THE PRODUCT OF TWO LEGENDRE POLYNOMIALS

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1. It is known that any polynomial in μ can be expanded as a linear function of Legendre polynomials [**1**]. In particular, we have

$$P_p(\mu)P_q(\mu)=A_0P_{p+q}(\mu)+A_2P_{p+q-2}(\mu)+\ldots+A_{2k}P_{p+q-2k}(\mu)+\ldots \qquad (1)$$

The earlier coefficients, say A_0, A_2, A_4, may easily be found by equating the coefficients of μ^{p+q}, μ^{p+q-2}, μ^{p+q-4} on the two sides of (1). The general coefficient A_{2k} might then be surmised, and the value verified by induction. This may have been the method followed by Ferrers, who stated the result as an exercise in his *Spherical Harmonics* (1877). A proof was published by J. C. Adams [**2**]. The proof now to be given follows different lines from his.

The problem has been discussed from the point of view of differential equations by Hobson [**3**] and Bailey [**4**].

2. If $r^2=x^2+y^2+z^2$, the product $r^nP_n(\mu)$, or the solid Zonal Harmonic Z_n, where n is a positive integer, is a rational integral function of z and r^2; viz.

$$r^nP_n(\mu)=Z_n=\lambda_n\left\{z^n-\frac{n(n-1)}{2(2n-1)}z^{n-2}r^2+\frac{n(n-1)(n-2)(n-3)}{2\,.\,4(2n-1)(2n-3)}z^{n-4}r^4+\ldots\right\}, \qquad (2)$$

where

$$\lambda_n=\frac{1\,.\,3\,.\,5\ldots(2n-1)}{1\,.\,2\ldots n}=\frac{(2n)!}{2^n\,n!\,n!}. \qquad (3)$$

We also have

$$Z_n=\frac{1}{2\pi}\int_0^{2\pi}(z+ix\cos\alpha+iy\sin\alpha)^n\,d\alpha. \qquad (4)$$

On multiplying (1) by r^{p+q}, we obtain

$$Z_pZ_q=A_0Z_{p+q}+A_2r^2Z_{p+q-2}+A_4r^4Z_{p+q-4}+\ldots+A_{2k}r^{2k}Z_{p+q-2k}+\ldots \qquad (5)$$

3. In (5) give x, y, z values such that $r^2=0$ while z is not 0; e.g. $y=0$, $x=iz$. Then (5) becomes

$$(Z_pZ_q)_{r=0}=A_0(Z_{p+q})_{r=0},$$

or, by (2), on cancelling z^{p+q},

$$\lambda_p\lambda_q=A_0\lambda_{p+q}, \qquad (6)$$

which gives A_0.

We shall obtain A_2, A_4, ..., A_{2k} in a similar way after first taking ∇^2, ∇^4, ... , ∇^{2k} of (5).

Now

$$\nabla^2(r^mZ_n)=m(2n+m+1)r^{m-2}Z_n,$$

so that, in turn,

$$\nabla^4(r^mZ_n)=m(m-2)(2n+m+1)(2n+m-1)r^{m-4}Z_n,$$
$$\nabla^6(r^mZ_n)=m(m-2)(m-4)(2n+m+1)(2n+m-1)(2n+m-3)r^{m-6}Z_n,$$

and so on.

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In particular, with a view to (5),

$$\nabla^2(r^2 Z_{p+q-2}) = 2(2p+2q-1) Z_{p+q-2},$$
$$\nabla^4(r^4 Z_{p+q-4}) = 2 \,.\, 4(2p+2q-3)(2p+2q-5) Z_{p+q-4},$$
$$\nabla^6(r^6 Z_{p+q-6}) = 2 \,.\, 4 \,.\, 6(2p+2q-5)(2p+2q-7)(2p+2q-9) Z_{p+q-6},$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\nabla^{2k}(r^{2k} Z_{p+q-2k}) = 2 \,.\, 4 \dots (2k)(2p+2q-2k+1)(2p+2q-2k-1) \dots (2p+2q-4k+3) Z_{p+q-2k}. \quad \dots\dots\dots(7)$$

