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Semistable reduction for overconvergent F-isocrystals I: Unipotence and logarithmic extensions

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Dedicated to Pierre Berthelot

Abstract

Let X be a smooth variety over a field k of characteristic $p > 0$, and let $\mathcal{E}$ be an overconvergent isocrystal on X. We establish a criterion for the existence of a 'canonical logarithmic extension' of $\mathcal{E}$ to a smooth compactification $\overline{X}$ of X whose complement is a strict normal crossings divisor. We also obtain some related results, including a form of Zariski–Nagata purity for isocrystals.

1. Introduction

This paper is intended as the first in a series in which we pursue a 'semistable reduction' theorem for overconvergent F-isocrystals, a class of p-adic analytic objects associated to schemes of finite type over a field of characteristic $p > 0$. Such a theorem would have consequences for the theory of rigid cohomology, in which overconvergent F-isocrystals play the role of coefficient objects of locally constant rank. In this introduction, we give a high-level description of a complex analytic model situation and the p-adic situation that imitates it, a bit about intended applications, and the structure of the paper. For a more detailed description of the questions we will be considering in subsequent papers, see § 7.

1.1 An analogy: complex local systems

Let $X \hookrightarrow \overline{X}$ be an open immersion of smooth varieties over $\mathbb{C}$, with $\overline{X}$ proper and $Z = \overline{X} \setminus X$ a strict normal crossings divisor. (Here and throughout, 'variety' will be used as shorthand for 'reduced, separated scheme of finite type' over some field.) A ∇-*module* on the complex analytic space X^{an} consists of a coherent locally free sheaf $\mathcal{E}$ of $\mathcal{O}_{X^{\mathrm{an}}}$-modules (or equivalently, a holomorphic vector bundle) equipped with an integrable connection. The integrability condition means that $\mathcal{E}$ admits a basis of horizontal sections on any contractible open subset; these fit together to form a local system of finite-dimensional $\mathbb{C}$-vector spaces on X^{an}. (In fact, the categories of ∇-modules and of local systems of finite-dimensional $\mathbb{C}$-vector spaces are equivalent, by the easy part of the Riemann–Hilbert correspondence.)

Suppose that X is connected, so that $\mathcal{E}$ has some rank n everywhere. Associated to $\mathcal{E}$ (or rather, from its associated local system) is a monodromy representation $\rho : \pi_1(X^{\mathrm{an}}) \to \mathrm{GL}_n(\mathbb{C})$ of the (topological) fundamental group of X^{an}. Specifically, given a pointed loop, one analytically continues a basis of local horizontal sections along the loop, and compares the basis before and after this parallel transport.

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Given a component D of Z, one obtains from ρ a new representation by restriction to the subgroup of $\pi_1(X^{\mathrm{an}})$ generated by some loop winding once around D (with the correct orientation). Of course this subgroup depends on the choice of the loop, but that choice acts on the loop by a conjugation in $\pi_1(X^{\mathrm{an}})$, and so does not alter the isomorphism class of the restricted representation. That restriction is called the *local monodromy representation* associated to D.

The local monodromy representation measures the 'badness' of the singularities of the connection along D. For instance, if the connection extends without singularities across D, the local monodromy representation is a trivial representation. More interestingly, by a theorem of Deligne [Del70, Proposition II.5.2], the local monodromy representation is unipotent (i.e., its semisimplification is a direct sum of trivial representations) if and only if the ∇-module extends to a log-∇-module with logarithmic singularities and nilpotent residues along D; such an extension is unique if it exists. (This uniqueness relies crucially on the nilpotent residue condition; otherwise many distinct extensions are possible.) In particular, the existence of such a 'canonical logarithmic extension' (the 'prolongement canonique' of [Del70]) is determined by a codimension 1 criterion, so its existence on $\overline{X}$ minus a codimension 2 subscheme implies its existence on $\overline{X}$ [Del70, Corollaire II.5.8].

For local systems of 'algebro-geometric origin', e.g., the ith relative Betti cohomology of a smooth proper morphism to X, one typically obtains a canonical logarithmic extension after pulling back along a suitable finite cover of $\overline{X}$. This can be shown 'extrinsically', using semistable reduction of varieties, but a more intrinsic approach involves recognizing such local systems as analytic objects equipped with extra data, namely variations of Hodge structures. (At this point our discussion, being purely of motivational nature, will turn unabashedly cursory; see [Gri70] for a more comprehensive overview.)

A *polarized variation of Hodge structures* on X consists of a local system of finitely generated $\mathbb{Z}$-modules on X^{an}, plus some additional Hodge-theoretic data which we will not describe here, save to mention the principal example (arising from a theorem of Griffiths): the ith cohomology of a family of smooth projective complex analytic varieties. A basic fact about polarized variations of Hodge structures is the *monodromy theorem*, due in this form to Borel [Sch73, Lemma 4.5]: the local monodromy representation associated to any component of Z is *quasi-unipotent*, i.e., becomes unipotent upon further restriction to a subgroup of finite index.

From the monodromy theorem, one easily deduces the following. Given a ∇-module $\mathcal{E}$ on X^{an} whose associated local system can be obtained from a polarized variation of Hodge structures (by tensoring over $\mathbb{Z}$ with $\mathbb{C}$), for any closed point x of $\overline{X}$, one can find an open neighborhood U of x in $\overline{X}$ and a finite cover $f : V \to U$ such that V is étale over $U \cap X$, V is smooth, $f^{-1}(U \cap Z)$ is a strict normal crossings divisor on V, and $f^*\mathcal{E}$ extends to a log-∇-module on V with logarithmic singularities and nilpotent residues along $f^{-1}(U \cap Z)$.

It is a bit less clear how to patch things together globally without further analysis of the local situations, but using resolution of singularities, one can at least assert that there is a proper, dominant, generically finite morphism $f : \overline{Y} \to \overline{X}$ with $\overline{Y}$ smooth and $f^{-1}(Z)$ a strict normal crossings divisor, such that $f^*\mathcal{E}$ extends to a log-∇-module everywhere on $\overline{Y}$, with logarithmic singularities and nilpotent residues along $f^{-1}(Z)$. We summarize this situation by saying that $\mathcal{E}$ 'admits semistable reduction'. (The reason for this terminological choice is that when $\mathcal{E}$ comes from the cohomology of a family of varieties, one is guaranteed to have the desired property if the family pulls back to a semistable family over $\overline{Y}$.)

1.2 Extension of overconvergent isocrystals

We now consider a p-adic analogue of the situation of the previous subsection. This will be appropriately vague for an introduction; see § 7 for a summary in more precise language.

Let $X \hookrightarrow \overline{X}$ be an open immersion of smooth k-varieties, for k a field of characteristic $p > 0$, such that $Z = \overline{X} \setminus X$ is a strict normal crossings divisor. Let $\mathcal{E}$ be an isocrystal on X which is overconvergent along Z; this is a positive characteristic analogue of a ∇-module with some additional convergence conditions, constructed using p-adic rigid analytic geometry. Although it is not so easy to define a p-adic local monodromy group, one can at least give meaning to the assertion that '$\mathcal{E}$ has constant/unipotent local monodromy along Z'. We show (Theorem 6.4.5) that again $\mathcal{E}$ has unipotent local monodromy if and only if $\mathcal{E}$ admits a 'canonical logarithmic extension' to $\overline{X}$; that extension will be a convergent log-isocrystal in the sense of Shiho [Shi00, Shi02]. This in particular implies a form of Zariski–Nagata purity for isocrystals on smooth varieties.

Continuing the analogy, one can then ask whether one can associate to $\mathcal{E}$ of 'algebro-geometric origin' a certain global analytic object that will ensure that $\mathcal{E}$ admits a canonical logarithmic extension. The object that provides this control is a Frobenius structure: the analogue of the monodromy theorem is that the semisimplified local monodromy representations, being equipped with Frobenius structures, necessarily have finite image when restricted to an inertia subgroup. This is 'Crew's conjecture', now the p-adic local monodromy theorem of André [And02], Mebkhout [Meb02], and the present author [Ked04a].

Thus one expects that one can pull back $\mathcal{E}$ along a generically finite cover and get a canonical logarithmic extension. Note that this is not at all a trivial consequence of Theorem 6.4.5, despite that the fact of an isocrystal having unipotent monodromy can be checked in codimension 1! The problem arises because of wild ramification in positive characteristic: the analogue of the local construction in the complex case produces a singular $\overline{Y}$, to which Theorem 6.4.5 does not (and should not) apply. Resolving the resulting singularities (using an alteration in the manner of de Jong [DJ96]) produces new components whose local monodromy is not *a priori* under control. We describe the situation in more detail in § 7.

It should also be noted that the failure to obtain a canonical logarithmic extension on a finite (not just generically finite) cover is also not merely an artifact of the proof technique. One can exhibit examples of overconvergent isocrystals with Frobenius structure that cannot admit a canonical logarithmic extension after pullback along *any* finite cover; obstructions to this can be exhibited using the Newton polygons of the Frobenius action at various points. We plan to include an example of this in a subsequent paper.

1.3 Applications in rigid cohomology

In the theory of algebraic de Rham cohomology of varieties over a field of characteristic 0, the ability to 'compactify coefficients' makes it possible to prove various finiteness theorems by passing to smooth proper varieties. With a semistable reduction theorem for overconvergent F-isocrystals, one would hope to obtain analogous results in rigid cohomology; we now describe some possible such results.

Shiho [Shi02] has shown that semistable reduction implies the finite dimensionality of rigid cohomology with coefficients in an overconvergent F-isocrystal. Although one can also prove this more directly [Ked06], Shiho's construction may yield insight into the relative setting, where a direct argument seems more difficult.

