

ON THE ARC INDEX OF AN ADEQUATE LINK

CHAN-YOUNG PARK AND MYOUNGSOO SEO

In 1996, Cromwell and Nutt conjectured that $\alpha(L) - c(L) = 2$ for a link L if and only if L is alternating. In this paper we calculate that $\alpha(L) = c(L)$ for some non-alternating pretzel links L , define a new invariant $\rho(L)$ of adequate links L and show that for each non-negative integer n , there is a prime adequate knot K such that $\alpha(K) - c(K) = -2n$. We conjecture that $\alpha(L) - c(L) = 2\rho(L)$ for any adequate link L .

1. INTRODUCTION

An open book decomposition of the 3-sphere S^3 consists of open discs as pages and an unknotted circle as a binding circle. Every link L can be embedded in an open book with finitely many pages so that it meets each page in a simple arc. Such an embedding is called an *arc presentation* of L . The minimum number of pages required to present a given link L in this manner is a knot invariant and it is called the *arc index* of L which is denoted by $\alpha(L)$.

In 1994, the notion of an arc index was introduced by Birman and Menasco [3] in their study of the braid index of satellite links. In 1995, Cromwell [4] gave basic properties on the arc index of links and in 1996, Cromwell and Nutt [6] introduced a method to construct binding circles on some link diagrams and found bounds on the arc index and conjectured that $\alpha(L) \leq c(L) + 2$ for a link L and that equality holds if and only if L is alternating, where $c(L)$ is the crossing number. Recently, this conjecture was proved affirmatively by Bae and Park [1]. In 1998, Morton and Beltrami [9] showed that $\alpha(L) \geq \text{breadth}_a F_L(a, z) + 2$, using the Dubrovnik framed version of the Kauffman polynomial and the notion of a stacked tangle, where $F_L(a, z)$ is the Kauffman polynomial of L and $\text{breadth}_a F_L(a, z)$ is the a -spread of the polynomial $F_L(a, z)$. In 1997, Beltrami and Cromwell [2] proved that for a semi-alternating link L , $\text{breadth}_a F_L(a, z) \geq c(L) - 2$

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and in 1998 Cromwell [5] showed that for an alternating link L , $\text{breadth}_a F_L(a, z) \geq c(L)$, using the Kauffman polynomial of an adequate link.

In this paper we calculate that $\alpha(L) = c(L)$ for some non-alternating pretzel links L , define a new invariant of adequate links and show that for each non-negative integer n , there is a prime adequate knot K such that $\alpha(K) - c(K) = -2n$.

Unless otherwise stated, all links considered in this paper are assumed to be non-split. Note that all diagrams of non-split links are connected.

2. AN INVARIANT OF ADEQUATE LINKS

In this section, we define an invariant $\rho(L)$ of an adequate link L .

We start by introducing some notations. Let D be a connected link diagram in a plane. Then the diagram D divides the plane into several regions. Starting with the unbounded region, we can colour the regions either black or white so that neighbouring regions never have the same colour. We colour the unbounded region white. Let G be the connected planar graph associated with the diagram D in which the vertices of G correspond to the black regions of D and the edges of G correspond to the crossings of D . Assign a sign to each edge of G as depicted in Figure 2.1 so that the graph G is completely determined by the diagram D .

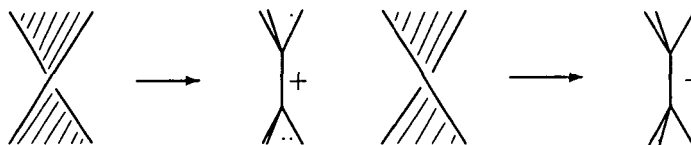


Figure 2.1

Given a graph G associated with a connected link diagram D , let G_+ be the subgraph of G consisting of all vertices of G and the positive edges of G and let \overline{G}_+ be the quotient graph obtained from G_+ by identifying those pairs of vertices which are ends of a path consisting of negative edges of G . The graphs G_- and \overline{G}_- are defined likewise. For any graph G , $p_0(G)$ denotes the number of components of G and $p_1(G)$ denotes the least number of edges of G whose deletion destroys all the circuits of G . Notice that $p_0(G)$ and $p_1(G)$ are the ranks of homology groups $H_0(G)$ and $H_1(G)$ respectively.