On putting $r=0$ in these, as at (6) above, we find :

$$\{\nabla^2(Z_p Z_q)\}_{r=0} = A_2 \,.\, 2(2p+2q-1)(Z_{p+q-2})_{r=0}$$
$$= A_2 \,.\, 2(2p+2q-1)\lambda_{p+q-2} z^{p+q-2};$$
$$\{\nabla^4(Z_p Z_q)\}_{r=0} = A_4 \,.\, 2 \,.\, 4(2p+2q-3)(2p+2q-5)\lambda_{p+q-4} z^{p+q-4};$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\{\nabla^{2k}(Z_p Z_q)\}_{r=0} = A_{2k} \,.\, 2 \,.\, 4 \,.\, 6 \,.\dots (2k) \times (2p+2q-2k+1)(2p+2q-2k-1) \dots (2p+2q-4k+3)\lambda_{p+q-2k} z^{p+q-2k}$$
$$= A_{2k} 2^k k! \frac{1 \,.\, 3 \,.\, 5 \dots (2p+2q-2k+1)}{1 \,.\, 3 \,.\, 5 \dots (2p+2q-4k+1)} \lambda_{p+q-2k} z^{p+q-2k}$$
$$= A_{2k} 2^k k! \frac{2p+2q-2k+1}{2p+2q-4k+1} \frac{\lambda_{p+q-k}(p+q-k)!}{\lambda_{p+q-2k}(p+q-2k)!} \lambda_{p+q-2k} z^{p+q-2k}$$
$$= A_{2k} 2^k k! \frac{2p+2q-2k+1}{2p+2q-4k+1} \lambda_{p+q-k} \frac{(p+q-k)!}{(p+q-2k)!} z^{p+q-2k}. \quad \dots\dots\dots(8)$$

4. To complete the determination of $A_2, \dots, A_{2k}$, we have to find the values of

$$\{\nabla^2(Z_p Z_q)\}_{r=0}, \dots, \{\nabla^{2k}(Z_p Z_q)\}_{r=0}.$$

We first express these as double integrals, and then determine the integrals numerically.

From (4)

$$Z_p Z_q = \frac{1}{2\pi}\int_0^{2\pi} (z + ix\cos\alpha + iy\sin\alpha)^p \, d\alpha \times \frac{1}{2\pi}\int_0^{2\pi} (z + ix\cos\beta + iy\sin\beta)^q \, d\beta$$
$$= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} (z + ix\cos\alpha + iy\sin\alpha)^p (z + ix\cos\beta + iy\sin\beta)^q \, d\alpha \, d\beta$$

Now
$$\nabla^2(uv) = u\nabla^2 v + v\nabla^2 u + 2\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial z} + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right),$$

so that, when $\nabla^2 u = 0$ and $\nabla^2 v = 0$, we have

$$\nabla^2(uv) = 2\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial z} + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right).$$

Thus

$$\nabla^2(Z_p Z_q) = \frac{1}{4\pi^2} \,.\, 2pq \int_0^{2\pi}\int_0^{2\pi} \{1 - \cos(\alpha - \beta)\} \times (z + ix\cos\alpha + iy\sin\alpha)^{p-1}(z + ix\cos\beta + iy\sin\beta)^{q-1} \, d\alpha \, d\beta.$$

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Here take $y=0$ and $x=iz$ so that $r=0$, and get

$$\{\nabla^2(Z_pZ_q)\}_{r=0}=\frac{1}{4\pi^2}\cdot 2pq\cdot z^{p+q-2}$$

$$\times\int_0^{2\pi}\int_0^{2\pi}\{1-\cos(\alpha-\beta)\}(1-\cos\alpha)^{p-1}(1-\cos\beta)^{q-1}\,d\alpha\,d\beta.$$

Similarly

$$\{\nabla^{2k}(Z_pZ_q)\}_{r=0}=\frac{1}{4\pi^2}\cdot 2^k p(p-1)\dots(p-k+1)\cdot q(q-1)\dots(q-k+1)z^{p+q-2k}$$

$$\times\int_0^{2\pi}\int_0^{2\pi}\{1-\cos(\alpha-\beta)\}^k(1-\cos\alpha)^{p-k}(1-\cos\beta)^{q-k}\,d\alpha\,d\beta. \quad \text{.........(9)}$$

From (8) and (9) A_{2k} is determined in terms of the integral

$$I_k=\int_0^{2\pi}\int_0^{2\pi}\{1-\cos(\alpha-\beta)\}^k(1-\cos\alpha)^{p-k}(1-\cos\beta)^{q-k}\,d\alpha\,d\beta. \quad \text{..............(10)}$$

We shall now show that I_k can be found in terms of simple factorial functions involving p, q, k. The proof depends on the theorem called by Hardy the Dougall-Ramanujan Theorem [**5**, **6**, **7**].