Nakkajima [Nak04] has shown that semistable reduction implies the existence of complexes, constructed from log-crystalline cohomology, that compute the rigid cohomology of an arbitrary scheme of finite type (not even separated!) over k. These complexes may shed some light on the rigid weight-monodromy conjecture of Mokrane [Mok93].

Berthelot (private communication) has suggested that semistable reduction may be of value in the theory of arithmetic $\mathcal{D}$-modules. In particular, one currently does not know that the restriction of a holonomic $\mathcal{D}$-module to a closed subscheme is again holonomic; possibly this can be proved by 'approximating' the $\mathcal{D}$-module with overconvergent log-isocrystals. Ongoing work of Caro may provide a workaround for this problem, but we still expect semistable reduction to intervene ultimately.

Some of our side results may have their own relevance. For instance, the fact that a convergent isocrystal admits an overconvergent structure if the same is true after restriction to an open dense subset (Proposition 5.3.7) can be used to prove some results in the direction of Berthelot's conjecture [Brt86] on overconvergence of direct images of smooth proper morphisms. The point is that the direct images one is trying to construct exist in the convergent category by arguments of Ogus [Ogu84], and can be shown to exist 'generically' (on an open dense subset of the original base) in the overconvergent category using the techniques of [Ked06]. We intend to amplify these comments elsewhere.

1.4 Structure of the paper

We conclude this introduction with a summary of the structure of the paper. Note that (unlike the rest of this introduction) these comments only summarize the structure of the present paper; the structures of subsequent papers in this series will be described therein.

In § 2, we review some notions from rigid analytic geometry. In particular, we introduce modules with connection and log-connection, as well as Berthelot's notions of tubes and strict neighborhoods, and define overconvergent isocrystals following Berthelot's treatment in [Brt96].

In § 3, we analyze modules with connection over the product of a polyannulus with another space. This amounts to recalling some results from the local theory of p-adic differential equations. In particular, we define the notion of a unipotent ∇-module in this context and analyze its relationship with log-connections.

In § 4, we specify what we mean for an isocrystal on a smooth variety to have 'constant monodromy' or 'unipotent monodromy' along the boundary in some partial compactification.

In § 5, we state several results to the effect that the obstruction to extending an isocrystal over a boundary subvariety is precisely its failure to have constant monodromy along the subvariety. Although this sort of result is not really needed for semistable reduction, such assertions may be of independent interest.

In § 6, we state a result to the effect that the obstruction to the existence of a canonical logarithmic extension of an isocrystal is precisely its failure to have unipotent monodromy. Our canonical logarithmic extensions will be convergent log-isocrystals in the sense of Shiho [Shi02], and some effort is expended to relate our construction to his.

In § 7, we conclude by articulating the questions we intend to address in subsequent papers in this series, fleshing out the discussion initiated in this introduction.

2. Rigid analytic setup

In this section, we recall briefly the construction of overconvergent isocrystals on schemes over a field of positive characteristic. Our reference for notation and terminology in rigid analytic geometry is [BGR84]; see also [FvP04]. Also see [Brt96, ch. 1] for more details on the construction of isocrystals.

2.1 Initial notation

We first set some notation and terminology conventions, which will hold throughout the paper unless otherwise specified.

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Convention 2.1.1. Throughout this paper, let k be an arbitrary field of characteristic $p > 0$. When we speak of a 'k-variety', we will mean a reduced separated (but not necessarily irreducible) scheme of finite type over k; any additional modifiers are to be passed through to the underlying scheme (e.g., connected, irreducible) or to the structural morphism (e.g., smooth, affine, proper) as appropriate.

Convention 2.1.2. Until further notice (specifically, until § 6), let K be a field of characteristic 0 complete with respect to a nonarchimedean absolute value $|\cdot| : K^* \to \mathbb{R}^+$, with residue field k. Let Γ^* denote the divisible closure of the image of $|\cdot|$. Let $\mathfrak{o} = \mathfrak{o}_K$ denote the ring of integers of K. Any norm or seminorm on a K-algebra will be assumed to be compatible with the given norm on K; in particular, any finite extension of K carries a unique such norm, which we also denote by $|\cdot|$.

Remark 2.1.3. The fact that K will start the paper being any field complete for a nonarchimedean absolute value, and end the paper being discretely valued, reflects a certain ambivalence in the p-adic cohomological community. It seems that if one's perspective is informed by crystals or formal-scheme constructions (like Monsky–Washnitzer cohomology), discretely valued fields are the ones that arise most naturally, whereas if one's perspective is informed by p-adic analysis, then fields like $\mathbb{C}_p$ and its spherical completion also arise naturally. We have decided to split the difference, by carrying along a general K as far as possible, namely until we begin to invoke Shiho's papers [Shi00, Shi02].

Convention 2.1.4. When forming an i-fold product or fibered product in any category, we use $\pi_1, \ldots, \pi_i$ to denote the projections onto the respective factors.

Convention 2.1.5. When any sort of norm is applied to a matrix, we mean this to be the maximum of the values of the norm on the individual elements of the matrix, and not any sort of spectral/operator norm.

2.2 Tubes, frames, and strict neighborhoods

We now set up some of the rigid geometry needed to construct isocrystals, in order to fix notation.

We start with Raynaud's notion of the 'generic fibre' of an affine formal scheme [Brt96, 0.2.2]. This construction provides the ambient rigid spaces inside which we will work.

Definition 2.2.1. Let $P = \operatorname{Spf} A$ be an affine formal scheme of finite type over $\mathfrak{o}_K$, and put $A_K = A \otimes_{\mathfrak{o}_K} K$ and $P_K = \operatorname{Max} A_K$. Then P_K is an affinoid space, called the *generic fibre* of P. The points of P_K correspond to quotients of A which are integral and finite flat over $\mathfrak{o}_K$; under this interpretation, we get a map $\mathrm{sp} : P_K \to P_k$ by tensoring these quotients with k. This is called the *specialization map*. For any subvariety U of P_k, define the *tube of* U *(within* P_K*)*, denoted $]U[_P$, as the inverse image $\mathrm{sp}^{-1}(U)$ within P_K; we drop the subscript P in if it is to be understood.

Remark 2.2.2. One could relax the restriction that P be affine; see Remark 2.2.6 for more discussion.

Definition 2.2.3. Suppose X is a closed subscheme of P_k cut out by the reductions of $g_1, \ldots, g_n \in \Gamma(P, \mathcal{O}_P)$. Then

$$]X[_P = \{x \in P_K : |g_i(x)| < 1 \ (i = 1, \ldots, n)\}.$$

As in [Brt96, 1.1.8], for $\lambda \in (0,1) \cap \Gamma^*$, put

$$[X]_{P\lambda} = \{x \in P_K : |g_i(x)| \leqslant \lambda \ (i = 1, \ldots, n)\}$$

and

$$]X[_{P\lambda} = \{x \in P_K : |g_i(x)| < \lambda \ (i = 1, \ldots, n)\};$$

then each $[X]_{P\lambda}$ is rational, and each of the collections $\{[X]_{P\lambda}\}$ and $\{]X[_{P\lambda}\}$, for λ running over a sequence in $(0,1) \cap \Gamma^*$ converging to 1, forms an admissible covering of $]X[_P$ [Brt96, Proposition 1.1.9]. Again, we drop the subscript P if it is to be understood.

We now specify a geometric setup we will be using repeatedly; the terminology is not standard, but will be rather convenient for us.

Definition 2.2.4. A *frame* (or *affine frame*) is a tuple (X, Y, P, i, j), in which:

(i) P is an affine formal scheme of finite type over $\mathfrak{o}_K$;

(ii) Y is a k-variety and $i : Y \hookrightarrow P_k$ is a closed immersion;

(iii) X is a k-variety and $j : X \hookrightarrow Y$ is an open immersion;

(iv) P is smooth over $\mathfrak{o}_K$ in a neighborhood of X.

We say that the frame *encloses* the variety Y and/or the pair (X, Y). Given two frames $F = (X, Y, P, i, j)$ and $F' = (X', Y', P', i', j')$, a *morphism* $F' \to F$ is a diagram of the form

$$\begin{tikzcd} X' \arrow[r, hook, "j'"] \arrow[d, "w"'] & Y' \arrow[r, hook, "i'"] \arrow[d, "v"'] & P' \arrow[d, "u"'] \\ X \arrow[r, hook, "j"] & Y \arrow[r, hook, "i"] & P \end{tikzcd} \tag{2.2.5}$$

in which u is smooth in a neighborhood of X. Define the *product frame* $F \times F'$ as the frame $(X \times_k X', Y \times_k Y', P \times_{\mathfrak{o}_K} P', i \times i', j \times j')$; it is equipped with the obvious projection morphisms $\pi_1 : F \times F' \to F$ and $\pi_2 : F \times F' \to F'$.

Remark 2.2.6. Berthelot considers also the analogous situation in which P is not necessarily affine. However, since our work here is entirely 'pre-cohomological', allowing nonaffine P would not really add any generality, since one can always cover such a P with affines, work locally, and keep track of glueing maps. (This is basically what Definition 2.6.4 does.) In fact, one is forced to do this anyway in order to deal with varieties which do not lift to characteristic 0. Thus, for simplicity, we have decided to use only affine frames throughout. (By contrast, when one passes to cohomological considerations, it is necessary to consider the case where P is proper in order to invoke Kiehl's finiteness theorem.)

We next introduce strict neighborhoods, following [Brt96, 1.2].

Definition 2.2.7. Let (X, Y, P, i, j) be a frame. An admissible open subset V of $]Y[_P$ containing $]X[_P$ is a *strict neighborhood* of $]X[_P$ within $]Y[_P$ if the covering $\{V,]Y \setminus X[_P\}$ of $]Y[_P$ is admissible. (Note that the covering $\{]X[_P,]Y \setminus X[_P\}$ of $]Y[_P$ is typically not admissible.)

