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A VARIATIONAL INEQUALITY IN NON-COMPACT SETS AND ITS APPLICATIONS

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In this note, we shall prove a new variational inequality in non-compact sets and as an application, we prove a generalisation of the Schauder-Tychonoff fixed point theorem.

Let E be a Hausdorff topological vector space. Denote the dual space of E by E^* and the pairing between E^* and E by $\langle w, x\rangle$ for each $w \in E^*$ and $x \in E$. If A is a subset of E, we shall denote by 2^A the family of all non-empty subsets of A and by $cl\,A$ the closure of A in E, and $co\,A$ the convex hull of A.

The following Fan-Browder fixed point theorem [2] is essential in convex analysis and is also the basic tool in proving many variational inequalities and intersection theorems in nonlinear functional analysis:

THEOREM. [2] *Let X be a non-empty compact convex subset of a Hausdorff topological vector space and let $T : X \to 2^X$ be a multimap satisfying the following:*

(1) *for each $x \in X$, $T(x)$ is convex,*
(2) *for each $y \in X$, $T^{-1}(y)$ is open.*

Then T has a fixed point $\widehat{x} \in X$, that is $\widehat{x} \in T(\widehat{x})$.

The Fan-Browder theorem can be proved by using Brouwer's fixed point theorem or the KKM-theorem. Until now, there have been numerous generalisations and applications of this Theorem by several authors; for example, see [4, 7] and the references there.

In a recent paper [4], Ding, Kim and Tan further generalise the above result to non-compact sets in locally convex spaces and the following is a special case of the fixed point version of their Theorem 1:

LEMMA 1. [4] *Let X be a non-empty convex subset of a locally convex Hausdorff topological vector space and D be a non-empty compact subset of X. Let $T : X \to 2^D$ be a multimap satisfying the following:*

(1) *for each $x \in X$, $co\,T(x) \subset D$,*
(2) *for each $y \in X$, $T^{-1}(y)$ is open in X.*

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Then there exists a point $\widehat{x} \in X$ *such that* $\widehat{x} \in co\,T(\widehat{x})$.

Recall that for a topological vector space E, the strong topology on its dual space E^* is the topology on E^* generated by the family $\{U(B;\varepsilon) : B$ is a non-empty bounded subset of E and $\varepsilon > 0\}$ as a base for the neighbourhood system at 0, where $U(B;\varepsilon) = \{f \in E^* : \sup_{x \in B} |\langle f, x\rangle| < \varepsilon\}$.

We begin with the following

LEMMA 2. *Let* E *be a topological vector space and* E^* *be the dual space of* E *equipped with the strong topology. Let* X *be a non-empty bounded subset of* E *and* $T : X \to 2^{E^*}$ *be an upper semicontinuous multimap such that each* $T(x)$ *is (strongly) compact. Then for each* $y \in E$, *the real-valued function* $g_y : X \to R$ *defined by*

$$g_y(x) = \inf_{w \in T(x)} Re\,\langle w, x - y\rangle, \quad \textit{for each } x \in X,$$

is lower semicontinuous.

PROOF: Let $x_0 \in X$ be given. For any $\varepsilon > 0$, we shall show that there exists an open neighbourhood $N(x_0)$ of x_0 such that

$$g_y(x) \geqslant g_y(x_0) - \varepsilon \quad \text{for each } x \in N(x_0).$$

Indeed, let $V := \{p \in E^* : \sup_{t \in X - y} |p(t)| < \varepsilon/3\}$, where $X - y = \{x - y : x \in X\}$. Then V is a strongly open neighbourhood of 0 in E^* since $X - y$ is a bounded set in E. Since T is upper semicontinuous at x_0 and $T(x_0) + V$ is a strongly open set containing $T(x_0)$, there exists an open neighbourhood N_0 of x_0 in X such that $T(x) \subset T(x_0) + V$ for each $x \in N_0$.

Next, for each $u \in T(x_0)$, we let

$$V_u := \left\{p \in E^* : \sup_{t \in X - X} |p(t) - u(t)| < \frac{\varepsilon}{3}\right\},$$

where $X - X = \{x - z : x, z \in X\}$; then V_u is also a strongly open neighbourhood of u in E^* since $X - X$ is a bounded set in E. Since $T(x_0)$ is strongly compact and $T(x_0) \subset \cup_{u \in T(x_0)} V_u$, there exists a finite subset $\{u_1, \ldots, u_n\}$ of $T(x_0)$ with $T(x_0) \subset \cup_{i=1}^n V_{u_i}$. For each $i = 1, \ldots, n$, since u_i is a continuous linear functional, there exists an open neighbourhood N_i of x_0 in X such that $|u_i(x) - u_i(x_0)| < \varepsilon/3$ for each $x \in N_i$.

