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AN EXISTENCE THEOREM FOR ORDINARY DIFFERENTIAL EQUATIONS IN BANACH SPACES

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The Cauchy problem $x' = f(t, x)$, $x(0) = x_0$, is considered in a non-reflexive Banach space E , where f is weakly continuous. A local existence theorem is proved using the measure of weak noncompactness.

Let E be a real Banach space and E^* its dual. Norms in both E and E^* are denoted by $\|\cdot\|$. Let $x_0 \in E$ and $a, b > 0$. We set $I = [0, a]$ and $D = \{x \in E : \|x-x_0\| \leq b\}$.

We consider the ordinary differential equation in E ,

$$x' = f(t, x) ,\quad x(0) = x_0 . \tag{1}$$

If $f \in C(I\times D, E)$, local existence theorems for (1) can be proved through compactness type conditions, such as f being α-Lipschitzian, where α denotes the measure of non-compactness (for example, [5]).

It is our purpose to examine the case that f is weakly continuous and ω-Lipschitzian, where ω is the measure of noncompactness in the weak topology (as introduced by De Blasi [4]). To be specific, given any bounded subset A of a Banach space X , we define

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$$\omega(A) = \inf\{\varepsilon > 0 : \text{there exists a weakly compact } C \subset X \text{ such that } A \subset C+\varepsilon S\} ,$$

where S is the unit closed ball in X. It follows from a well known characterization of reflexivity that $\omega(A) = 0$ for any bounded subset A of a reflexive Banach space X. Almost all of the following properties of ω were proved in [4] (for the proof of (10) we used the Weak Ascoli Theorem of [3]).

LEMMA 1. *If* A, B *are bounded subsets of the Banach space* X, *then*

(1) $A \subset B$ *implies* $\omega(A) \le \omega(B)$,

(2) $\omega(A) = \omega\left(\bar{A}^w\right)$, *where* $\bar{A}^w$ *denotes the weak closure of* A,

(3) $\omega(A) = 0$ *if and only if* $\bar{A}^w$ *is weakly compact,*

(4) $\omega(A \cup B) = \max\{\omega(A), \omega(B)\}$,

(5) $\omega(A) = \omega(\text{co } A)$,

(6) $\omega(\{x\}+A) = \omega(A)$, *for any* $x \in X$,

(7) $\omega(A+B) \le \omega(A) + \omega(B)$,

(8) $\omega(\lambda A) = |\lambda|\omega(A)$, *for all* $\lambda \in R$,

(9) $\omega\left(\bigcup_{0\le\lambda\le h} \lambda A\right) = h\omega(A)$.

If $M \subset C(I, E)$ *(strongly) bounded and equicontinuous, then*

(10) $\omega(M) = \sup\{\omega(M(t)) : t \in I\}$.

We now state our main result.

THEOREM 2. *Let* $f : I \times D \to E$ *be weakly continuous, (strongly) bounded with* $M = \sup\{\|f(t, x)\| : (t, x) \in I \times D\}$ *and* ω*-Lipschitzian, that is, there exists a* $k \ge 0$ *such that*

$$\omega\big(f(I \times B)\big) \le k\omega(B) , \quad B \subset D .$$

Then (1) *has a solution on* $J = [0, h]$, *where*

$$h \le \min\{a, b/M\} \text{ and } hk < 1 .$$

By a solution we mean a strongly continuous, once weakly differentiable function $x : J \to E$ satisfying (1) in J with x' denoting the weak derivative. Of course, such an x is strongly differentiable almost everywhere in J and satisfies (1) almost everywhere in J with x' denoting its strong derivative (see [7] and [9]).

If f is weakly continuous and $\overline{f(I \times D)}^{\omega}$ is weakly compact, then obviously f is ω-Lipschitzian. Furthermore, if f is weakly continuous and E is a reflexive Banach space, f is trivially ω-Lipschitzian. Thus the results of Kato [8] and Browder [2] (see also [5] and [10]) are special cases of Theorem 2. We state them as separate corollaries.

COROLLARY 3. *Let* $f : I \times D \to E$ *be weakly continuous and* $\overline{f(I \times D)}^{\omega}$ *be weakly compact. Then* (1) *has a solution on* $J = [0, h]$ *, where* $h = \min\{a, b/M\}$ *and* $M = \sup\{\|f(t, x)\| : (t, x) \in I \times D\}$.

COROLLARY 4. *Let* E *be reflexive and* $f : I \times D \to E$ *be weakly continuous and (strongly) bounded with*

$$M = \sup\{\|f(t, x)\| : (t, x) \in I \times D\} .$$

Then (1) *has a solution on* $J = [0, h]$ *, where* $h = \min\{a, b/M\}$.

Proof of Theorem 1. We are going to employ Euler's method of polygonal lines as it was developed by Szufla in [11] (also in [1]). The proof proceeds in three steps.