To test locally whether an open set is a strict neighborhood, one may use the following lemma, which is the variant of [Brt96, Proposition 1.2.2] described in [Brt96, Remarques 1.2.3(iii)].

Lemma 2.2.8. *Let (X, Y, P, i, j) be a frame and choose $g_1, \dots, g_n \in \Gamma(P, \mathcal{O}_P)$ whose reductions cut out $Y \setminus X$ within Y. For $\lambda \in (0, 1) \cap \Gamma^*$, put*

$$\begin{aligned} U_\lambda &=]Y[_P \setminus]Y \setminus V(g_1, \dots, g_n)[_{P\lambda} \\ &= \left\{ y \in\ ]Y[_P : \max_i \{|g_i(y)|\} \geqslant \lambda \right\} \end{aligned}$$

as in Definition 2.2.3. Let V be an admissible open subset of $]Y[_P$ containing $]X[_P$. Then V is a strict neighborhood of $]X[_P$ if and only if for any admissible affinoid $W \subseteq\ ]Y[_P$, there exists $\lambda_0 \in (0, 1) \cap \Gamma^*$ such that for all $\lambda \in [\lambda_0, 1) \cap \Gamma^*$, $U_\lambda \cap W \subseteq V$.

A key tool in the construction and study of isocrystals is Berthelot's 'strong fibration theorem' [Brt96, Théorème 1.3.7], which constructs analogues of tubular neighborhoods (of a closed subset) in ordinary topology.

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Proposition 2.2.9 (Strong fibration theorem). *Let $F' \to F$ be a morphism of frames as in (2.2.5) with $X' = X$ and $w = \mathrm{id}_X$. Let $\overline{X}$ be the closure of X in $P'_Y = P' \times_P Y$, and suppose that $\overline{X} \to Y$ is proper (e.g., if $P' \to P$ is proper). Let $\mathcal{I}' \subset \mathcal{O}_{P'}$ be the defining ideal of Y' within P', and let $\overline{\mathcal{I}}'$ be the defining ideal of Y' within P'_Y; suppose further that there exist sections $t_1, \dots, t_d \in \Gamma(P', \mathcal{I}')$ whose reductions induce a basis of the conormal sheaf $\overline{\mathcal{I}}'/(\overline{\mathcal{I}}')^2$ on X. Put*

$$P'' = P \times_{\mathfrak{o}_K} \widehat{\mathbb{A}^d_{\mathfrak{o}_K}} = \operatorname{Spf} \mathcal{O}_{P'}\langle t_1, \dots, t_d \rangle;$$

then the morphism $\phi : P' \to P''$ defined by $t_1, \dots, t_d$ is an isomorphism on X, and induces an isomorphism of some strict neighborhood of $]X[_{P'}$ within $]Y[_{P'}$ with some strict neighborhood of $]X[_{P''}$ within $]Y[_{P''}$.

Remark 2.2.10. The strong fibration theorem is crucial to the independence under pullback properties of isocrystals (Propositions 2.6.1 and 2.6.2). It also intervenes in the definition of constant/unipotent monodromy (§ 4.3).

2.3 Connections and log-connections

Convention 2.3.1. When some construction is made relative to a morphism $f : V \to W$ of rigid spaces, in case we omit mention of this morphism we take it to be the structure morphism $f : V \to \operatorname{Max} K$ of a rigid space V over K.

Definition 2.3.2. For A an affinoid algebra, let $\Omega^1_{A/K}$ denote the module of continuous differentials of A over K, as in [FvP04, Theorem 3.6.1]. Likewise, for X a rigid space, let $\Omega^1_{X/K}$ denote the sheaf of continuous differentials on X over K; this sheaf is coherent, and is locally free if X is smooth over K (see [FvP04, Theorem 3.6.3]). If $f : V \to W$ is a morphism of rigid spaces, we define $\Omega^1_{V/W} = \Omega^1_{V/K}/f^*\Omega^1_{W/K}$. Write $\Omega^i_{V/W} = \bigwedge^i_{\mathcal{O}_V} \Omega^1_{V/W}$.

Remark 2.3.3. Note that if V is smooth over K, then for any point $x \in V$, we can find an affinoid subdomain W of V containing x and some $t_1, \dots, t_n \in \mathcal{O}(W)$ such that $dt_1, \dots, dt_n$ freely generate $\Omega^1_{V/K}$ on W. If x is a K-rational point, we can further ensure that $t_1, \dots, t_n$ all vanish at x. For such a choice, we obtain an étale map $W \to \mathbb{A}^n_K$ defined by $t_1, \dots, t_n$, sending x to the origin; this map can be shown (as is done in the proof of [GK04, Proposition 1.3]) to induce an isomorphism of an affinoid subdomain of V containing x with some affinoid subdomain of $\mathbb{A}^n_K$ containing the origin. In particular, we obtain a cofinal set of affinoid subdomains of V containing x of the form

$$\{y \in V : |t_i(y)| \leqslant \epsilon \ (i = 1, \dots, n)\}$$

for $\epsilon \in (0, +\infty) \cap \Gamma^*$ sufficiently small.

Definition 2.3.4. Let $f : V \to W$ be a morphism of rigid spaces. A ∇-*module* on V, relative to W, is a coherent sheaf $\mathcal{E}$ of $\mathcal{O}_V$-modules on V, equipped with an integrable $f^{-1}\mathcal{O}_W$-linear connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_V} \Omega^1_{V/W}$. If V is smooth over K, then any ∇-module on V (relative to $\operatorname{Max} K$) is automatically locally free, as in [Brt96, Proposition 2.2.3].

One can also make a logarithmic analogue of this construction; we will not use it again in this section, but it will become crucially important later on.

Definition 2.3.5. Let $f : V \to W$ be a morphism of rigid spaces, and fix $x_1, \dots, x_m \in \Gamma(V, \mathcal{O})$. Let $\Omega^{1,\log}_{V/W}$ be the coherent sheaf on V given as the quotient of

$$\Omega^1_{V/W} \oplus \mathcal{O}_V s_1 \oplus \cdots \oplus \mathcal{O}_V s_m$$

by the relations $x_i s_i - dx_i$ for $i = 1, \dots, m$. We call $\Omega^{1,\log}_{V/W}$ the *module of (continuous) logarithmic differentials* with respect to the x_i.

Remark 2.3.6. A better way to make this definition would be to first define logarithmic structures on rigid spaces, then define $\Omega^{1,\log}_{V/W}$ to be the module of differentials of V equipped with the log structure generated by $x_1, \dots, x_m$, relative to W. Rather than do that here, we stick to the *ad hoc* construction; however, we will discuss logarithmic structures on schemes and formal schemes in § 6.

Definition 2.3.7. With notation as in Definition 2.3.5, a *log-∇-module* on V with respect to the x_i, relative to W, is a coherent locally free sheaf $\mathcal{E}$ of $\mathcal{O}$-modules on V, equipped with an integrable $f^{-1}\mathcal{O}_W$-linear connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^{1,\log}_{V/W}$.

We will also need the notion of horizontal sections.

Definition 2.3.8. With notation as in Definition 2.3.7, a section $\mathbf{v} \in \Gamma(V, \mathcal{E})$ is said to be *horizontal relative to* W if $\nabla \mathbf{v} = 0$. Let $H^0_W(V, \mathcal{E})$ denote the set of horizontal sections relative to W; it is a $\Gamma(W, \mathcal{O})$-module.

As in the complex analytic setting, a logarithmic connection has a residue map associated to it.

Definition 2.3.9. With notation as in Definition 2.3.7, note that over the zero locus $V(x_i)$, ∇ induces an $\mathcal{O}$-linear map from $\mathcal{E}$ to $\mathcal{E} \otimes \mathcal{O}_V s_i$, after quotienting $\mathcal{E} \otimes \Omega^{1,\log}_{V/W}$ by the image of $\mathcal{E} \otimes (\Omega^1_{V/W} \oplus \bigoplus_{j \neq i} \mathcal{O}_V s_j)$ and then reducing modulo x_i. Identifying $\mathcal{E} \otimes \mathcal{O}_V s_i$ with $\mathcal{E}$ yields an $\mathcal{O}$-linear endomorphism of $\mathcal{E}$ over $V(x_i)$; we call this the *residue* of ∇ along $V(x_i)$.

Remark 2.3.10. Beware that unlike in the ∇-module case, we built the locally free hypothesis into the definition of a log-∇-module: otherwise we could take for instance $\mathcal{O}/t\mathcal{O}$ on the affine t-line to be a log-∇-module with respect to t. By the same token, a log-∇-submodule $\mathcal{F}$ of a log-∇-module $\mathcal{E}$ need not have locally free quotient. However, one can get around these issues by inserting hypotheses about nilpotence of residues; see § 3.2.

2.4 Convergence of Taylor series

The construction of overconvergent isocrystals can be described in terms of a Taylor series associated to a connection; here is a relevant constraint.

Definition 2.4.1. Let X be an affinoid space and let $\mathcal{E}$ be a coherent $\mathcal{O}$-module on X. For $\eta_1, \dots, \eta_n \in [0, +\infty)$, we say that a multisequence $\{\mathbf{v}_I\}$ of elements of $\Gamma(X, \mathcal{E})$, indexed by n-tuples $I = (i_1, \dots, i_n)$ of nonnegative integers, is $(\eta_1, \dots, \eta_n)$-*null* if, for any multisequence $\{c_I\}$ of elements of K with $|c_I| \leqslant \eta_1^{i_1} \cdots \eta_n^{i_n}$, the multisequence $\{c_I \mathbf{v}_I\}$ converges to zero in $\Gamma(X, \mathcal{E})$ (for the canonical topology induced on this module from the affinoid topology on $\mathcal{O}(X)$). If $\eta_1 = \cdots = \eta_n = \eta$, we simply say that the multisequence is η-*null*. Note that it suffices to check the convergence on each element of an admissible affinoid cover of X.