Now let $N(x_0) := \cap_{i=0}^n N_i$; then $N(x_0)$ is an open neighbourhood of x_0 in X. We shall show that this open neighbourhood $N(x_0)$ of x_0 is the desired one. For each $x \in N(x_0)$ and each $w \in T(x)$, since $x \in N_0$, there exists $u \in T(x_0)$ such that $w - u \in V$. Also, since $u \in T(x_0) \subset \cup_{i=1}^{n} V_{u_i}$, there exists $i_0 \in \{1, \ldots, n\}$ such that $u \in V_{u_{i_0}}$. Therefore we have

$$|Re\langle w, x-y\rangle - Re\langle u, x-y\rangle| \leqslant |\langle w-u, x-y\rangle| < \frac{\varepsilon}{3},$$

so that

$$\begin{aligned} Re\langle w, x-y\rangle &> Re\langle u, x-y\rangle - \frac{\varepsilon}{3} \\ &= Re\langle u, x_0 - y\rangle + Re\langle u, x - x_0\rangle - \frac{\varepsilon}{3} \\ &= Re\langle u, x_0 - y\rangle + Re\langle u - u_{i_0}, x - x_0\rangle \\ &\quad + Re\langle u_{i_0}, x - x_0\rangle - \frac{\varepsilon}{3} \\ &> Re\langle u, x_0 - y\rangle - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} \\ &\geqslant \inf_{v \in T(x_0)} Re\langle v, x_0 - y\rangle - \varepsilon \\ &= g_y(x_0) - \varepsilon. \end{aligned}$$

Since $w \in T(x)$ is arbitrary, we have $g_y(x) = \inf_{w \in T(x)} Re\langle w, x-y\rangle \geqslant g_y(x_0) - \varepsilon$, which completes the proof. □

Lemma 2 is a multivalued generalisation of Lemma 1 in [**2**] (see also [**10**, Lemma 1] where it was observed that the result holds for X being bounded instead of compact).

Now we shall prove the following new variational inequality in non-compact sets.

THEOREM 1. *Let X be a bounded convex subset of a locally convex Hausdorff topological vector space E and D be a non-empty compact subset of X. Let $T : X \to 2^{E^*}$ be an upper semicontinuous multimap from the relative topology of X to the strong topology of E^* such that each $T(x)$ is (strongly) compact. Suppose further that for each $x \in X \setminus D$,*

$$(*) \qquad \inf_{w \in T(y)} Re\langle w, y - x\rangle \leqslant 0 \quad \text{for all } y \in X.$$

Then there exists a point $\hat{x} \in X$ such that

$$\inf_{w \in T(\hat{x})} Re\langle w, \hat{x} - x\rangle \leqslant 0 \quad \text{for all } x \in X.$$

Furthermore, if $T(\hat{x})$ is also convex, then there exists a point $\hat{w} \in T(\hat{x})$ such that

$$Re\langle \hat{w}, \hat{x} - x\rangle \leqslant 0 \quad \text{for all } x \in X.$$

PROOF: Suppose that for each $x \in X$ there exists a point $\widetilde{x} \in X$ such that $\inf_{w \in T(x)} Re\langle w, x - \widetilde{x}\rangle > 0$. Then by the assumption $(*)$, $\widetilde{x} \in D$. Now we define a multimap $P : X \to 2^D$ by

$$P(x) := \{y \in D : \inf_{w \in T(x)} Re\langle w, x - y\rangle > 0\} \quad \text{for all } x \in X.$$

Then for each $x \in X, P(x)$ is non-empty. For each $x \in X$, we shall show that $co\, P(x) \subset D$. Indeed, let $n \in N$, $y_1, \ldots, y_n \in P(x)$ and $t_1, \ldots, t_n \in [0,1]$ with $\sum_{i=1}^{n} t_i = 1$; then for each $i = 1, \ldots, n$,

$$\inf_{w \in T(x)} Re\langle w, x - y_i\rangle > 0;$$

it follows that

$$\inf_{w \in T(x)} Re\langle w, x - \sum_{i=1}^{n} t_i y_i\rangle \geqslant \sum_{i=1}^{n} t_i \inf_{w \in T(x)} Re\langle w, x - y_i\rangle > 0.$$

Since $\sum_{i=1}^{n} t_i y_i \in X$, by the assumption $(*)$ again, $\sum_{i=1}^{n} t_i y_i \in D$. Hence $co\, P(x) \subset D$.

