same corollary also follows from the generalization of Lin's result by P. Friis and M. Rordam (see [7], Theorem 4.4). In that paper the authors proved that the relations $\mathcal{R}$ above are weakly stable for the class of C^*-algebras with the property (IR). AF-algebras have the property (IR). (See [7], Definition 3.1.)

From Theorem 3.4 above and Theorem 4.4 in [7] we have the following Corollary, which appears to be a new result. (It may be true that the (IR) class is closed under inductive limits, but this doesn't seem obvious.)

Corollary 3.5 *The set of relations* $\mathcal{R} = \{x - x^*, y - y^*, xy - yx\} \subset \mathcal{F}_2^{(1)}$ *is weakly stable in inductive limits of* C^**-algebras with the property (*IR*).*

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References

[1] B. Blackadar, *Shape Theory for* C^**-Algebras*. Math. Scand. **56**(1985), 249–275.
[2] L. Brown, Ext *of certain free product* C^**-algebras*. J. Operator Theory **6**(1981), 135–141.
[3] C. Cerri, *Non-commutative Deformation of* $C(T^2)$ *and* K*-theory*. Internat. J. Math. (5) **8**(1997), 555–571.
[4] L. A. Coburn, *The* C^**-algebra generated by an Isometry*. Bull. Amer. Math. Soc. **73**(1967), 722–726.
[5] J. Cuntz, K*-Theory for certain* C^**-algebras*. Ann. of Math. **113**(1981), 181–197.
[6] S. Eilers and T. A. Loring and G. K. Pedersen, *Stability of Anticommutation Relations. An Application of Noncommutative* CW *complexes*. J. Reine Angew. Math. **499**(1998), 101–143.
[7] P. Friis and M. Rørdam, *Almost commuting self-adjoint matrices—a short proof of Huaxin Lin's theorem*. J. Reine Angew. Math. **479**(1996), 121–131.
[8] D. W. Hadwin, *Closures of direct sums of classes of operators*. Proc. Amer. Math. Soc. **121**(1994), 697–701.
[9] H. Lin, *Almost commuting selfadjoint matrices and applications*. Operator algebras and their applications, Amer. Math. Soc. **13**, Fields Inst. Commun., 193–233.
[10] T. A. Loring, C^**-Algebras generated by stable relations*. J. Funct. Anal. **112**(1993), 159–201.
[11] ———, *Lifting Solutions to Perturbing Problem in* C^**-Algebras*. Amer. Math. Soc. **8**1997, Fields Institute Monographs.
[12] M. S. Monteiro, *Weakly Stable Relations and Inductive Limits of* C^**-algebras*. University of New Mexico, July, 2000.
[13] G. J. Murphy, C^**-Algebras and Operator Theory*. Academic Press, New York, 1990.
[14] N. C. Phillips, *Inverse limit of* C^**-algebras and applications*. Operator algebras and applications **1**, Cambridge Univ. Press **135**, London Math. Soc. Lecture Note. Ser. 1988, 127–185.

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