Definition 2.4.2. Let $h : V \to X$ be a morphism of affinoid spaces, and suppose that $x_1, \dots, x_m \in \mathcal{O}(V)$ have the property that $dx_1, \dots, dx_m$ freely generate $\Omega^1_{V/X}$ (so that in particular the morphism h is smooth). Let $\mathcal{E}$ be a ∇-module over V relative to X; we may then view $\mathcal{E}$ as being equipped with commuting actions of the partial differential operators $\partial/\partial x_i$ for $i = 1, \dots, m$. For $\eta \in [0, +\infty)$ and $\mathbf{v} \in \Gamma(V, \mathcal{E})$, we say that $\mathcal{E}$ (or its connection) is η-*convergent at* $\mathbf{v}$ (with respect to $x_1, \dots, x_m$) if the multisequence

$$\frac{1}{i_1! \cdots i_m!} \frac{\partial^{i_1}}{\partial x_1^{i_1}} \cdots \frac{\partial^{i_m}}{\partial x_m^{i_m}} \mathbf{v}$$

is η-null; if $\mathcal{E}$ is η-convergent at all $\mathbf{v} \in \Gamma(V, \mathcal{E})$, we simply say that $\mathcal{E}$ is η-convergent.

Definition 2.4.3. With notation as in Definition 2.4.2, we say that $x_1, \dots, x_m$ form an η-*admissible coordinate system* on V (relative to X) if the trivial ∇-module with $\mathcal{E} = \mathcal{O}$ and $\nabla = d$ is η-convergent.

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In this case, by the Leibniz rule, any $\mathcal{E}$ is η-convergent if and only if it is η-convergent at each of a set of generators of $\Gamma(V, \mathcal{E})$.

Remark 2.4.4. With notation as in Definitions 2.4.2 and 2.4.3, suppose that $y_1, \ldots, y_m \in \Gamma(V, \mathcal{O})$ form another η-admissible coordinate system, and suppose that the $m \times m$ matrix A defined by $A_{ij} = \partial y_i / \partial x_j$ is invertible over $\Gamma(V, \mathfrak{o})$. Then the criterion of η-convergence with respect to $y_1, \ldots, y_m$ is equivalent to the criterion with respect to $x_1, \ldots, x_m$.

Remark 2.4.5. Retain notation as in Definitions 2.4.2 and 2.4.3. If $0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$ is a short exact sequence of ∇-modules, then $\mathcal{E}$ is η-convergent (with respect to a particular η-admissible coordinate system) if and only if $\mathcal{E}_1$ and $\mathcal{E}_2$ are η-convergent. In particular, the η-convergent ∇-modules on a given V form an abelian category.

2.5 Overconvergent sections

We recall the 'overconvergent sections' functor from [Brt96, 2.1.1].

DEFINITION 2.5.1. Let (X, Y, P, i, j) be a frame. For $V' \subset V$ two strict neighborhoods of $]X[$ within $]Y[$, let α_V (respectively, $\alpha_{VV'}$) denote the open immersion of V into $]Y[$ (respectively, of V' into V). Given an $\mathcal{O}_V$-module $\mathcal{E}$ on V, define

$$j_V^\dagger \mathcal{E} = \varinjlim \alpha_{VV'*} \alpha_{VV'}^* \mathcal{E},$$

the limit taken over strict neighborhoods V' of $]X[$ within $]Y[$ which are contained in V. The functors $\alpha_{VV'*}$ and $\alpha_{VV'}^*$ induce equivalences of categories between $j_V^\dagger \mathcal{O}$-modules and $j_{V'}^\dagger \mathcal{O}$-modules. The functor $\alpha_{V*} j_V^\dagger \alpha_V^*$ on $\mathcal{O}_{]Y[}$-modules does not depend on the choice of V, so we denote it simply as $j^\dagger$.

Remark 2.5.2. By [Brt96, Proposition 2.2.10], any coherent $j^\dagger \mathcal{O}_{]Y[}$-module is the pullback of a coherent $\mathcal{O}$-module on a strict neighborhood of $]X[$ in $]Y[$. Moreover, if two such modules are given, any morphism between them is obtained from a morphism between them on a strict neighborhood where they are both defined. In practice, then, we will write down coherent $j^\dagger \mathcal{O}_{]Y[}$-modules by writing down coherent $\mathcal{O}$-modules on strict neighborhoods of $]X[$, with the understanding that the strict neighborhood is to be shrunk as needed.

DEFINITION 2.5.3. Let (X, Y, P, i, j) be a frame. Let $\delta : P_K \to P_K \times_K P_K$ be the diagonal, put $j' = \delta \circ j$, let $\mathcal{I} \subset \mathcal{O}_{P_K \times P_K}$ be the ideal of the image of δ, and put $\mathcal{P}^n = \mathcal{O}_{P_K \times P_K} / \mathcal{I}^{n+1}$. Let $\mathcal{E}$ be a coherent $j^\dagger \mathcal{O}_{]Y[}$-module equipped with an integrable K-linear connection ∇. Then in the usual fashion [Brt96, 2.2.2], the connection gives rise to isomorphisms

$$\epsilon_n : j^\dagger \mathcal{P}^n \otimes_{j^\dagger \mathcal{O}_{]Y[}} \mathcal{E} \xrightarrow{\sim} \mathcal{E} \otimes_{j^\dagger \mathcal{O}_{]Y[}} j^\dagger \mathcal{P}^n.$$

We say that $\mathcal{E}$ is *overconvergent along* $Y \setminus X$ if there exists an isomorphism $\epsilon : \pi_2^* \mathcal{E} \xrightarrow{\sim} \pi_1^* \mathcal{E}$ which induces each ϵ_n by reducing modulo $(j')^\dagger \mathcal{I}^{n+1}$ and using the canonical identification $\delta^{-1}(j')^\dagger \cong j^\dagger \delta^{-1}$ of [Brt96, (2.1.4.4)]. If $Y \setminus X = \emptyset$, we say instead that $\mathcal{E}$ is *convergent*.

Remark 2.5.4. By [Brt96, Proposition 2.2.3] (as in Remark 2.5.2), any coherent $j^\dagger \mathcal{O}_{]Y[}$-module equipped with an integrable K-linear connection ∇ is the pullback of a ∇-module $\mathcal{E}$ on some strict neighborhood of $]X[$ in $]Y[$, and likewise any morphism between such modules extends to some strict neighborhood of $]X[$ in $]Y[$. By [Brt96, Proposition 2.2.6], the connection is overconvergent along $Y \setminus X$ if and only if there exists $\epsilon : \pi_2^* \mathcal{E} \xrightarrow{\sim} \pi_1^* \mathcal{E}$ of the desired form over some strict neighborhood of $]X[_{P^2}$ in $]Y[_{P^2}$. By abuse of language, we will say that '$\mathcal{E}$ is overconvergent along $Y \setminus X$' to mean that $j^\dagger \mathcal{E}$ is overconvergent along $Y \setminus X$.

The condition of overconvergence can also be interpreted in terms of the convergence of the Taylor series associated to the connection, as follows.

Remark 2.5.5. Let (X, Y, P, i, j) be a frame and suppose that the differentials of $x_1, \dots, x_n \in \Gamma(P, \mathcal{O}_P)$ generate $\Omega^1_{P/\mathfrak{o}_K}$ over a neighborhood of X. Then $dx_1, \dots, dx_n$ also generate $\Omega^1_{P_K/K}$ over a strict neighborhood of $]X[$ in $]Y[$ (see [Brt96, Proposition 2.2.13]).

By [Brt96, Proposition 2.2.13], we have the following. (Note that the statement of [Brt96, Proposition 2.2.13] only includes the equivalence between (a) and (b) below; however, the fact that (b) holds for all sufficiently large λ is evident in the proof of [Brt96, Proposition 2.2.13].)

PROPOSITION 2.5.6. *Let (X, Y, P, i, j) be a frame; define the sets $[Y]_\eta$ as in Definition 2.2.3 (using any set of generators). Suppose further that there exists $g \in \Gamma(P, \mathcal{O}_P)$ which cuts out $Y \setminus X$ within Y; define the sets U_λ as in Lemma 2.2.8 using g. Suppose further that the differentials of $x_1, \dots, x_n \in \Gamma(P, \mathcal{O}_P)$ generate $\Omega^1_{P/\mathfrak{o}_K}$ over a neighborhood of X. Let V be a strict neighborhood of $]X[$ in $]Y[$, and let $\mathcal{E}$ be a ∇-module on V. Then the following conditions are equivalent:*

(a) *$j^\dagger\mathcal{E}$ is overconvergent;*

(b) *for each $\eta \in (0,1)\cap\Gamma^*$, there exists $\lambda \in (0,1)\cap\Gamma^*$ such that $[Y]_\eta \cap U_\lambda \subseteq V$ and $\mathcal{E}$ is η-convergent with respect to $x_1, \dots, x_n$ over $[Y]_\eta \cap U_\lambda$.*

Moreover, if these hold, then for each $\eta \in (0,1)\cap\Gamma^*$, the conclusion of condition (b) holds for all $\lambda \in (0,1)\cap\Gamma^*$ sufficiently large.