Next for each $y \in D$, we shall show that $P^{-1}(y)$ is open in X. Let $(x_\alpha)_{\alpha \in \Gamma}$ be a net in $X \setminus P^{-1}(y)$, which converges to some $x_0 \in X$. Then we have

$$\inf_{w \in T(x_\alpha)} Re\langle w, x_\alpha - y\rangle \leqslant 0 \quad \text{for all } \alpha \in \Gamma.$$

By Lemma 2, the real-valued function

$$x \to \inf_{w \in T(x)} Re\langle w, x - y\rangle$$

is lower semicontinuous, it follows that

$$\inf_{w \in T(x_0)} Re\langle w, x_0 - y\rangle \leqslant 0.$$

Therefore $X \setminus P^{-1}(y)$ is closed, and hence $P^{-1}(y)$ is open in X. Thus all the hypotheses of Lemma 1 are satisfied, so that by Lemma 1 there exists a point $\widehat{x} \in X$ such that $\widehat{x} \in co\, P(\widehat{x})$. But then there exist $y_1, \ldots, y_m \in P(\widehat{x})$ and $\lambda_1, \ldots, \lambda_m \geqslant 0$ with $\sum_{i=1}^{m} \lambda_i = 1$ such that $\widehat{x} = \sum_{i=1}^{m} \lambda_i y_i$. Therefore we have

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$$
\begin{aligned}
0 &= \inf_{w\in T(\widehat{x})} Re\langle w, \widehat{x}-\widehat{x}\rangle \\
&= \inf_{w\in T(\widehat{x})} Re\langle w, \widehat{x}-\sum_{i=1}^{m}\lambda_i y_i\rangle \\
&= \inf_{w\in T(\widehat{x})} \sum_{i=1}^{m}\lambda_i Re\langle w, \widehat{x}-y_i\rangle \\
&\geqslant \sum_{i=1}^{m}\lambda_i \inf_{w\in T(\widehat{x})} Re\langle w, \widehat{x}-y_i\rangle \;>\; 0,
\end{aligned}
$$

which is a contradiction. Hence there must exist a point $\widehat{x}\in X$ such that

$$
(1) \qquad \inf_{w\in T(\widehat{x})} Re\langle w, \widehat{x}-x\rangle \leqslant 0 \quad \text{for all } x\in X.
$$

To prove the second assertion, suppose that $T(\widehat{x})$ is convex. Then we define $f: X\times T(\widehat{x})\to R$ by

$$
f(x,w) := Re\langle w, \widehat{x}-x\rangle \quad \text{for each } (x,w)\in X\times T(\widehat{x}).
$$

Note that for each fixed $x\in X$, $x\to Re\langle w,\widehat{x}-x\rangle$ is continuous affine, and for each $w\in T(\widehat{x})$, $x\to Re\langle w,\widehat{x}-x\rangle$ is affine. Thus, by Kneser's minimax theorem [8], we have

$$
\min_{w\in T(\widehat{x})}\sup_{x\in X} f(x,w) = \sup_{x\in X}\min_{w\in T(\widehat{x})} f(x,w).
$$

Thus

$$
\min_{w\in T(\widehat{x})}\sup_{x\in X} Re\langle w,\widehat{x}-x\rangle \leqslant 0 \quad \text{by (1)}.
$$

Since $T(\widehat{x})$ is compact, there exists $\widehat{w}\in T(\widehat{x})$ such that

$$
\sup_{x\in X} Re\langle \widehat{w},\widehat{x}-x\rangle = \min_{w\in T(\widehat{x})}\sup_{x\in X} Re\langle w,\widehat{x}-x\rangle.
$$

Therefore $Re\langle \widehat{w},\widehat{x}-x\rangle \leqslant 0$ for all $x\in X$. This completes the proof. □

When $X = D$ is compact convex, we obtain the following generalisation of Hartman-Stampacchia's variational inequality [6] due to Browder [3, Theorem 6]:

COROLLARY 1. *Let X be a non-empty compact convex subset of a locally convex Hausdorff topological vector space E and let $T: X\to 2^{E^*}$ be an upper semicontinuous multimap from the relative topology of X to the strong topology of E^* such that each $T(x)$ is a (strongly) compact convex subset of E^*.*