For a diagram D of an oriented link L with writhe $w(D)$, the regular isotopy invariant $\Lambda_D(a, z)$ is defined by $\Lambda_D(a, z) = a^{w(D)} F_L(a, z)$, where $F_L(a, z)$ is the Kauffman polynomial of L . Alternatively, the polynomial $\Lambda_D(a, z)$ can be defined by the following four axioms:

- (1) $\Lambda_D(a, z)$ is a regular isotopy invariant of diagrams of L ,
- (2) $\Lambda_D = 1$ if D is a simple closed curve,
- (3) $\Lambda_{D_1} = a\Lambda_{D_3}$ and $\Lambda_{D_2} = a^{-1}\Lambda_{D_3}$, where D_1, D_2 and D_3 are the same diagrams except within the neighbourhoods depicted as in Figure 2.2, and
- (4) $\Lambda_{D_+} + \Lambda_{D_-} = z(\Lambda_{D_0} + \Lambda_{D_\infty})$ where D_+, D_-, D_0 and D_∞ are the same except within the neighbourhoods depicted as in Figure 2.3.

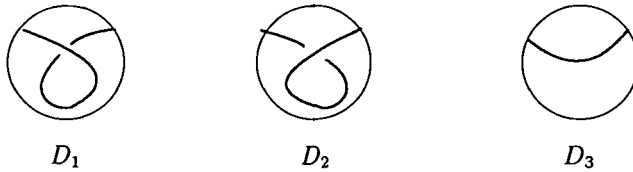


Figure 2.2

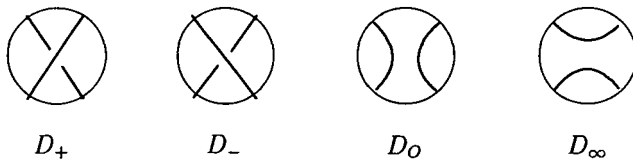


Figure 2.3

Write $\Lambda_D(a, z) = \sum u_{r,s} a^r z^s$ and let $c(D) = n$ where $c(D)$ denotes the crossing number of the link diagram D . Then $r + s \leq n$ and $-r + s \leq n$ if $u_{r,s}$ is nonzero [11]. Put $\phi_D^+(t) = \sum_i u_{i,n-i} t^i$ and $\phi_D^-(t) = \sum_i u_{-i,n-i} t^i$. Then there are no negative powers of t occurring in $\phi_D^+(t)$ or $\phi_D^-(t)$. In 1988, Thistlethwaite showed the following proposition by using a relation between the regular isotopy invariant $\Lambda_D(a, z)$ of a link diagram D and the Tutte polynomial of the associated graph G of D .

PROPOSITION 2.1. [12] *Let D be a connected link diagram with $c(D) \geq 1$. Then the following hold.*

- (1) $\phi_D^+(t) \neq 0$ if and only if G_+ has no isthmus and $\overline{G_-}$ has no loop.
- (2) $\phi_D^-(t) \neq 0$ if and only if G_- has no isthmus and $\overline{G_+}$ has no loop.
- (3) If $\phi_D^+(t) \neq 0$, then $\max \deg \phi_D^+(t) = p_1(G_+) + |V(\overline{G_-})| - 1$.
- (4) If $\phi_D^-(t) \neq 0$, then $\max \deg \phi_D^-(t) = p_1(G_-) + |V(\overline{G_+})| - 1$.
- (5) $\phi_D^+(t) \neq 0$ and $\phi_D^-(t) \neq 0$ if and only if D is adequate.

Here $|S|$ denotes the cardinality of a set S .

A link diagram D is said to be + adequate if D has at least one crossing, and for each crossing in the Kauffman state $s_+ D$ formed by nullifying each crossing according to rule P

in Figure 2.4, the two segments of the image part in rule P belong to different components of s_+D . If D satisfies the corresponding condition for the state s_-D obtained by nullifying each crossing according to rule N in Figure 2.4, then D is said to be $-$ *adequate*. A link diagram D is said to be *adequate* if it is both $+$ adequate and $-$ adequate and a link is called an *adequate link* if it has an adequate diagram.