Remark 2.5.7. Note that the trivial ∇-module $\mathcal{O}_V$ evidently satisfies the definition of overconvergence given in Definition 2.5.3. Hence with conditions as in Proposition 2.5.6, for each $\eta \in (0,1)\cap\Gamma^*$, $x_1, \dots, x_n$ necessarily form an η-admissible coordinate system (in the sense of Definition 2.4.3) on $[Y]_\eta \cap U_\lambda$ for all $\lambda \in (0,1)\cap\Gamma^*$ sufficiently large. In particular, the property of η-convergence may be checked at each element of a set of generators of $\Gamma([Y]_\eta \cap U_\lambda, \mathcal{E})$.

Remark 2.5.8. Note that the criterion for overconvergence in Proposition 2.5.6 simplifies somewhat in the case $Y = P_K$, as in that case $[Y]_\eta = P_K$ for all $\eta \in (0,1)$. This will be a great help as we work with 'small frames' in § 4.

Remark 2.5.9. In a previous version of this paper, the restriction that $Y \setminus X$ must be a divisor in Y was omitted from Proposition 2.5.6; thanks to Bernard le Stum for pointing this out. That restriction will be harmless in practice, as we will be able to blow up in $Y \setminus X$ without disturbing the concept of overconvergence; see Definition 2.6.7 below.

2.6 Isocrystals

Given a morphism of frames as in (2.2.5), one obtains a pullback functor u_K^* from the category of $j^\dagger\mathcal{O}_{]Y[}$-modules with integrable overconvergent connection to the analogous category of $(j')^\dagger\mathcal{O}_{]Y'[}$-modules. The key consequences of overconvergence are the following two 'homotopy invariance' results for the pullback functors, which are [Brt96, Proposition 2.2.17] and [Brt96, Théorème 2.3.1], respectively.

PROPOSITION 2.6.1. *Given two morphisms of frames as in (2.2.5) factoring through the same map $Y' \to Y$ with $u = u_1$ and $u = u_2$, respectively, there is a canonical isomorphism ϵ_{u_1,u_2} between the functors u_{1K}^* and u_{2K}^*. Moreover, for any horizontal section s, one has $\epsilon_{u_1,u_2}(u_{1K}^*(s)) = u_{2K}^*(s)$.*

PROPOSITION 2.6.2. *Given a morphism of frames as in (2.2.5) in which $X = X', Y = Y'$, and v and w are the identity maps, the functor u_K^* is an equivalence of categories.*

Using Proposition 2.6.2, one can define a category of isocrystals. This is done somewhat informally in [Brt96]; a more 'crystalline' presentation is given by ongoing work of le Stum (see [LS04] for a report, and [LS06] for further details). Here we take a middle road.

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Definition 2.6.3. Given an open immersion $j : X \hookrightarrow Y$ of k-varieties, define the site $\mathcal{C}_{X,Y}$ as follows. The objects of $\mathcal{C}_{X,Y}$ are tuples (U, P, i), where U is an open subscheme of Y and $(X \cap U, U, P, i, j)$ form a frame. A morphism $(U, P, i) \to (U', P', i')$ consists of an inclusion $U \subseteq U'$ and a morphism $f : P \to P'$ of formal schemes such that $f \circ i$ equals the restriction of i' to U. A covering $\{(U_i, P_i, i_i) \to (U, P, i)\}$ is admissible if $\{U_i \to U\}$ is surjective.

Definition 2.6.4. With notation as in Definition 2.6.3, put $Z = Y \setminus X$. An *isocrystal on X overconvergent along Z (over K)* is a crystal on $\mathcal{C}_{X,Y}$ of coherent locally free $j^\dagger\mathcal{O}$-modules with overconvergent connection: i.e., one specifies for each $(U, P, i) \in \mathcal{C}_{X,Y}$ a coherent locally free $j^\dagger\mathcal{O}_{]U[}$-module $\mathcal{E}_U$ equipped with an integrable connection overconvergent along $U \cap Z$, and for each morphism $u : (U, P, i) \to (U', P', i')$ an isomorphism $\mathcal{E}_U \xrightarrow{\sim} u^*\mathcal{E}_{U'}$ of modules with connection, such that the isomorphisms satisfy the obvious cocycle condition. Let $\mathrm{Isoc}^\dagger(X, Y/K)$ denote the category of these objects. In the case $X = Y$, we call the category $\mathrm{Isoc}(X/K)$ and call its elements *convergent isocrystals on X*.

Definition 2.6.5. For $F = (X, Y, P, i, j)$ a frame, there is an obvious restriction functor from isocrystals on X overconvergent along $Y \setminus X$ to coherent locally free $j^\dagger\mathcal{O}_{]Y[}$-modules with integrable overconvergent connection; this is called the *realization* functor for the frame F. Using Proposition 2.6.2, one can show that each realization functor is itself an equivalence of categories. Namely, to construct an isocrystal with any given realization, given $j : Y \hookrightarrow P_k$ and $j' : U \hookrightarrow P'_k$, we restrict j to U, pull back along the first projection of $P \times P'$, then apply Proposition 2.6.2 to 'push forward' along the second projection. (One can also speak of realizations on frames enclosing open subvarieties of X, but of course those will not typically be equivalences of categories.)

Remark 2.6.6. In fact, carrying the connection around in this construction is superfluous; as happens for the infinitesimal and crystalline sites, the connection data are already captured in the structure of a crystal of $j^\dagger\mathcal{O}$-modules. This is the point of view adopted in [LS04, LS06]. Another approach is to state the definition in terms of simplicial schemes, as in [Shi00, 1.3.1].

Definition 2.6.7. Given a diagram of the form

$$\begin{tikzcd} X' \arrow[r, hook, "j'"] \arrow[d, "w"'] & Y' \arrow[r] \arrow[d, "v"'] & \operatorname{Spec} \mathfrak{o}_{K'} \arrow[d] \\ X \arrow[r, hook, "j"] & Y \arrow[r] & \operatorname{Spec} \mathfrak{o}_K \end{tikzcd}$$

one obtains by pullback (as in [Brt96, 2.3.2.2]) an inverse image functor

$$v^* : \mathrm{Isoc}^\dagger(X, Y/K) \to \mathrm{Isoc}^\dagger(X', Y'/K').$$

In the case $X = X'$, $w = \mathrm{id}_X$, $K = K'$, and v is proper, then v^* is an equivalence of categories [Brt96, Théorème 2.3.5]. In particular, if Y itself is proper, then the category $\mathrm{Isoc}^\dagger(X, Y/K)$ is independent of Y; it is thus denoted $\mathrm{Isoc}^\dagger(X/K)$ and its objects are called *overconvergent isocrystals on X (over K)*. This category is abelian [Brt96, Remarques 2.3.3].

Definition 2.6.8. Suppose that

$$\begin{tikzcd} X' \arrow[r, hook, "j'"] \arrow[d, "w"'] & Y' \arrow[d, "v"'] \\ X \arrow[r, hook, "j"] & Y \end{tikzcd}$$

is a commutative diagram of k-varieties with j, j' open immersions, v finite, and w finite étale. Then one obtains a pushforward functor

$$v_* : \mathrm{Isoc}^\dagger(X', Y'/K) \to \mathrm{Isoc}^\dagger(X, Y/K)$$

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from the pushforward along a finite étale morphism of rigid spaces. As shown by Tsuzuki (see [Tsu02, 5.1]), for $\mathcal{E}, \mathcal{F} \in \mathrm{Isoc}^\dagger(X', Y'/K)$, we have a canonical bijection

$$\mathrm{Hom}(\mathcal{E}, \mathcal{F}) \to \mathrm{Hom}(v_*\mathcal{E}, v_*\mathcal{F}); \tag{2.6.9}$$

in addition, for $\mathcal{E} \in \mathrm{Isoc}^\dagger(X, Y/K)$ and $\mathcal{F} \in \mathrm{Isoc}^\dagger(X', Y'/K)$, one has adjunction and trace morphisms

$$\mathcal{E} \xrightarrow{\mathrm{ad}} v_*v^*\mathcal{E} \xrightarrow{\mathrm{tr}} \mathcal{E}, \quad \mathcal{F} \xrightarrow{\mathrm{ad}} v^*v_*\mathcal{F} \xrightarrow{\mathrm{tr}} \mathcal{F},$$

such that the displayed compositions are multiplication by the degree of v. (Tsuzuki explicitly constructs the first sequence; the second sequence is obtained from the first by putting $\mathcal{E} = v_*\mathcal{F}$ and invoking (2.6.9).)

3. Local monodromy of p-adic differential equations

We next gather some facts about differential modules on p-adic annuli. Various aspects of this theory have been treated previously, e.g., by Crew [Cre98], Tsuzuki [Tsu98], de Jong [DJ98a], and this author [Ked04b]. New features here include the systematic presentation in terms of rigid analytic spaces (which obviates the need to restrict to discretely valued or even spherically complete coefficient fields), the treatment of multidimensional annuli, the consideration of families of annuli (based partly on [Ked06]), and the introduction of logarithmic singularities. However, we restrict here to cases of unipotent monodromy; we will consider 'quasi-unipotent' differential modules later in the series.

Throughout this section, we retain the conventions introduced in § 2.1.

3.1 Polyannuli

DEFINITION 3.1.1. We say that a subinterval I of $[0, +\infty)$ is *aligned* if any endpoint at which it is closed is either equal to zero or contained in Γ^* (the divisible closure of the image of $|\cdot|$ on K^*). In particular, any open interval is aligned, and any aligned interval can be written as the union of a weakly increasing sequence of aligned closed subintervals. We say that I is *quasi-open* if it is open at each *nonzero* endpoint, i.e., it is of one of the forms (a, b) or $[0, b)$; any quasi-open interval is aligned.