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Then there exists a point $\widehat{x} \in X$ *and* $\widehat{w} \in T(\widehat{x})$ *such that*

$$Re\langle \widehat{w}, \widehat{x} - x\rangle \leqslant 0 \quad \textit{for all } x \in X.$$

The following is a single-valued version of Theorem 1:

COROLLARY 2. *Let* X *be a bounded convex subset of a locally convex Hausdorff topological vector space* E *and* D *be a non-empty compact subset of* X. *Let* $T : X \to E^*$ *be a continuous mapping from the relative topology of* X *to the strong topology of* E^* *satisfying the following condition:*

$$\textit{for each } x \in X \setminus D,\ Re\langle T(y), y - x\rangle \leqslant 0 \quad \textit{for all } y \in X.$$

Then there exists a point $\widehat{x} \in X$ such that

$$Re\langle T(\widehat{x}), \widehat{x} - x\rangle \leqslant 0 \quad \textit{for all } x \in X.$$

Let E be a topological vector space and M be a topological space. Recall that a multimap $F : M \to 2^E$ is *upper hemicontinuous* (for example, see [**1**, p.122]) if for each $p \in E^*$ and for each $\lambda \in R$, the set $\{x \in M : \sup_{u \in F(x)} Re\langle p, u\rangle < \lambda\}$ is open in M. For relationships among upper semicontinuity, upper demicontinuity and upper hemicontinuity, we refer to [**11**, Propositions 1 and 2 and Examples 1 and 2].

As an application of Corollary 2, we prove the following fixed point theorem:

THEOREM 2. *Let* X *be a non-empty paracompact bounded convex subset of a locally convex Hausdorff topological vector space* E, D *be a non-empty compact subset of* X. *Let* $F : X \to 2^E$ *be an upper hemicontinuous multimap satisfying the following:*

(1) *for each* $x \in X$, $F(x)$ *is non-empty closed convex,*

(2) *for each* $x \in X$, $F(x) \cap cl\,(x + \cup_{\lambda > 0} \lambda(X - x)) \neq \phi$,

(3) *for each* $x \in X \setminus D$, $y \in X$ *and* $p \in E^*$, *if* $\inf\{Re\langle p, y - z\rangle : z \in F(y)\} > 0$, *then* $Re\langle p, y - x\rangle \leqslant 0$.

Then there exists $\widehat{x} \in X$ *such that* $\widehat{x} \in F(\widehat{x})$.

PROOF: Since F is upper hemicontinuous, for each $p \in E^*$, the set

$$\begin{aligned} U(p) &= \{x \in X : \sup_{z \in F(x)} Re\langle p, z\rangle < Re\langle p, x\rangle\} \\ &= \cup_{\lambda \in R}[\{x \in X : \sup_{z \in F(x)} Re\langle p, z\rangle < \lambda\} \\ &\qquad \cap \{x \in X : Re\langle p, x\rangle > \lambda\}] \end{aligned}$$

is open in X. Suppose $x \notin F(x)$ for each $x \in X$. Then for each $x \in X$, there exists $p \in E^*$ such that $\sup_{z \in F(x)} Re\langle p, z\rangle < Re\langle p, x\rangle$ so that $x \in U(p)$. Thus $\{U(p) : p \in E^*\}$

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is an open cover of the paracompact space X. Let $\{V(p) : p \in E^*\}$ be a locally finite open precise refinement of $\{U(p) : p \in E^*\}$ and $\{\beta_p : p \in E^*\}$ be the continuous partition of unity subordinated to this refinement $\{V(p) : p \in E^*\}$. Define a mapping $T : X \to E^*$ by

$$T(x) = \sum_{p \in E^*} \beta_p(x) p \quad \text{for all } x \in X.$$

Let $x \in X$ be given. If $p \in E^*$ and $\beta_p(x) > 0$, then $x \in V(p) \subset U(p)$ so that $\sup_{z \in F(x)} Re\langle p, z\rangle < Re\langle p, x\rangle$; it follows that $\inf_{z \in F(x)} Re\langle p, x - z\rangle > 0$. Therefore for each $x \in X$,

$$(*) \qquad \begin{aligned} \inf_{z \in F(x)} Re\langle T(x), x - z\rangle &= \inf_{z \in F(x)} \sum_{p \in E^*} \beta_p(x) Re\langle p, x - z\rangle \\ &\geqslant \sum_{p \in E^*} \beta_p(x) \inf_{z \in F(x)} Re\langle p, x - z\rangle \\ &> 0. \end{aligned}$$