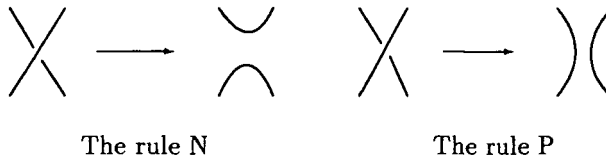


Figure 2.4

For a directed graph G , the subgraphs G_+ and G_- intersect only in vertices of G . Thus $|G_+ \cap G_-|$ is the number of vertices of $G_+ \cap G_-$.

LEMMA 2.2. *Let D be a connected adequate diagram of an adequate link L and let G be the associated graph of D . Then the following hold.*

- (1) $p_1(G_+) = p_0(G_+) + |E(G_+)| - |V(G_+)|$ and $p_1(G_-) = p_0(G_-) + |E(G_-)| - |V(G_-)|$.
- (2) $|V(\overline{G_+})| = |V(G_+)| - |G_+ \cap G_-| + p_0(G_-)$ and $|V(\overline{G_-})| = |V(G_-)| - |G_+ \cap G_-| + p_0(G_+)$.
- (3) $\max \deg \phi_D^+(t) + \max \deg \phi_D^-(t) = |E(G_+)| + |E(G_-)| + 2(p_0(G_+) + p_0(G_-) - |G_+ \cap G_-|) - 2$.

PROOF: (1) Assume that G_+ has n components G_1, G_2, \dots, G_n . Then $p_0(G_+) = n$. Since each G_i is planar, $p_1(G_i)$ equals the number of bounded regions of G_i . Thus $p_1(G_i) = |E(G_i)| - |V(G_i)| + 1$. Since $|E(G_+)| = \sum_{i=1}^n |E(G_i)|$ and $|V(G_+)| = \sum_{i=1}^n |V(G_i)|$, $p_1(G_+) = \sum_{i=1}^n p_1(G_i) = \sum_{i=1}^n |E(G_i)| - \sum_{i=1}^n |V(G_i)| + n = p_0(G_+) + |E(G_+)| - |V(G_+)|$. Similarly, $p_1(G_-) = p_0(G_-) + |E(G_-)| - |V(G_-)|$.

(2) Note that $\overline{G_+}$ is the quotient graph obtained from G_+ by identifying those pairs of vertices which are ends of a path consisting of negative edges of G , while $G_+ \cap G_-$ is a vertex graph and the vertices of $G_+ \cap G_-$ are ends of a path consisting of negative edges of G . Since the number of identified vertices in $\overline{G_+}$ is $p_0(G_-)$, $|V(\overline{G_+})| = |V(G_+)| - |G_+ \cap G_-| + p_0(G_-)$. Similarly, $|V(\overline{G_-})| = |V(G_-)| - |G_+ \cap G_-| + p_0(G_+)$.

(3) The equality follows from (1), (2) and [12]. □

Notice that there is a one-to-one correspondence between the crossings of L and the edges of G , which implies $c(D) = |E(G_+)| + |E(G_-)|$.

Now we put $\rho(D) = p_0(G_+) + p_0(G_-) - |G_+ \cap G_-|$ for a connected diagram D of a link L . Then we get the following

THEOREM 2.3. *Let D be an adequate diagram of an adequate link L and let G be the associated graph of D . Then*

$$\text{breadth}_a \Lambda_D(a, z) \geq c(D) + 2\rho(D) - 2.$$

PROOF: Since

$$\max \deg_a \Lambda_D(a, z) \geq \max \deg \phi_D^+(t) \quad \text{and} \quad \min \deg_a \Lambda_D(a, z) \leq -\max \deg \phi_D^-(t),$$

from Lemma 2.2 we have

$$\begin{aligned} \text{breadth}_a \Lambda_D(a, z) &= \max \deg_a \Lambda_D(a, z) - \min \deg_a \Lambda_D(a, z) \\ &\geq \max \deg \phi_D^+(t) + \max \deg \phi_D^-(t) \\ &= |E(G_+)| + |E(G_-)| \\ &\quad + 2(p_0(G_+) + p_0(G_-) - |G_+ \cap G_-|) - 2 \\ &= c(D) + 2\rho(D) - 2. \end{aligned}$$

□

Now for an adequate link L , we define $\rho(L) = \rho(D) = p_0(G_+) + p_0(G_-) - |G_+ \cap G_-|$ for some adequate diagram D of L .