DEFINITION 3.1.2. For I an aligned subinterval of $[0, +\infty)$, we define the *polyannulus* $A_K^n(I)$ as

$$A_K^n(I) = \{(t_1, \ldots, t_n) \in \mathbb{A}_K^n : |t_i| \in I \ (i = 1, \ldots, n)\}.$$

CONVENTION 3.1.3. In the notation $A_K^n(I)$, we drop the parentheses around the interval I if it is being written out explicitly, e.g., we write $A_K^n[0, 1)$ instead of $A_K^n([0, 1))$.

Remark 3.1.4. Note that if $0 \notin I$ and $n > 1$, then $A_K^n(I)$ is not the same as a punctured polydisc; if $I = J_1 \setminus J_2$, where J_1 and J_2 are aligned intervals both containing 0, the latter would be

$$A_K^n(J_1) \setminus A_K^n(J_2),$$

which unlike $A_K^n(I)$ is not an affinoid space if I is closed.

DEFINITION 3.1.5. For X an affinoid space and I an aligned subinterval of $[0, +\infty)$, the ring $\Gamma(X \times A_K^n(I), \mathcal{O})$ consists of Laurent series

$$\sum_{J \in \mathbb{Z}^n} c_J t^J = \sum_{J = (j_1, \ldots, j_n)} c_J t_1^{j_1} \cdots t_n^{j_n}$$

with coefficients in $\Gamma(X, \mathcal{O})$, such that $|c_J|_X \rho_1^{j_1} \cdots \rho_n^{j_n} \to 0$ as $J \to \infty$ (that is, $|c_J|_X \rho_1^{j_1} \cdots \rho_n^{j_n}$ exceeds any particular positive number for only finitely many J) for each $\rho_1, \ldots, \rho_n \in I$.

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For $R = (r_1, \dots, r_n) \in I^n$, let $|\cdot|_{X,R}$ denote the function on $\mathcal{O}(X \times A_K^n(I))$ given by

$$\left|\sum_J c_J t^J\right|_{X,R} = \sup_J \{|c_J|_X r_1^{j_1} \cdots r_n^{j_n}\};$$

note that the supremum is achieved by at least one, but only finitely many, tuples $(j_1, \dots, j_n)$. If $R = (r, \dots, r)$, we also write $|\cdot|_{X,r}$ for $|\cdot|_{X,R}$.

One has analogues of the maximum modulus principle and the Hadamard three circles theorem for $|\cdot|_{X,R}$.

LEMMA 3.1.6. *Let X be an affinoid space.*

(a) *For $x \in \Gamma(X \times A_K^n[0,b], \mathcal{O})$ with $b \in [0, +\infty) \cap \Gamma^*$, and $R \in [0,b]^n$, we have $|x|_{X,R} \leqslant |x|_{X,b}$.*
(b) *For $x \in \Gamma(X \times A_K^n(I), \mathcal{O})$ with I an aligned subinterval of $[0, +\infty)$, $A, B \in I^n$, and $c \in [0,1]$, put $r_i = a_i^c b_i^{1-c}$; then $|x|_{X,R} \leqslant |x|_{X,A}^c |x|_{X,B}^{1-c}$.*

Proof. (a) If $x = \sum c_J t^J \in \Gamma(X \times A_K^n[0,b], \mathcal{O})$, then $c_J = 0$ unless $j_1, \dots, j_n \geqslant 0$. Hence if $R \in [0,b]^n$, then

$$|c_J|_X r_1^{j_1} \cdots r_n^{j_n} \leqslant |c_J|_X b^{j_1 + \cdots + j_n};$$

taking suprema yields $|x|_{X,R} \leqslant |x|_{X,b}$.

(b) Note that the desired inequality holds with equality if $x = c_J t^J$ is a monomial. For a general $x = \sum c_J t^J$, we then have

$$\begin{aligned} |x|_{X,R} &= \sup_J \{|c_J t^J|_{X,R}\} \\ &= \sup_J \{|c_J t^J|_{X,A}^c |c_J t^J|_{X,B}^{1-c}\} \\ &\leqslant \sup_J \{|c_J t^J|_{X,A}\}^c \sup_J \{|c_J t^J|_{X,B}\}^{1-c} \\ &= |x|_{X,A}^c |x|_{X,B}^{1-c}, \end{aligned}$$

as desired. ∎

COROLLARY 3.1.7. *For $x \in \mathcal{O}(X \times A_K^n[a,b])$, the maximum of $|x|_{X,R}$ over all $R \in [a,b]^n$ is achieved by a tuple $R \in \{a,b\}^n$.*

LEMMA 3.1.8. *For X an affinoid space, I an aligned subinterval of $[0, +\infty)$, and $A = (a_i) \in I^n \cap (\Gamma^*)^n$, the norm $|\cdot|_{X,A}$ coincides with the supremum seminorm on the affinoid space*

$$X \times \{(x_1, \dots, x_n) \in A_K^n(I) : |x_i| = a_i \ (i = 1, \dots, n)\}.$$

Proof. There is no loss of generality in enlarging K so that $a_1, \dots, a_n$ land in the image of $|\cdot|$ itself. Given $\sum c_J t^J \in \mathcal{O}(X \times A_K^n(I))$, the supremum defining $|\sum c_J t^J|_{X,A}$ is achieved by finitely many tuples J. Let S be the set of these tuples; by enlarging K again, we can ensure that there exist $x_1, \dots, x_n \in K$, with $|x_i| = a_i$ for each i, such that the evaluation of $\sum_{J \in S} c_J t^J$ at $t_i = x_i$ has norm equal to $|\sum c_J t^J|_{X,A}$. ∎

COROLLARY 3.1.9. *For $[a,b]$ aligned, the affinoid topology on $\mathcal{O}(X \times A_K^n[0,b])$ coincides with the subspace topology induced by the affinoid topology on $\mathcal{O}(X \times A_K^n[a,b])$.*

COROLLARY 3.1.10. *For any aligned subinterval I of $[0, +\infty)$, the space $A_K^n(I)$ is a quasi-Stein space. (In particular, if X is also quasi-Stein, then so is $X \times A_K^n(I)$.)*

Proof. Let $I_1 \subseteq I_2 \subseteq \cdots$ be a weakly increasing sequence of closed aligned intervals with union I. Then $A_K^n(I)$ is the union of the $A_K^n(I_j)$; moreover, if $0 \in I$, then the polynomial ring $K[t_1, \dots, t_n]$ is

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dense in each $\mathcal{O}(A_K^n(I_j))$, since the Laurent series $\sum c_J t^J$ is the limit under each $|\cdot|_{X,R}$ of its finite partial sums. By the same token, if $0 \notin I$, then the Laurent polynomial ring $K[t_1, \dots, t_n, t_1^{-1}, \dots, t_n^{-1}]$ is dense in each $\mathcal{O}(A_K^n(I_j))$. In either case, $A_K^n(I)$ is quasi-Stein. $\square$

We will need a refinement of the argument of Corollary 3.1.10 for quasi-open intervals.

Lemma 3.1.11. *Let I be a quasi-open subinterval of $[0, +\infty)$, and let $x = \sum_J c_J t^J$ be an element of $\mathcal{O}(A_K^n(I))$. For $l = 1, 2, \dots,$ put*

$$x_l = \sum_{J:|j_1|,\dots,|j_n| \geqslant l} c_J t^J.$$

Then for any $R \in I^n$, there exists $\eta > 1$ such that $\lim_{l\to\infty} \eta^l |x_l|_{K,R} = 0$.

Proof. First suppose that $I = (a, b)$. Pick $a', b' \in \Gamma^*$ with $a < a' < r_i < b' < b$ for $i = 1, \dots, n$; then the supremum seminorm of x_l on $A_K^n[a', b']$ tends to 0 as $l \to \infty$. That is,

$$\lim_{l\to\infty} \max_{S \in \{a',b'\}^n} \{|x_l|_{K,S}\} = 0.$$

Given J, put

$$s_i = \begin{cases} b' & \text{for } j_i \geqslant 0, \\ a' & \text{for } j_i < 0, \end{cases}$$

for $i = 1, \dots, n$, and put $S = (s_1, \dots, s_n)$. Then

$$|c_J t^J|_{X,R} \leqslant |c_J t^J|_{X,S} \prod_{i=1}^n \max\{a'/r_i, r_i/b'\}^{|j_i|}.$$

We may thus take $\eta = \prod_{i=1}^n \min\{r_i/a', b'/r_i\} > 1$.

In the case $I = [0, b)$, the argument is similar but easier: for any b' with $r_i < b' < b$ for $i = 1, \dots, n$, we have

$$|c_J t^J|_{X,R} \leqslant |c_J t^J|_{X,S} \prod_{i=1}^n (r_i/b')^{j_i}.$$

and so we may take $\eta = \prod_{i=1}^n (b'/r_i) > 1$. $\square$

3.2 Constant and unipotent connections

We now start considering ∇-modules on the product of a smooth rigid space with a polyannulus. For convenience, we encapsulate a running hypothesis.

Hypothesis 3.2.1. Let $f : V \to W$ be a morphism of smooth rigid spaces, and suppose that $x_1, \dots, x_m \in \Gamma(V, \mathcal{O})$ have zero loci which are smooth and meet transversely.

Definition 3.2.2. Under Hypothesis 3.2.1, let n be a positive integer, and let X be an admissible open subset of $V \times A_K^n[0, 1)$. Define the category $\mathrm{LNM}_{X/W}$ to be the category of log-∇-modules over X relative to W with respect to $t_1, \dots, t_n, x_1, \dots, x_m$, having nilpotent residues.