Now we shall show that T satisfies all hypotheses of Corollary 2. To show that T is continuous from the relative topology of X to the strong topology of E^*, let $(x_\alpha)_{\alpha \in \Gamma}$ be a net in X which converges to some $x_0 \in X$. Since $\{V(p) : p \in E^*\}$ is locally finite, there is an open neighbourhood U_0 of x_0 in X such that $\{p \in E^* : V(p) \cap U_0 \neq \phi\}$ is finite, so we let $\{p \in E^* : V(p) \cap U_0 \neq \phi\} = \{p_1, \ldots, p_n\}$. Let B be any non-empty bounded subset of E; then by Theorem 1.18 [**9**], $M = \max_{1 \leqslant i \leqslant n} \sup\{|p_i(x)| : x \in B\} < \infty$. Since each β_{p_i} is continuous, there exists $\alpha_1 \in \Gamma$ such that for each $\alpha \geqslant \alpha_1$,

$$\left|\beta_{p_i}(x_\alpha) - \beta_{p_i}(x_0)\right| < \frac{\varepsilon}{Mn} \quad \text{for all } i = 1, \ldots, n.$$

Also since (x_α) converges to x_0 and U_0 is an open neighbourhood of x_0, there exists $\alpha_2 \in \Gamma$ such that for each $\alpha \geqslant \alpha_2$, $x_\alpha \in U_0$. Let $\alpha_0 \geqslant \max\{\alpha_1, \alpha_2\}$. Then for each $\alpha \geqslant \alpha_0$, we have

$$\begin{aligned} &\sup_{z \in B} |\langle T(x_\alpha) - T(x_0), z\rangle| \\ &= \sup_{z \in B} \left| \sum_{p \in E^*} (\beta_p(x_\alpha) - \beta_p(x_0)) p(z) \right| \\ &= \sup_{z \in B} \left| \sum_{i=1}^{n} \left(\beta_{p_i}(x_\alpha) - \beta_{p_i}(x_0)\right) p_i(z) \right| \\ &\leqslant \sum_{i=1}^{n} \left|\left(\beta_{p_i}(x_\alpha) - \beta_{p_i}(x_0)\right)\right| \sup_{z \in B} |p_i(z)| \\ &< \sum_{i=1}^{n} \frac{\varepsilon}{Mn} M = \varepsilon, \end{aligned}$$

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and hence $(T(x_\alpha))$ converges to $T(x_0)$ in the strong topology of E^*.

Next suppose there exists $x_1 \in X \setminus D$ such that for some $y \in X$,

$$(**) \qquad Re\langle T(y), y - x_1\rangle = \sum_{p\in E^*} \beta_p(y)\, Re\langle p, y - x_1\rangle \ > \ 0.$$

If $\beta_p(y) > 0$, then $\inf_{z\in F(y)} Re\langle p, y - z\rangle \ > \ 0$ so that by (3), $Re\langle p, y - x_1\rangle \leqslant 0$, which contradicts $(**)$.

Therefore by Corollary 2, there exists $\widehat{x} \in X$ such that

$$(***) \qquad Re\langle T(\widehat{x}), \widehat{x} - y\rangle \ \leqslant \ 0 \quad \text{for all } y \in X.$$

By the assumption (2), $F(\widehat{x}) \cap cl\,(\widehat{x} + \cup_{\lambda>0}\lambda(X - \widehat{x})) \neq \phi$. Let $\widehat{y} \in F(\widehat{x}), (\lambda_\alpha)_{\alpha\in\Gamma}$ be a net in $(0, \infty)$ and $(u_\alpha)_{\alpha\in\Gamma}$ be a net in X such that $(\widehat{x} + \lambda_\alpha(u_\alpha - \widehat{x})) \to \widehat{y}$. Then we have

$$\begin{aligned} Re\langle T(\widehat{x}), \widehat{x} - \widehat{y}\rangle &= \lim_\alpha \ Re\langle T(\widehat{x}), \widehat{x} - (\widehat{x} + \lambda_\alpha(u_\alpha - \widehat{x}))\rangle \\ &= \lim_\alpha \lambda_\alpha \ Re\langle T(\widehat{x}), \widehat{x} - u_\alpha\rangle \\ &\leqslant 0 \qquad \text{by } (***). \end{aligned}$$

Hence $\inf_{z\in F(\widehat{x})} Re\langle T(\widehat{x}), \widehat{x} - z\rangle \ \leqslant \ 0$, which contradicts $(*)$. This completes the proof. $\square$

Theorem 2 generalises Theorem 2 of Halpern [**5**, p.88] in the following ways: (i) X need not be compact and (ii) F is upper hemicontinuous instead of upper semicontinuous.