THEOREM 2.4. *$\rho(L)$ is an invariant of an adequate link L .*

PROOF: Let L be an adequate link and let D_1, D_2 be adequate diagrams of L with writhes $w(D_1)$ and $w(D_2)$, respectively. Then $w(D_1) = w(D_2)$ [12]. Since $\Lambda_{D_1}(a, z) = a^{w(D_1)} F_L(a, z) = a^{w(D_2)} F_L(a, z) = \Lambda_{D_2}(a, z)$, $\phi_{D_1}^+(t) = \phi_{D_2}^+(t)$ and $\phi_{D_1}^-(t) = \phi_{D_2}^-(t)$. Thus $\max \deg \phi_{D_1}^+(t) + \max \deg \phi_{D_1}^-(t) = \max \deg \phi_{D_2}^+(t) + \max \deg \phi_{D_2}^-(t)$. Since $\max \deg \phi_{D_1}^+(t) + \max \deg \phi_{D_1}^-(t) = c(D_1) + 2\rho(D_1) - 2$, in the proof of Theorem 2.3, and $c(D_1) = c(D_2)$ [12], we have $\rho(D_1) = \rho(D_2)$. Thus $\rho(L)$ is well-defined and $\rho(L)$ is a link invariant for adequate links. □

Since $\text{breadth}_a \Lambda_D(a, z) = \text{breadth}_a F_D(a, z)$ and $\alpha(L) \geq \text{breadth}_a F_D(a, z) + 2$, we have the following

COROLLARY 2.5. *Let L be an adequate link. Then $\text{breadth}_a F_L(a, z) \geq c(L) + 2\rho(L) - 2$ and hence $\alpha(L) - c(L) \geq 2\rho(L)$.*

COROLLARY 2.6. *Let L be an alternating link. Then $\rho(L) = 1$.*

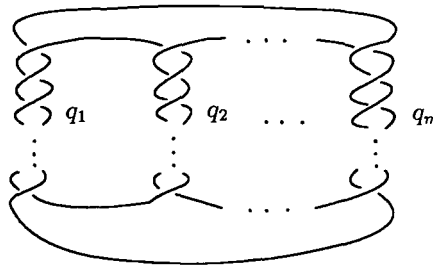
PROOF: Let D be an alternating diagram of an alternating link L and let G be the associated graph of D . Then D is an adequate diagram. Since D is alternating, the signs of all edges of G are the same. That is, one of the subgraphs G_+ and G_- is empty. Suppose G_+ is empty. Since D is connected and $G = G_-$, $p_0(G_-) = 1$ and $p_0(G_+) = 0$. Hence $\rho(L) = p_0(G_+) + p_0(G_-) - |G_+ \cap G_-| = 1$. □

In [1], Bae and Park showed that $\alpha(L) - c(L) = 2$ for an alternating link L . If L is an alternating link, then L is adequate, $\rho(L) = 1$ and $\alpha(L) - c(L) = 2\rho(L)$. We conjecture that $\alpha(L) - c(L) = 2\rho(L)$ for any adequate link L .

3. PRETZEL LINKS WITH ADEQUATE DIAGRAMS

For non-zero integers q_1, q_2, \dots, q_m , let $P(q_1, q_2, \dots, q_m)$, called a *pretzel link*, be the link with the regular diagram shown in Figure 3.1.

If $(q'_1, q'_2, \dots, q'_m)$ is a cyclic permutation of (q_1, q_2, \dots, q_m) , then $P(q'_1, q'_2, \dots, q'_m)$ and $P(q_1, q_2, \dots, q_m)$ are equivalent. If $q_i = \pm 1$, then $P(q_1, q_2, \dots, q_i, \dots, q_m)$ is equivalent to $P(q_i, q_1, q_2, \dots, \widehat{q}_i, \dots, q_m)$ where \widehat{q}_i means the deletion of q_i . So any pretzel link can be deformed into the form $P(\varepsilon, \dots, \varepsilon, p_1, \dots, p_n)$ where $\varepsilon = \pm 1$ and $|p_i| > 1$. We denote $P(\varepsilon, \dots, \varepsilon, p_1, \dots, p_n)$ by $P(k\varepsilon; p_1, \dots, p_n)$ where k is the number of ε 's. If $k > 0$ and $p_i = -2\varepsilon$, then $P(k\varepsilon; p_1, \dots, p_i, \dots, p_n)$ has the same link type as $P((k - 1)\varepsilon; p_1, \dots, -p_i, \dots, p_n)$. Hence we may assume that none of the p_i 's are equal to -2ε , when $k > 0$. Then we have that $P(k\varepsilon; p_1, \dots, p_n)$ is a knot if and only if either (1) $n \geq 0$ and $p_1, \dots, p_n, n + k$ are odd or (2) $n \geq 1$ and just one of the p_i 's is even. We call the knot $P(k\varepsilon; p_1, \dots, p_n)$ a *pretzel knot of odd type* in the case of (1) and a *pretzel knot of even type* in the case of (2).



