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SIMPLE MODULES OVER THE COORDINATE RING OF QUANTUM AFFINE SPACE

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The simple modules of $\mathcal{O}_q(\mathbb{C}^n)$, the coordinate ring of quantum affine space, are classified in the case when q is a root of unity.

The coordinate ring of quantum affine space, $\mathcal{O}_q(\mathbb{C}^n)$, is the algebra generated by $x_1, \cdots, x_n$ satisfing the relations $x_j x_i = q x_i x_j$, $i < j$, $0 \neq q \in \mathbb{C}$. Thus $\mathcal{O}_q(\mathbb{C}^n)$ is the iterated skew polynomial ring $\mathbb{C}[x_1][x_2;\tau_2]\cdots[x_n;\tau_n]$, where automorphisms τ_k are defined by $\tau_k(x_i) = q x_i$, $i < k$. Therefore it is a Noetherian domain of Gelfand-Kirillov dimension n, and has a $\mathbb{C}$-basis given by the monomials X^I where $I = (i_1, \cdots, i_n)$ is a multi-index with each $i_j \geqslant 0$.

The quantum matrices $\mathcal{O}_q M_n(\mathbb{C})$ and the quantised universal enveloping algebra $U_q(\mathfrak{sl}(n, \mathbb{C}))$ act on $\mathcal{O}_q(\mathbb{C}^n)$. The reader is referred to the articles [1] and [5] for further background and actions on $\mathcal{O}_q(\mathbb{C}^n)$.

The prime ideals and the primitive ideals of $\mathcal{O}_q(\mathbb{C}^n)$ are classified in [4] and [5, Section 3], in the case when q is not a root of unity. In this note, we prove that there is a surjective map Ψ from $\mathbb{C}^n$ onto the set of all the simple modules of $\mathcal{O}_q(\mathbb{C}^n)$, in the case when q is a primitive m-th root of unity, such that

$$\dim_{\mathbb{C}} \Psi(\underline{\alpha}) = m^{[p/2]},$$

where p is the number of nonzero α_i in $\underline{\alpha} = (\alpha_1, \cdots, \alpha_n)$ and $[x]$ denotes the greatest integer $\leqslant x$. Henceforce, assume throughout that q *is a primitive* m*-th root of unity unless stated otherwise.*

PROPOSITION 1. *Let R be an algebra over a field k, Z a finitely generated subalgebra contained in the center of R and let R be finitely generated as a Z-module. For a simple right R-module M, the following hold.*

(i) *R is Noetherian.*
(ii) *$\dim_k(M)$ is finite.*
(iii) *$\operatorname{ann}_R(M)$ is a maximal ideal of R.*
(iv) *$\operatorname{ann}_R(M) \cap Z$ is a maximal ideal of Z.*

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PROOF: (i) Since Z is Noetherian by Hilbert's basis theorem and R is finitely generated as a Z-module, R is also Noetherian.

(ii) [3, 9.5.5 (ii)].

(iii) and (iv) Look at the following monomorphisms:

$$Z/(\mathrm{ann}_R(M)\cap Z) \xrightarrow{\alpha} R/\mathrm{ann}_R(M), \quad R/\mathrm{ann}_R(M) \xrightarrow{\beta} \mathrm{End}_k(M)$$

where α is induced from the inclusion map from Z into R and β is induced from right module structure map on M. Since $\mathrm{End}_k(M)$ is finite dimensional by (ii), $R/\mathrm{ann}_R(M)$ is Artinian and prime. Hence it is simple. Moreover, $Z/(\mathrm{ann}_R(M)\cap Z)$ is integral domain and Artinian. Therefore, $\mathrm{ann}_R(M)\cap Z$ and $\mathrm{ann}_R(M)$ are maximal ideals of Z and R, respectively. ∎

COROLLARY 2. *Every simple right $\mathcal{O}_q(\mathbb{C}^n)$-module is finite dimensional.*

PROOF: The subalgebra Z of $\mathcal{O}_q(\mathbb{C}^n)$ generated by x_i^m, $i = 1, \cdots, n$, is contained in the center of $\mathcal{O}_q(\mathbb{C}^n)$ and $\mathcal{O}_q(\mathbb{C}^n)$ is finitely generated as a Z-module. This completes the proof by Proposition 1 (ii). ∎

For convenience, set $i' = n+1-i$ for $1 \leqslant i < (n+1)/2$. If $n = 2k+1$ or $n = 2k$ then, for nonzero $\alpha_1, \cdots, \alpha_n \in \mathbb{C}$, let $M(\underline{\alpha}) = M(\alpha_1, \cdots, \alpha_n)$ be the $\mathbb{C}$-vector space with basis $e(a_1, \cdots, a_k)$, $0 \leqslant a_i \leqslant m-1$. Then $M(\underline{\alpha})$ has a right $\mathcal{O}_q(\mathbb{C}^n)$-module structure defined as follows :