Under the integral signs in I_k (10) there are three factors each of the form $(1-\cos\psi)^h$. This may be written

$$2^h\sin^{2h}(\tfrac{1}{2}\psi),$$

or

$$2^h(e^{\frac{1}{2}i\psi}-e^{-\frac{1}{2}i\psi})^{2h}(1/2i)^{2h},$$

or

$$(-\tfrac{1}{2})^h(e^{\frac{1}{2}i\psi}-e^{-\frac{1}{2}i\psi})^{2h};$$

i.e.

$$(-\tfrac{1}{2})^h\left\{\begin{array}{c} e^{hi\psi}-2he^{(h-1)i\psi}+\dfrac{2h(2h-1)}{1\,.\,2}e^{(h-2)i\psi} \\ -\dots+(-1)^h\dfrac{2h(2h-1)\dots(h+1)}{1\,.\,2\dots h}\pm\dots+e^{-hi\psi}\end{array}\right\}.$$

If we take the middle term out as a factor this becomes, when the terms are arranged to left and right from the middle,

$$\frac{1}{2^h}\frac{2h(2h-1)\dots(h+1)}{1\,.\,2\dots h}\times\left\{\begin{array}{c}1-\dfrac{h}{h+1}e^{i\psi}+\dfrac{h(h-1)}{(h+1)(h+2)}e^{2i\psi}-\dots \\ -\dfrac{h}{h+1}e^{-i\psi}+\dfrac{h(h-1)}{(h+1)(h+2)}e^{-2i\psi}-\dots\end{array}\right\}. \quad \text{................(11)}$$

When each of the three factors under the integral signs in I_k is treated in this way, and their product formed by multiplying the individual terms of each factor by those of the other two, and adding, the only terms in the sum which contribute anything to the double integral are those containing factors of the type

$$e^{ni(\alpha-\beta)}e^{-ni\alpha}e^{ni\beta},$$

with n a positive or negative integer.

Now

$$\int_0^{2\pi}\int_0^{2\pi}e^{ni(\alpha-\beta)}\,e^{-ni\alpha}\,e^{ni\beta}\,d\alpha\,d\beta=4\pi^2.$$

Thus, from (10) and (11),

$$I_k = \frac{1}{2^k}\frac{1}{2^{p-k}}\cdot\frac{1}{2^{q-k}}\cdot 4\pi^2\cdot\frac{2k(2k-1)\ldots(k+1)}{1\,.\,2\,.\ldots.\,k}$$

$$\times\frac{(2p-2k)(2p-2k-1)\ldots(p-k+1)}{1\,.\,2\,\ldots.\,(p-k)}\cdot\frac{(2q-2k)(2q-2k-1)\ldots(q-k+1)}{1\,.\,2\ldots(q-k)}$$

$$\times\left\{\begin{array}{l}1-2\dfrac{k}{k+1}\dfrac{p-k}{p-k+1}\dfrac{q-k}{q-k+1}\\ \qquad +2\dfrac{k(k-1)}{(k+1)(k+2)}\dfrac{(p-k)(p-k-1)}{(p-k+1)(p-k+2)}\dfrac{(q-k)(q-k-1)}{(q-k+1)(q-k+2)}-\ldots\end{array}\right\}.\quad\ldots(12)$$

The series within the brackets here ; viz.

$$\left\{1-2\frac{k}{k+1}\frac{p-k}{p-k+1}\frac{q-k}{q-k+1}+\ldots\right\}.\quad\ldots\ldots(13)$$

can be summed as a special case of the Dougall-Ramanujan Theorem. We have, in fact, [**8**, **9**],

$$1-2\frac{a}{a+1}\frac{b}{b+1}\frac{c}{c+1}+2\frac{a(a-1)}{(a+1)(a+2)}\frac{b(b-1)}{(b+1)(b+2)}\frac{c(c-1)}{(c+1)(c+2)}-\ldots$$

$$=\frac{\Pi(a)\,\Pi(b)\,\Pi(c)\,\Pi(a+b+c)}{\Pi(b+c)\,\Pi(c+a)\,\Pi(a+b)},\quad\ldots\ldots(14)$$

where none of a, b, c is a negative integer, and the real part of $a+b+c$ is greater than -1.