$P(q_1, q_2, \dots, q_m)$

Figure 3.1

We denote $P(0\varepsilon; p_1, \dots, p_n)$ by $P(p_i; n)$. Then $P(p_i; n)$ is a pretzel link with $|p_i| > 1$, for all $i = 1, \dots, n$. Let $N_+P(p_i; n) = \{p_i \mid p_i > 0\}$ and $N_-P(p_i; n) = \{p_i \mid p_i < 0\}$ and let $n_+P(p_i; n)$ and $n_-P(p_i; n)$ denote the number of elements of $N_+P(p_i; n)$ and $N_-P(p_i; n)$ respectively. We can easily get the following

PROPOSITION 3.1. *Let $P(p_i; n)$ be a pretzel link. If $n_+P(p_i; n) = 0$ or $n_-P(p_i; n) = 0$, then we have*

- (1) $P(p_i; n)$ is an alternating link and hence is an adequate link,

- (2) $\rho(P(p_i; n)) = 1$ and
- (3) $\alpha(P(p_i; n)) = |p_1| + \dots + |p_n| + 2 = c(P(p_i; n)) + 2$.

Now we consider the special class of pretzel links such that $n_+P(p_i; n) \geq 2$ and $n_-P(p_i; n) \geq 2$. Let $D(p_i; n)$ denote the regular diagram of $P(p_i; n)$ in Figure 3.1. We call the diagram $D(p_i; n)$ the *pretzel diagram* of $P(p_i; n)$.

LEMMA 3.2. *Let $D(p_i; n)$ be the pretzel diagram of a pretzel link $P(p_i; n)$ and let G be the associated graph of $D(p_i; n)$. Assume that $n_+P(p_i; n) \geq 2$ and $n_-P(p_i; n) \geq 2$. Then we have*

- (1) G_+ and G_- have no isthmus,
- (2) $\overline{G_+}$ and $\overline{G_-}$ have no loop and
- (3) $\rho(D(p_i; n)) = 0$.

PROOF: Since $n_+P(p_i; n) \geq 2$ and $n_-P(p_i; n) \geq 2$, G_+ and G_- have no isthmus. And since $|p_i| \geq 2$, $i = 1, \dots, n$, $\overline{G_+}$ and $\overline{G_-}$ have no loop. Since G_+ and G_- are connected and $G_+ \cap G_-$ has two vertices, $p_0(G_+) = 1$, $p_0(G_-) = 1$ and $|G_+ \cap G_-| = 2$. Thus $\rho(D(p_i; n)) = p_0(G_+) + p_0(G_-) - |G_+ \cap G_-| = 0$. □

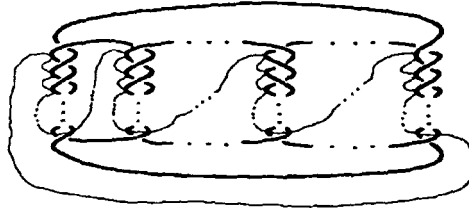


Figure 3.2

THEOREM 3.3. *Let $P(p_i; n)$ be a pretzel link. Assume that $n_+P(p_i; n) \geq 2$ and $n_-P(p_i; n) \geq 2$. Then we have*

- (1) $P(p_i; n)$ is an adequate link,
- (2) $\rho(P(p_i; n)) = 0$ and
- (3) $\alpha(P(p_i; n)) = c(P(p_i; n))$.

PROOF: (1) follows from Proposition 2.1 and Lemma 3.2 and (2) follows from Theorem 2.4 and Lemma 3.2.

