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REMARKS ON QUASI-LINDELÖF SPACES

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Abstract

In this paper, we show that there exist a Tychonoff quasi-Lindelöf space X and a compact space Y such that $X \times Y$ is not quasi-Lindelöf. This answers negatively an open question of Petra Staynova.

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1. Introduction

By a space we mean a topological space. Let us recall that a space X is *Lindelöf* if every open cover of X has a countable subcover. As a generalisation of Lindelöfness, Frolik [3] defined a space X to be *weakly Lindelöf* if for every open cover $\mathcal{U}$ of X there exists a countable subset $\mathcal{V}$ of $\mathcal{U}$ such that $\overline{\cup\{V : V \in \mathcal{V}\}} = X$. Unfortunately, this property is not inherited by closed subspaces. Thus Arhangel'skiĭ [1] defined a space X to be *quasi-Lindelöf* if every closed subspace of X is weakly Lindelöf. Recently, Staynova [5, 6] studied the relationships between quasi-Lindelöf spaces and related spaces and investigated topological properties of quasi-Lindelöf spaces. In [7], Song and Zhang stated that the product of a weakly Lindelöf space and a compact space is weakly Lindelöf, for which a proof was provided by Staynova [5]. Thus Staynova [5, 6] asked the following question.

Problem 1.1. Is the product of a quasi-Lindelöf space and a compact space quasi-Lindelöf?

The purpose of this paper is to show that there exist a Tychonoff quasi-Lindelöf space X and a compact space Y such that $X \times Y$ is not quasi-Lindelöf, which gives a negative answer to the question.

Throughout this paper, the cardinality of a set A is denoted by $|A|$. Let ω be the first infinite cardinal and $\mathfrak{c}$ the cardinality of the set of all real numbers. As usual, a cardinal

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is the initial ordinal and an ordinal is the set of smaller ordinals. Every cardinal is often viewed as a space with the usual order topology. Other terms and symbols that we do not define follow [2].

2. Main result

In the following, we give an example showing that the product of a Tychonoff quasi-Lindelöf space X and a compact space Y need not be quasi-Lindelöf. We need the following lemma from [5].

LEMMA 2.1. *If X is a separable space, then X is quasi-Lindelöf.*

EXAMPLE 2.2. There exist a Tychonoff quasi-Lindelöf space X and a compact space Y such that $X \times Y$ is not quasi-Lindelöf.

PROOF. Let $\mathcal{R}$ be a maximal almost disjoint family of infinite subsets of ω with $|\mathcal{R}| = \mathfrak{c}$.

Let $X = \mathcal{R} \cup \omega$ be the Isbell–Mrówka space [4]. Then X is quasi-Lindelöf by Lemma 2.1, since ω is a countable dense subset of X.

Let $D = \{d_\alpha : \alpha < \mathfrak{c}\}$ be the discrete space of cardinality $\mathfrak{c}$ and let $Y = D \cup \{d^*\}$ be the one-point compactification of D.

Now we show that $X \times Y$ is not quasi-Lindelöf. Since $|\mathcal{R}| = \mathfrak{c}$, we can enumerate $\mathcal{R}$ as $\{r_\alpha : \alpha < \mathfrak{c}\}$. Let $A = \{\langle r_\alpha, d_\alpha\rangle : \alpha < \mathfrak{c}\}$. Then A is a closed subset of $X \times Y$ by the construction of the topology of $X \times Y$ with $|A| = \mathfrak{c}$. For each $\alpha < \mathfrak{c}$, let

$$U_\alpha = X \times \{d_\alpha\}.$$

Then U_α is an open subset of $X \times Y$. Let $\mathcal{U} = \{U_\alpha : \alpha < \mathfrak{c}\}$. Then $\mathcal{U}$ is a family of open subsets of $X \times Y$ such that $A \subseteq \bigcup \mathcal{U}$. Let $\mathcal{V}$ be any finite or countably infinite subset of $\mathcal{U}$. It is not difficult to see that

$$\overline{\bigcup \mathcal{V}} = \begin{cases} \bigcup \mathcal{V}, & \mathcal{V} \text{ finite}, \\ \bigcup \mathcal{V} \cup (X \times \{d^*\}, & \mathcal{V} \text{ infinite}. \end{cases}$$

Let $\alpha_0 = \sup\{\alpha : U_\alpha \in \mathcal{V}\}$. Then $\alpha_0 < \mathfrak{c}$, since $\mathcal{V}$ is finite or countably infinite. If we pick $\alpha' > \alpha_0$, then $U_{\alpha'} \notin \mathcal{V}$. Thus

$$\langle r_{\alpha'}, d_{\alpha'}\rangle \notin \overline{\bigcup \mathcal{V}},$$

since $U_{\alpha'}$ is the only element of $\mathcal{U}$ containing $\langle r_{\alpha'}, d_{\alpha'}\rangle$ and $\overline{\bigcup \mathcal{V}} \cap U_{\alpha'} = \emptyset$, which completes the proof. □

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