Case 1. $n = 2k+1$:

$$e(a_1,\cdots,a_k)x_i = \alpha_i q^{-(a_1+\cdots+a_{i-1})} e(a_1,\cdots,a_{i-1},a_i \dot{+} 1, a_{i+1},\cdots,a_k), \quad 1 \leqslant i \leqslant k$$

$$e(a_1,\cdots,a_k)x_{i'} = \alpha_i^{-1}\alpha_{i'} q^{-(a_1+\cdots+a_i)+1} e(a_1,\cdots,a_{i-1},a_i \dot{+} (-1), a_{i+1},\cdots,a_k), \quad 1 \leqslant i \leqslant k$$

$$e(a_1,\cdots,a_k)x_{k+1} = \alpha_{k+1} q^{-(a_1+\cdots+a_k)} e(a_1,\cdots,a_k);$$

Case 2. $n = 2k$:

$$e(a_1,\cdots,a_k)x_i = \alpha_i q^{-(a_1+\cdots+a_{i-1})} e(a_1,\cdots,a_{i-1},a_i \dot{+} 1, a_{i+1},\cdots,a_k), \quad 1 \leqslant i \leqslant k$$

$$e(a_1,\cdots,a_k)x_{i'} = \alpha_i^{-1}\alpha_{i'} q^{-(a_1+\cdots+a_i)+1} e(a_1,\cdots,a_{i-1},a_i \dot{+} (-1), a_{i+1},\cdots,a_k), \quad 1 \leqslant i \leqslant k-1$$

$$e(a_1,\cdots,a_k)x_{k+1} = \alpha_{k+1} q^{-(a_1+\cdots+a_k)} e(a_1,\cdots,a_k);$$

where $\dot{+}$ is addition in the additive group $\mathbb{Z}_m$. To confirm the well-definedness of these rules, it suffices to check that

$$e(a_1,\cdots,a_k)x_j x_i = q e(a_1,\cdots,a_k)x_i x_j, \ 1 \leqslant i < j \leqslant n.$$

These are all routinely verified.

Proposition 3. *The right $\mathcal{O}_q(\mathbb{C}^n)$-module $M(\underline{\alpha})$ is simple.*

Proof: If $n = 2k+1$ and $e(a_1, \cdots, a_k) \neq e(a'_1, \cdots, a'_k)$, choose an index i such that $a_1 = a'_1, \cdots, a_{i-1} = a'_{i-1}$, $a_i \neq a'_i$. The vectors $e(a_1, \cdots, a_k)$ and $e(a'_1, \cdots, a'_k)$ are eigenvectors of $x_i x_{i'}$ associated with the distinct eigenvalues $\alpha_{i'} q^{-2(a_1+\cdots+a_{i-1})-a_i}$ and $\alpha_{i'} q^{-2(a'_1+\cdots+a'_{i-1})-a'_i}$, respectively.

If $n = 2k$, let $e(a_1, \cdots, a_k)$, $e(a'_1, \cdots, a'_k)$ and i be as in the case $n = 2k+1$. If $i < k$ then $e(a_1, \cdots, a_k)$ and $e(a'_1, \cdots, a'_k)$ are eigenvectors of $x_i x_{i'}$ associated with the distinct eigenvalues $\alpha_{i'} q^{-2(a_1+\cdots+a_{i-1})-a_i}$ and $\alpha_{i'} q^{-2(a'_1+\cdots+a'_{i-1})-a'_i}$, respectively, and if $i = k$ then $e(a_1, \cdots, a_k)$ and $e(a'_1, \cdots, a'_k)$ are eigenvectors of x_{k+1} associated with the distinct eigenvalues $\alpha_{k+1} q^{-(a_1+\cdots+a_k)}$ and $\alpha_{i'} q^{-(a'_1+\cdots+a'_k)}$, respectively.

Hence every nonzero submodule of $M(\underline{\alpha})$ contains a vector $e(a_1, \cdots, a_k)$, thus $M(\underline{\alpha})$ is simple by the action of x_i, $1 \leqslant i \leqslant k$. ∎

Proposition 4. *Let a simple right $\mathcal{O}_q(\mathbb{C}^n)$-module N be x_i-torsion free for each $i = 0, \cdots, n$. Then N is isomorphic to $M(\underline{\alpha})$ for some $\underline{\alpha} = (\alpha_i) \in (\mathbb{C}^*)^n$.*