(3) Since the graphs G_+ and G_- are connected, $p_0(G_+) = p_0(G_-) = 1$. Since G_+ and G_- intersect in two vertices, $|G_+ \cap G_-| = 2$. By Corollary 2.5, $\alpha(P(p_i; n)) \geq c(P(p_i; n)) + 2\rho(P(p_i; n)) = c(P(p_i; n))$. Since $n_+P(p_i; n) \neq 0$ and $n_-P(p_i; n) \neq 0$, the pretzel diagram $D(p_i; n)$ of the adequate link $P(p_i; n)$ has a binding circle which intersects $D(p_i; n)$ in $c(P(p_i; n))$ points as in Figure 3.2. Hence $\alpha(P(p_i; n)) = c(P(p_i; n))$. □

For a pretzel link $P(p_i; n)$ with $n_+P(p_i; n) = 1$ or $n_-P(p_i; n) = 1$, we know that

$\alpha(P(p_i; n)) \leq c(P(p_i; n))$. But we don't know yet whether $\alpha(P(p_i; n)) = c(P(p_i; n))$ or not.

4. PRIME TANGLES

By a *tangle* we mean a 2-string tangle, which is a pair (B, t) of a 3-ball B and two arcs t properly embedded in B . A tangle without local knots is said to be *locally trivial*. Two tangles (A, s) and (B, t) are said to be *equivalent* if there is a homeomorphism from (A, s) to (B, t) . If two arcs in a tangle (B, t) can be separated by a 2-disc properly embedded in a 3-ball, the tangle (B, t) is said to be *separable*. Otherwise, the tangle (B, t) is said to be *non-separable*. A locally trivial separable tangle is said to be *trivial*. A regular diagram of a trivial tangle, or its spatial realisation, is said to be *rational*. A tangle is called a *prime tangle* if it is locally trivial and non-separable. A *tangle sum* of two tangles (A, s) and (B, t) is defined by the link $(A, s) \cup_{\pi} (B, t)$ obtained by gluing them together via a homeomorphism $\pi : \partial(B, t) \rightarrow \partial(A, s)$. The following proposition is well-known.

PROPOSITION 4.1 [7, 8, 10]

- (1) A link obtained from two prime tangles by any tangle sum is prime.
- (2) Let (C, v) be a tangle and D a disc properly embedded in C such that D separates (C, v) into two tangles (A, s) and (B, t) and that $(\partial A \setminus D) \cap v, (\partial B \setminus D) \cap v, D \cap v$ consist of two points. Suppose that the two tangles (A, s) and (B, t) are prime. Then (C, v) is a prime tangle.

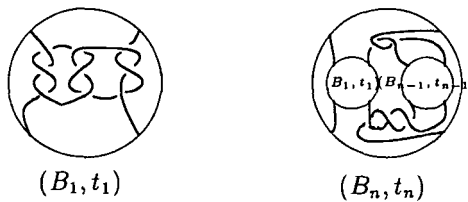


Figure 4.1

Let $(B_1, t_1), \dots, (B_n, t_n)$ be the tangles defined inductively as in Figure 4.1 and let P_n be the knot obtained from the tangle sum of two tangles (B_1, t_1) and (B_n, t_n) defined as in Figure 4.2.

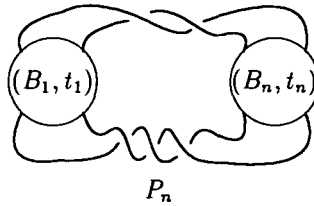


Figure 4.2

PROPOSITION 4.2. For each positive integer n , we have

- (1) The tangle (B_i, t_i) is a prime tangle, for $1 \leq i \leq n$ and
- (2) The knot P_n is a prime knot.

PROOF: (1) First, we claim that the tangle (B_1, t_1) is prime. Since each of two arcs in the tangle (B_1, t_1) is unknotted, the tangle (B_1, t_1) is locally trivial. If it is separable, then it must be equivalent to the trivial tangle. Since a trivial tangle diagram induces a two bridge knot, the tangle (B_1, t_1) induces a two bridge knot. But the tangle (B_1, t_1) induces the pretzel knot $P(3, -3, 3)$ which is not a two bridge knot. Thus the tangle (B_1, t_1) is non-separable. Therefore it is prime. It follows from Proposition 4.1 (2) that the tangles (B_i, t_i) , $i = 1, \dots, n$, are prime tangles.