Remark 3.2.3. In Definition 3.2.2, we omit W if f coincides with the structural morphism $V \to \operatorname{Max} K$ (which we will more briefly describe hereafter by saying 'if $W = \operatorname{Max} K$'). If we are in this case and $m = n = 0$, then LNM_X is an abelian category; this will also turn out to be true if $m > 0$ or $n > 0$, by virtue of Lemma 3.2.14.

Convention 3.2.4. If $I = [0, 0]$, we will regard $\Omega^{1,\log}$ over $A_K^n[0, 0]$ as being freely generated by $dt_1/t_1, \dots, dt_n/t_n$. That is, for $U = V \times A_K^n[0, 0]$, the elements of $\mathrm{LNM}_{U/W}$ will be log-∇-modules over V relative to W with respect to $x_1, \dots, x_m$, equipped with n commuting endomorphisms, which for consistency we view as the actions of the operators $\partial_i = t_i\, \partial/\partial t_i$ for $i = 1, \dots, n$.

Definition 3.2.5. Under Hypothesis 3.2.1, take $\mathcal{E} \in \mathrm{LNM}_{X/W}$ for $X = V \times A_K^n(I)$. We say that $\mathcal{E}$ is *constant* if $\mathcal{E} \cong \pi_1^* \mathcal{F}$ for some log-∇-module $\mathcal{F}$ on V relative to W with respect to $x_1, \dots, x_m$ (necessarily having nilpotent residues). Note that if $\mathcal{E}$ is constant, then for any affinoid subspace U of V, the restriction of $\mathcal{E}$ to $X \cap (U \times A_K^n[0,1))$ is spanned by finitely many sections which are horizontal relative to V. We say that $\mathcal{E}$ is *unipotent* if $\mathcal{E}$ admits an exhaustive filtration by log-∇-submodules whose successive quotients are constant; we call such a filtration a *unipotent filtration*. Let $\mathrm{ULNM}_{X/W}$ be the subcategory of $\mathrm{LNM}_{X/W}$ consisting of unipotent objects.

We will ultimately see (Theorem 3.3.4) that the following construction produces all unipotent ∇-modules.

Definition 3.2.6. Under Hypothesis 3.2.1, let I be an aligned subinterval of $[0, +\infty)$. We define the functor

$$\mathcal{U}_I : \mathrm{LNM}_{V \times A_K^n[0,0]/W} \to \mathrm{LNM}_{V \times A_K^n(I)/W}$$

as follows. Given a log-∇-module $\mathcal{E}$ over V relative to W with respect to $x_1, \dots, x_m$, equipped with n commuting nilpotent endomorphisms $N_1, \dots, N_n$, define $\mathcal{U}_I(\mathcal{E})$ to be the sheaf $\pi_1^* \mathcal{E}$ equipped with the connection

$$\mathbf{v} \mapsto \pi_1^*(\nabla)\mathbf{v} + \sum_{i=1}^n \pi_1^*(N_i)(\mathbf{v}) \otimes \frac{dt_i}{t_i}.$$

This connection is integrable because the $\pi_1^*(N_i)$ commute with each other and with the action of the connection on the base.

Remark 3.2.7. Suppose that $V = W = \mathrm{Max}\, K$. In Definition 3.2.5, if $X = V \times A_K^n[0,0]$, then $\mathcal{E}$ is constant if and only if the ∂_i all act via the zero map, and $\mathcal{E}$ is unipotent if and only if the ∂_i are all nilpotent. In Definition 3.2.6, any nilpotent filtration of $\mathcal{E}$ with respect to $N_1, \dots, N_n$ lifts to a unipotent filtration of $\mathcal{U}_I(\mathcal{E})$, so $\mathcal{U}_I(\mathcal{E})$ is unipotent. We will generalize this remark later (Remark 3.2.16), but beware that it is not true for V, W general.

In ordinary analysis, ∇-modules on open polydiscs are automatically constant, but this fails in rigid analysis without extra hypotheses; see Remark 3.6.5. However, one can at least salvage the following result.

Lemma 3.2.8. *Under Hypothesis 3.2.1, suppose that V is affinoid, that $X = V \times A_K^1[0,a]$, and that $\mathcal{E} \in \mathrm{LNM}_{X/V}$ is such that the restriction of $\mathcal{E}$ to $V \times \{0\}$ is free. Then there exists $b \in (0,a] \cap \Gamma^*$ such that the restriction of $\mathcal{E}$ to $V \times A_K^1[0,b]$ is in the essential image of the functor $\mathcal{U}_{[0,b]}$. In particular, if the residue of $\mathcal{E}$ along $V \times \{0\} = V(t_1)$ vanishes, then the restriction of $\mathcal{E}$ to $V \times A_K^1[0,b]$ is constant.*

Proof. Choose elements $\mathbf{e}_1, \dots, \mathbf{e}_n$ of $\Gamma(X, \mathcal{E})$ restricting to a basis of $\mathcal{E}$ on $V \times \{0\}$. The locus where these sections fail to be linearly independent or fail to span $\mathcal{E}$ is a closed analytic subspace of X not meeting V; by the maximum modulus principle, the values of t_1 on this subspace are bounded away from zero. Hence by making a smaller, we can ensure that $\mathbf{e}_1, \dots, \mathbf{e}_n$ form a basis of $\Gamma(X, \mathcal{E})$.

Define the $n \times n$ matrix N over $\mathcal{O}(X)$ by the formula

$$\partial_1 \mathbf{e}_l = \sum_j N_{jl} \mathbf{e}_j$$

and formally write $N = \sum_{i=0}^\infty N_i t_1^i$, where each N_i is an $n \times n$ matrix over $\mathcal{O}(V)$. We now verify that there is a unique $n \times n$ matrix M over $\mathcal{O}(V)[\![t_1]\!]$, congruent to the identity matrix modulo t_1, such that $NM + \partial_1 M = MN_0$. Namely, if we write $M = \sum_{i=0}^\infty M_i t_1^i$, for each $i > 0$ we then have

$$iM_i + N_0 M_i - M_i N_0 = -\sum_{j=0}^{i-1} N_{i-j} M_j. \tag{3.2.9}$$

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Let e be the nilpotency index of N_0; then the map g on the space of $n \times n$ matrices over $\mathcal{O}(V)$ defined by $g(M_i) = N_0 M_i - M_i N_0$ is itself nilpotent of index at most $2e - 1$. The map $M_i \mapsto iM_i + N_0 M_i - M_i N_0$ on $n \times n$ matrices is then the sum of an invertible linear map and a nilpotent linear map, hence is invertible. Thus M_i is uniquely determined by $M_0, \dots, M_{i-1}$, proving the existence and uniqueness of M.

We now analyze (3.2.9) to show that M converges on $V \times A_K^1[0, b]$ for some b. Put $\delta = \max\{1, |N|_X\}$; then for all i,

$$|N_i|_V \leqslant \delta a^{-i}.$$

In particular $|N_0|_V \leqslant \delta$. We now prove by induction that

$$|M_i|_V \leqslant |i!|^{-2e} a^{-i} \delta^{2ei}. \tag{3.2.10}$$

For $i = 0$, this is merely $1 \leqslant 1$. Given the result for all $j < i$, examining the right side of (3.2.9) yields the bound

$$|iM_i + N_0 M_i - M_i N_0|_V \leqslant |(i-1)!|^{-2e} a^{-i} \delta^{2e(i-1)+1}. \tag{3.2.11}$$

If $W = iM_i + N_0 M_i - M_i N_0$, we can then write

$$M_i = \sum_{j=0}^{2e-1} (-1)^j i^{-j-1} g^{(j)}(W),$$

where $g^{(j)}$ denotes the j-fold composition. In particular, we have

$$|M_i|_V \leqslant |i|^{-2e} \delta^{2e-1} |W|_V,$$

which combines with (3.2.11) to yield (3.2.10).

Finally, choose $b \in (0, 1) \cap \Gamma^*$ with

$$b < |p|^{2e/(p-1)} a \delta^{-2e}.$$

By virtue of (3.2.10) and the inequality $|i!| \leqslant |p|^{i/(p-1)}$, we have $|M_i|_V b^i < 1$ for all $i > 0$. Hence the matrix M gives rise to an invertible matrix over $\mathcal{O}(V \times A_K^1[0, b])$. Define the vectors $\mathbf{v}_1, \dots, \mathbf{v}_m \in \Gamma(V \times A_K^1[0, b], \mathcal{E})$ by

$$\mathbf{v}_l = \sum_j M_{jl} \mathbf{e}_j;$$

then we can write $\mathcal{E} = \mathcal{U}_{[0,b]}(\mathcal{F})$ for $\mathcal{F}$ equal to the $\mathcal{O}(V)$-span of $\mathbf{v}_1, \dots, \mathbf{v}_n$. □

LEMMA 3.2.12. *Let $\mathcal{E}$ be a ∇-module (respectively, a log-∇-module with nilpotent residues) on $A_K^n[0, a]$ for some $a \in (0, +\infty) \cap \Gamma^*$. Then there exists $b \in (0, a] \cap \Gamma^*$ such that $\mathcal{E}$ is constant (respectively, unipotent) on $A_K^n[0, b]$.*

Proof. We proceed by induction on n, with vacuous base case $n = 0$. Identify $V = A_K^{n-1}[0, a]$ with the zero locus of t_n in $A_K^n[0, a]$; by the induction hypothesis, by making a smaller, we can ensure that the restriction of $\mathcal{E}$ to V is constant (respectively, unipotent). In particular, we can choose sections $\mathbf{e}_1, \dots, \mathbf{e}_m \in \Gamma(A_K^n[0, a], \mathcal{E})$ restricting to sections on V which form a basis of $\Gamma(V, \mathcal{E})$ on which the ∂_i act trivially (respectively, act via commuting nilpotent matrices over K). The locus where these sections fail to be linearly independent or fail to span $\mathcal{E}$ is a closed analytic subspace of $A_K^n[0, a]$ not meeting V; by the maximum modulus principle, the values of t_n on this subspace are bounded away from zero. Hence by making a smaller, we can ensure that in fact $\mathbf{e}_1, \dots, \mathbf{e}_m$ form a basis of $\Gamma(A_K^n[0, a], \mathcal{E})$. We may then apply Lemma 3.2.8 to see that, for some b, the restriction of $\mathcal{E}$ to $A_K^n[0, b]$ can be pulled back from $A_K^{n-1}[0, b]$. By the induction hypothesis, $\mathcal{E}$ is in fact constant (respectively, unipotent). □

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Note that Lemma 3.2.12 has important consequences for connections on arbitrary smooth rigid spaces: it gives us a 'very local' criterion for checking local freeness of a module equipped with a logarithmic connection. (Here 'very local' means that the criterion can be checked in an affinoid neighborhood around each point, not just on an admissible affinoid covering.)