In our case a, b, c are positive integers, and the sum of the series (13) is

$$\frac{k!\,(p-k)!\,(q-k)!\,(p+q-k)!}{(p+q-2k)!\,p!\,q!}.\quad\ldots\ldots(15)$$

The part of the right-hand side of (12) preceding the series in brackets may be written

$$\frac{4\pi^2}{2^{p+q-k}}\frac{(2k)!}{k!\,k!}\frac{(2p-2k)!}{(p-k)!\,(p-k)!}\frac{(2q-2k)!}{(q-k)!\,(q-k)!}.\quad\ldots\ldots(16)$$

Thus, from the product of (15) and (16)

$$I_k=\frac{4\pi^2}{2^{p+q-k}}\frac{(2k)!}{k!}\frac{(2p-2k)!}{(p-k)!}\frac{(2q-2k)!}{(q-k)!}\frac{(p+q-k)!}{(p+q-2k)!\,p!\,q!}.\quad\ldots\ldots(17)$$

The value of A_{2k} in (1) and (5) can now be written down from (9), (10), (8) and (17). Thus

$$\frac{1}{4\pi^2}2^k\frac{p!}{(p-k)!}\frac{q!}{(q-k)!}I_k=A_{2k}\,2^k\,k!\,\frac{2p+2q-2k+1}{2p+2q-4k+1}\lambda_{p+q-k}\frac{(p+q-k)!}{(p+q-2k)!},$$

or

$$A_{2k}=\frac{1}{2^{p+q-k}}\frac{2p+2q-4k+1}{2p+2q-2k+1}\frac{1}{\lambda_{p+q-k}}\frac{(2k)!}{(k!)^2}\frac{(2p-2k)!}{\{(p-k)!\}^2}\frac{(2q-2k)!}{\{(q-k)!\}^2}$$

$$=\frac{2p+2q-4k+1}{2p+2q-2k+1}\frac{\lambda_k\,\lambda_{p-k}\,\lambda_{q-k}}{\lambda_{p+q-k}},\quad\ldots\ldots(18)$$

where, as at (3)

$$\lambda_n=\frac{1\,.\,3\,.\,5\ldots(2n-1)}{1\,.\,2\,.\,3\ldots n}=\frac{(2n)!}{2^n n!\,n!}.$$

This is the value of A_{2k} found by Adams.

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The integral (from $\mu = -1$ to $\mu = 1$) of the product of three of the functions P can be written down at once when the coefficients A in the series (1) are known. Thus in (1) multiply by $P_{p+q-2k}(\mu)$ and integrate from -1 to 1. Then

$$\int_{-1}^{1} P_{p+q-2k}(\mu)\, P_p(\mu)\, P_q(\mu)\, d\mu = A_{2k} \int_{-1}^{1} \{P_{p+q-2k}(\mu)\}^2\, d\mu$$

$$= A_{2k} \frac{2}{2p+2q-4k+1}.$$

If we write n for $p+q-2k$, this gives, by (18), if $p+q-n$, $n+p-q$, $n-q+p$ are even positive integers or zero,

$$\int_{-1}^{1} P_n(\mu)\, P_p(\mu)\, P_q(\mu)\, d\mu = \frac{2}{n+p+q+1} \frac{\lambda_{(p+q-n)/2}\, \lambda_{(n+q-p)/2}\, \lambda_{(n+p-q)/2}}{\lambda_{(n+p+q)/2}}$$

$$= \frac{2}{n+p+q+1} \cdot \frac{1\,.\,3\,\ldots\,(p+q-n-1)}{2\,.\,4\,\ldots\,(p+q-n)} \cdot \frac{1\,.\,3\,\ldots\,(n+q-p-1)}{2\,.\,4\,\ldots\,(n+q-p)}$$

$$\times \frac{1\,.\,3\,\ldots\,(n+p-q-1)}{2\,.\,4\,\ldots\,(n+p-q)} \cdot \frac{2\,.\,4\,\ldots\,(n+p+q)}{1\,.\,3\,\ldots\,(n+p+q-1)}.$$

This is the form given by Hobson, *l.c.*, p. 87. For other values of n, p, q, the value of the integral is, of course, zero.

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