(2) Since the knot P_n is a tangle sum of two tangles (B_1, t_1) and (B_n, t_n) , this follows from Proposition 4.1 (1). \square

5. ARC INDEXES AND ADEQUATE KNOTS

We have seen that there is an adequate knot K such that $\alpha(K) = c(K)$ in Theorem 3.3. In 1995, Cromwell [4] proved that $\alpha(L_1 \# L_2) = \alpha(L_1) + \alpha(L_2) - 2$. From this we have the following

PROPOSITION 5.1. For each non-negative integer n , there is an adequate knot K_n such that $\alpha(K_n) - c(K_n) = -2n$.

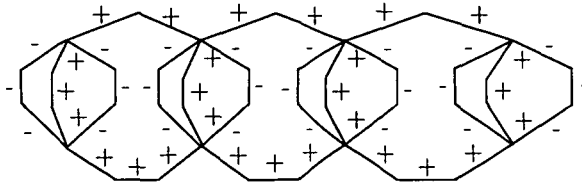
PROOF: Let K_0 be the pretzel knot $P(3, -3, 3, -2)$ and let K_n be the connected sum of $n+1$ copies of K . Then $\alpha(K_0) = c(K_0)$. Since a connected sum of adequate knots is again an adequate knot, K_n is an adequate knot. Thus $c(K_n) = (n+1)c(K_0)$. Since $\alpha(L_1 \# L_2) = \alpha(L_1) + \alpha(L_2) - 2$ for any links L_1 and L_2 , $\alpha(K_n) = (n+1)\alpha(K_0) - 2n = c(K_n) - 2n$. \square

For each non-negative integer n , we are interested in finding an adequate and prime knot K_n such that $\alpha(K_n) = c(K_n) - 2n$ and $\rho(K_n) = -n$. In the case $n = 3$, we have the following example whose generalisation is Theorem 5.3.

EXAMPLE 5.2. Let D_3 be the knot diagram of the knot P_3 introduced in Section 4 as depicted in Figure 5.1. Let G be the associated graph of the knot diagram D_3 as in Figure 5.2. Since G_+ and G_- have no isthmus and $\overline{G_+}$ and $\overline{G_-}$ have no loop, D_3 is an adequate diagram. Thus $p_0(G_+) = 1$, $p_0(G_-) = 4$ and $|G_+ \cap G_-| = 8$. Hence $\rho(D_3) = -3$. Since $c(D_3) = 51$, $\alpha(D_3) \geq c(D_3) + 2\rho(D_3) = 51 - 6 = 45$. The knot diagram D_3 has a binding circle which intersects D_3 in 45 points as in Figure 5.3 and hence $\alpha(D_3) = 45 = c(D_3) - 2 \cdot 3$.



Figure 5.1



D_3

Figure 5.2

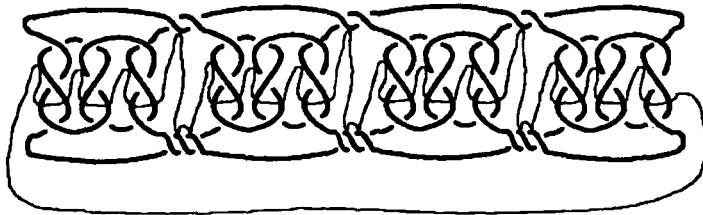


Figure 5.3

THEOREM 5.3. For each positive integer n , the prime knot P_n is an adequate knot such that $\alpha(P_n) - c(P_n) = -2n = 2\rho(P_n)$.

PROOF: The proof proceeds in exactly the same way as in Example 5.2. Let D_n be the knot diagram of the knot P_n introduced in Section 4 and let G be the associated graph of the knot diagram D_n . Since G_+ and G_- have no isthmus and $\overline{G_+}$ and $\overline{G_-}$ have no loop, D_n is an adequate diagram. Then $p_0(G_+) = 1$, $p_0(G_-) = n + 1$ and $|G_+ \cap G_-| = 2n + 2$. Hence $\rho(P_n) = \rho(D_n) = 1 + (n + 1) - (2n + 2) = -n$. Since $c(D_n) = 9(n + 1) + 5n = 14n + 9$ and the knot diagram D_n has a binding circle which intersects D_n in $12n + 9$ points, $\alpha(D_n) = 12n + 9 = (14n + 9) + 2(-n) = c(D_n) + 2\rho(D_n)$. Hence $\alpha(P_n) = c(P_n) - 2n$. □

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Department of Mathematics
College of Natural Sciences
Kyungpook National University
Taegu 702-701
South Korea