Proof: Let $n = 2k+1$. Since each $x_i^m, x_i x_{i'}$, $i = 1, \cdots, k$ and x_{k+1} commutes and N is finite dimensional, there is a common eigenvector v of $x_i^m, x_i x_{i'}$, $i = 1, \cdots, k$ and x_{k+1}. Put $v x_i^m = \nu_i v$, $v x_i x_{i'} = \alpha_{i'} v$, $v x_{k+1} = \alpha_{k+1} v$, $i = 1, \cdots, k$. For each $i = 1, \cdots, k$, let α_i be an m-th root of ν_i. Notice that the α_i are all nonzero and

$$v x_k^{a_k} \cdots x_1^{a_1} x_{i'} = \begin{cases} \alpha_{i'} q^{-(a_1+\cdots+a_i)+1} v x_k^{a_k} \cdots x_i^{a_i-1} \cdots x_1^{a_1}, & a_i > 0 \\ \nu_i^{-1} \alpha_{i'} q^{-(a_1+\cdots+a_i)+1} v x_k^{a_k} \cdots x_i^{m-1} \cdots x_1^{a_1}, & a_i = 0. \end{cases}$$

Define a linear transformation

$$\psi : M(\underline{\alpha}) \longrightarrow N, \quad \psi(e(a_1, \cdots, a_k)) = \alpha_1^{-a_1} \cdots \alpha_k^{-a_k} v x_k^{a_k} \cdots x_1^{a_1}.$$

It is routinely verified that

$$\psi(e(a_1, \cdots, a_k) x_i) = \psi(e(a_1, \cdots, a_k)) x_i, \quad i = 1, \cdots, n.$$

Hence ψ is an $\mathcal{O}_q(\mathbb{C}^n)$-homomorphism, and thus it is an isomorphism because $M(\underline{\alpha})$ and N are both simple. The proof of the case when $n = 2k$ is similar. ∎

Theorem 5. *There is a surjective map Ψ from $\mathbb{C}^n$ onto the set of all the simple right $\mathcal{O}_q(\mathbb{C}^n)$-modules such that $\dim_{\mathbb{C}} \Psi(\underline{\alpha}) = m^{[p/2]}$, where p is the number of nonzero α_i in $\underline{\alpha} = (\alpha_i)$ and $[x]$ is the greatest integer $\leqslant x$.*

Proof: Let M be a simple $\mathcal{O}_q(\mathbb{C}^n)$-module and let Z be the subalgebra generated by x_i^m, $i = 1, \cdots, n$. Then $\operatorname{ann}(M) \cap Z$ is a maximal ideal of Z by Proposition 1 (iv).

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Hence $x_i^m - \lambda_i \in$ ann(M) for some $\lambda_i \in \mathbb{C}$, $i = 1, \cdots, n$. If $\lambda_i = 0$ then $x_i \in$ ann(M) since x_i is a normal element of $\mathcal{O}_q(\mathbb{C}^n)$ and ann(M) is prime. For convenience, assume that $\lambda_1, \cdots, \lambda_p$ are all nonzero and $\lambda_{p+1} = \cdots = \lambda_n = 0$. Thus M is a simple $\mathcal{O}_q(\mathbb{C}^p)$-module and is x_i-torsion free for each $i = 1, \cdots, p$, since ann(M) contains $x_{p+1}, \cdots, x_n$ and $\mathcal{O}_q(\mathbb{C}^n)/\langle x_{p+1}, \cdots, x_n\rangle$ is isomorphic to $\mathcal{O}_q(\mathbb{C}^p)$. Hence the result follows from Proposition 4. ∎

REMARK 1. The map Ψ of Theorem 5 is not injective.

PROOF: Let $n = 2k+1$ and $\underline{\alpha}, \underline{\beta}$ be two elements in $(\mathbb{C}^*)^n$ such that $\alpha_i = \beta_i$, $i \neq k+1, k+2$ and $\alpha_{k+1} = q^{-1}\beta_{k+1}$, $\alpha_{k+2} = q^{-1}\beta_{k+2}$. Then, it is easy to see that the map $\psi : M(\underline{\alpha}) \longrightarrow M(\underline{\beta})$ given by $\psi(e(a_1, \cdots, a_k)) = e(a_1, \cdots, a_{k-1}, a_k + 1)$ is an isomorphism. ∎

REMARK 2. All primitive ideals of $\mathcal{O}_q(\mathbb{C}^n)$ are annihilators of $\Psi(\underline{\alpha}), \underline{\alpha} \in \mathbb{C}^n$.

REMARK 3. If q is not a root of unity then every finite dimensional simple $\mathcal{O}_q(\mathbb{C}^n)$-module is one-dimensional by [**2**, 1.3]. The classification of the one-dimensional simple modules is a fairly easy exercise.

REMARK 4. Smith classified all simple modules of $\mathcal{O}_q(\mathbb{C}^2)$ in [**6**, pp. 123].

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