First Step. For any $A \subset D$, set

$$R(A) = x_0 + \overline{\bigcup_{0 \le \lambda \le h} \lambda \text{ co } f(J \times A)} ,$$

$$H = \bigcap_{A \in \Omega} A \text{ , where } \Omega = \{A \subseteq D : R(A) \subset A\} .$$

It can be easily shown that $\Omega \neq \emptyset$, $H \neq \emptyset$ and $H = R(H)$, and so H is closed. Moreover H is weakly compact, since

$$\omega(H) = \omega\big(R(H)\big) \le \omega\Big(\bigcup_{0 \le \lambda \le h} \lambda \text{ co } f(J \times H)\Big) = h\omega\big(\text{co } f(J \times H)\big) = h\omega\big(f(J \times H)\big)$$
$$\le hk\omega(H)$$

and $hk < 1$ implies that $\omega(H) = 0$; that is, $H = \overline{H}^{\omega}$ weakly compact.

Define

$$S = \{x : J \to H \text{ such that } x(0) = x_0,\ \|x(t)-x(t')\| \le M|t-t'|,\ t,\ t' \in J\} .$$

As $S(t) \subset H$ implies $\omega\big(S(t)\big) \le \omega(H) = 0$, for all $t \in J$, and since S

is bounded and equicontinuous

$$\omega(S) = \sup\{\omega(S(t)) : t \in J\} = 0 ;$$

that is, $\overline{S}^{w}$ is weakly compact.

Second step. We claim that for any $\varepsilon > 0$ there exists a $u \in S$ such that, for all $t \in J$,

$$\left\| u(t) - x_0 - \int_0^t f(s, u(s)) ds \right\| < \varepsilon t .$$

Indeed, the weak continuity of f implies that f is weakly uniformly continuous on the weakly compact $J \times H$; that is, for any $\varepsilon > 0$ there is a $\delta > 0$ such that, for all $x^* \in E^*$,

$$|(f(t, x)-f(s, y), x^*)| \le \varepsilon ,$$

whenever $t, s \in J$, $x, y \in H$ such that

$$|t-s| \le \delta , \quad \|x-y\| \le \delta .$$

We divide J into n subintervals

$$0 = t_0 < t_1 < \ldots < t_n = t_0 + h$$

so that

$$\max_{i=1,\ldots,n} |t_i-t_{i-1}| \le \min\{\delta, \delta/M\}$$

and define a mapping $u : J \to E$ as

$$u(t_0) = x_0 ,$$

$$u(t) = u(t_i) + (t-t_i) f(t_i, u(t_i)) ,$$

for

$$t \in [t_i, t_{i+1}] , \quad i = 0, 1, \ldots, n-1 .$$

Clearly, for $t \in [t_i, t_{i+1}]$,

$$u(t) = x_0 + (t_1-t_0) f(t_0, x_0) + \ldots + (t_i-t_{i-1}) f(t_{i-1}, u(t_{i-1})) + (t-t_i) f(t_i, u(t_i)) .$$

Now a direct computation shows that, for any $t, t' \in J$ and for all

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$x^* \in E^*$,

$$|(u(t)-u(t'), x^*)| \le M|t-t'|$$

and by a well known consequence of the Hahn-Banach Theorem it is implied that

$$\|u(t)-u(t')\| \le M|t-t'| .$$

Moreover it is not hard to see that $u(t) \in H$ for all $t \in J$. In fact, $u(t_0) = x_0 \in H$ and if $u(t) \in H$ for all $t \in [t_0, t_i]$, then for any $t \in [t_i, t_{i+1}]$, $u(t) \in x_0 + (t-t_0) \text{ co } f(J \times H) \subset R(H) = H$.

Finally, if $t \in [t_i, t_{i+1}]$, then we can find, for all $x^* \in E^*$,

$$\left|\left(u(t) - x_0 - \int_0^t f(s, u(s))ds, x^*\right)\right| < \varepsilon t$$

and again as a consequence of the Hahn-Banach Theorem

$$\left\|u(t) - x_0 - \int_0^t f(s, u(s))ds\right\| < \varepsilon t ,$$

which proves the claim.

Third step. Let $\{\varepsilon_n\}$ be a decreasing sequence of real numbers converging to 0 . By the second step, there exists a sequence $\{u_n\} \subset S$ such that, for all n ,

$$\left\|u_n(t) - x_0 - \int_0^t f(s, u_n(s))ds\right\| < \varepsilon_n t , \quad t \in J .$$

By the Eberlein-Šmulian Theorem ([6]), $\bar{S}^w$ weakly compact implies that S is weakly relatively sequentially compact, that is, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ converging weakly in $C(J, E)$ to some $u \in S$. Hence from the weak uniform continuity of f on the weakly compact $J \times H$ it follows that $f(t, u(t)) = w - \lim_{k\to\infty} f(t, u_{n_k}(t))$ uniformly on J . Thus at the limit we obtain

$$u(t) = x_0 + \int_0^t f(s, u(s))ds , \text{ for all } t \in J .$$

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Since f is weakly continuous, u is strongly continuous, once weakly continuously differentiable on J, where it satisfies (1) with u' denoting the weak derivative of u.

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