The following is a reformulation of Proposition 3.1.21 of Aubin-Ekeland [**1**]:

Lemma 3. *Let X and Y be topological spaces and $\Phi : X \times Y \to R$ be a real-valued lower semicontinuous function on $X \times Y$ and $T : Y \to 2^X$ be upper semicontinuous at $y_0 \in Y$ and $T(y_0)$ is non-empty compact. Then a real-valued function $g : Y \to R$ defined by*

$$g(y) := \inf_{x\in T(y)} \Phi(x, y), \quad \textit{for all } y \in Y,$$

is lower semicontinuous at y_0.

Lemma 4. *Let E be a normed space, X be a non-empty subset of E and $T : X \to 2^{E^*}$ be an upper semicontinuous multimap such that each $T(x)$ is (norm-) compact. Then for each $y \in E$, the real-valued function $g_y : X \to R$ defined by*

$$g_y(x) := \inf_{w\in T(x)} Re\langle w, x - y\rangle, \quad \textit{for each } x \in X,$$

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is lower semicontinuous.

PROOF: Define $\Phi : X \times E^* \to R$ by

$$\Phi(x,w) = Re\langle w, x-y\rangle \quad \text{for each } (x,w) \in X \times E^*.$$

Let (x_n) be a sequence in X which converges to $x \in X$ and (w_n) be a sequence in E^* which converges to $w \in E^*$. Then we have

$$\begin{aligned} &|\Phi(x_n,w_n) - \Phi(x,w)| \\ &= |Re\langle w_n, x_n - y\rangle - Re\langle w, x-y\rangle| \\ &\leqslant |\langle w_n - w, x-y\rangle| + |\langle w_n, x_n - x\rangle| \\ &\leqslant \|w_n - w\| \, \|x-y\| + \|w_n\| \, \|x_n - x\| \to 0, \end{aligned}$$

since $\{\|w_n\| : n \geqslant 1\}$ is bounded.

Thus Φ is continuous. By Lemma 3, g_y is lower semicontinuous. This completes the proof. ∎

We remark that in the proof of Theorem 1, the condition "X is bounded" was never needed until Lemma 2 was quoted. In view of Lemma 4, the same proof of Theorem 1 gives the following

THEOREM 3. *Let X be a convex subset of a normed linear space E and D be a non-empty compact subset of X. Let $T : X \to 2^{E^*}$ be an upper semicontinuous multimap from the relative topology of X to the norm topology of E^* such that each $T(x)$ is (norm-) compact in E^*. Suppose further that for each $x \in X \setminus D$,*

$$(*) \qquad \inf_{w \in T(y)} Re\langle w, y-x\rangle \leqslant 0 \quad \text{for all } y \in X.$$

Then there exists a point $\widehat{x} \in X$ such that

$$\inf_{w \in T(\widehat{x})} Re\langle w, \widehat{x} - x\rangle \leqslant 0 \quad \text{for all } x \in X.$$

Furthermore, if $T(\widehat{x})$ is also convex, then there exists a point $\widehat{w} \in T(\widehat{x})$ such that

$$Re\langle \widehat{w}, \widehat{x} - x\rangle \leqslant 0 \quad \text{for all } x \in X.$$

For the same reason, in case E is a normed space, the condition that X be bounded in Corollary 2 can be deleted. As a result, we have the following norm-version of Theorem 2; recalling that every metric space is paracompact:

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Theorem 4. *Let X be a non-empty convex subset of a normed linear space E and D be a non-empty compact subset of X. Let $F : X \to 2^E$ be an upper hemicontinuous multimap satisfying the following:*

(1) *for each $x \in X$, $F(x)$ is non-empty closed convex,*

(2) *for each $x \in X$, $F(x) \cap cl(x + \cup_{\lambda > 0} \lambda(X - x)) \neq \phi$,*

(3) *for each $x \in X \setminus D$, $y \in X$ and $p \in E^*$, if $\inf\{Re\langle p, y - z\rangle : z \in F(y)\} > 0$, then $Re\langle p, y - x\rangle \leqslant 0$.*

Then there exists $\hat{x} \in X$ such that $\hat{x} \in F(\hat{x})$.

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