Lemma 3.2.13. *Let X be a rigid space, and let $\mathcal{E}$ be a coherent sheaf on X. Then $\mathcal{E}$ is locally free if and only if for each $x \in X$, there is an affinoid neighborhood of x on which $\mathcal{E}$ is free.*

Proof. By passing to an admissible affinoid cover, it suffices to check this in the case $X = \operatorname{Max} A$ is affinoid. In that case, by Kiehl's theorem [BGR84, Theorem 9.4.3/3], $M = \Gamma(X, \mathcal{E})$ is a finitely generated A-module and $\mathcal{E}$ is the coherent sheaf on X associated to M. Let Y denote the scheme $\operatorname{Spec} A$; then for each $x \in X$, we may also regard x as a point of Y, and the local ring $\mathcal{O}_{X,x}$ is flat over the local ring $\mathcal{O}_{Y,x}$ because both have the same completion [BGR84, Proposition 7.3.2/3]. Hence the coherent sheaf on Y associated to M has free stalks at each maximal ideal, and so M is locally free. □

Lemma 3.2.14. *Let X be a smooth rigid space, and suppose that the zero loci of $t_1, \dots, t_n \in \mathcal{O}(X)$ are smooth and meet transversely. Let $f : \mathcal{E} \to \mathcal{F}$ be a morphism of log-∇-modules with nilpotent residues on X with respect to $t_1, \dots, t_n$. Then the kernel and cokernel of f are also log-∇-modules with nilpotent residues.*

Proof. By Lemma 3.2.13, it suffices to check the local freeness pointwise; clearly the same is true of the nilpotence of residues. Moreover, there is no harm in enlarging K before checking these conditions at a given point. It thus suffices to check that if $x \in X$ is a K-rational point, then the kernel and cokernel of f have free stalks at x and have nilpotent residues there. There is no harm in assuming that $t_1, \dots, t_n$ vanish at x and generate $dt_1, \dots, dt_n$ there. (To get to this case, first drop the t_i which do not vanish at x, then add back additional ones to fill out a local coordinate system.) That done, by Remark 2.3.3, we may assume that in fact $X = A^n_K[0, a]$ for some $a \in (0, +\infty) \cap \Gamma^*$; then by Lemma 3.2.12, we may assume that $\mathcal{E}$ and $\mathcal{F}$ are unipotent.

We proceed by induction on the rank of $\mathcal{E} \oplus \mathcal{F}$. Choose bases $\mathbf{e}_1, \dots, \mathbf{e}_l$ and $\mathbf{f}_1, \dots, \mathbf{f}_m$ of $\mathcal{E}$ and $\mathcal{F}$, respectively, on which each ∂_i acts via a nilpotent matrix over K. In particular, $\nabla(\mathbf{e}_1) = 0$, and so $\nabla(f(\mathbf{e}_1)) = 0$. By a formal power series calculation, each element of the kernel of ∇ belongs to the K-span of $\mathbf{f}_1, \dots, \mathbf{f}_m$. Hence either $f(\mathbf{e}_1) = 0$, or $f(\mathbf{e}_1)$ generates a direct summand of $\mathcal{F}$. Quotienting by the spans of $\mathbf{e}_1$ and $f(\mathbf{e}_1)$ and repeating the argument, we deduce that the kernel and image of f are free with nilpotent residues, as desired. □

Remark 3.2.15. The local freeness in Lemma 3.2.14 can also be proved on the level of completed local rings, which does not require the use of Lemma 3.2.12. However, Lemma 3.2.12 will come in handy later; see Proposition 3.3.8.

Remark 3.2.16. We can now generalize both assertions of Remark 3.2.7 to the case $W = \operatorname{Max} K$ and V arbitrary; it suffices to treat the first of them, and this can be done as follows. For $\mathcal{E} \in \mathrm{LNM}_X$ with $X = V \times A^n_K[0, 0]$, the map $\partial_n : \mathcal{E} \to \mathcal{E}$ is a morphism in $\mathrm{LNM}_{V \times A^{n-1}_K[0,0]}$, so its kernel is an object in that category. Repeating the argument, we find that $\mathcal{E}_1 = \bigcap_i \ker(\partial_i)$ is an object in LNM_V, which is nonzero because the ∂_i are nilpotent. By Lemma 3.2.14, $\mathcal{E}/\mathcal{E}_1 \in \mathrm{LNM}_X$, so we may repeat to conclude that $\mathcal{E} \in \mathrm{ULNM}_X$.

Lemma 3.2.17. *Let X be a smooth rigid space, and suppose that the zero loci of $t_1, \dots, t_n \in \mathcal{O}(X)$ are smooth and meet transversely. Let $\mathcal{E}$ be a coherent $\mathcal{O}_X$-module equipped with an integrable log-connection with respect to $t_1, \dots, t_n$. Then the following conditions are equivalent:*

(a) *$\mathcal{E}$ is locally free (i.e., is a log-∇-module) and has nilpotent residues;*

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(b) *for each point $x \in X$, there is an affinoid subdomain of X containing x, on which $\mathcal{E}$ admits a filtration whose successive quotients are ∇-modules;*

(c) *for each point $x \in X$, there is an affinoid subdomain of X containing x, on which $\mathcal{E}$ admits a filtration whose successive quotients are* trivial *∇-modules.*

Proof. Note that condition (c) implies (b) trivially, and condition (b) implies (a) by Lemma 3.2.13. It thus remains to show that condition (a) implies (c).

Given (a), pick $x \in X$, and let K' be a finite Galois extension of K containing the residue field of x. By shrinking X, we may further assume that $dt_1, \dots, dt_n$ form a basis of $\Omega^1_{X/K}$ in a neighborhood of X, and that $t_1, \dots, t_n$ all vanish at x. Then by applying Remark 2.3.3 and shrinking X further, we may reduce to the case where X is a polydisc, in which case Lemma 3.2.12 yields the claim over K'. Since the filtration of $\mathcal{E}$ can be chosen canonically (by taking the first step to be the span of all horizontal sections, and so on), it descends from K' to K. Each successive quotient of the result is locally free by Lemma 3.2.14, and becomes trivial over K', hence is also trivial over K: given a spanning set of horizontal sections defined over K', we can decompose over a basis for K' over K to get a spanning set of horizontal sections defined over K. □

Remark 3.2.18. Note that the properties of being constant/unipotent are stable under formation of direct sums, tensor products, and duals; the property of being unipotent is also stable under extensions. When working with ULNM_X (i.e., with $W = \operatorname{Max} K$), one can say more, as follows.

LEMMA 3.2.19. *Let $\mathcal{E}$ be a ∇-module over $A^1_K(I)$ for some closed aligned interval I. If $\mathbf{v}_1, \dots, \mathbf{v}_n \in H^0(A^1_K(I), \mathcal{E})$ are linearly independent over K, then they are linearly independent over $\mathcal{O}(A^1_K(I))$.*

Proof. Suppose the contrary; choose a counterexample with n minimal. Take $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = 0$ with $c_1, \dots, c_n \in \mathcal{O}(A^1_K(I))$; then $c_1, \dots, c_n$ are all nonzero. Since $\mathcal{O}(A^1_K(I))$ is a principal ideal domain, we may divide through to ensure that $c_1, \dots, c_n$ generate the unit ideal; then they are uniquely determined up to a unit in $\mathcal{O}(A^1_K(I))$.

Now observe that

$$\frac{\partial c_1}{\partial t_1}\mathbf{v}_1 + \cdots + \frac{\partial c_n}{\partial t_1}\mathbf{v}_n = 0;$$

consequently, $\partial c_1/\partial t_1, \dots, \partial c_n/\partial t_1$ must equal $c_1, \dots, c_n$ times an element of $\mathcal{O}(A^1_K(I))$. If c_1 vanishes anywhere on $A^1_K(I)$, then $\partial c_i/\partial t_1$ vanishes there to lower order, yielding a contradiction.

Hence c_1 is a unit in $A^1_K(I)$, so we could have taken $c_1 = 1$ to begin with. But in that case, $\partial c_1/\partial t_1, \dots, \partial c_n/\partial t_1$ would all vanish, yielding $c_1, \dots, c_n \in K$ and contradicting the linear independence of $\mathbf{v}_1, \dots, \mathbf{v}_n$ over K. This proves the claim. □

PROPOSITION 3.2.20. *For any smooth rigid space X over K, LNM_X is an abelian tensor category. If $X = V \times A^n_K(I)$ for I a closed aligned interval, then ULNM_X is an abelian tensor subcategory of LNM_X.*

Proof. The fact that LNM_X is an abelian category follows from Lemma 3.2.14. To show that ULNM_X is an abelian tensor subcategory, we must check that the property of being constant/unipotent is preserved by formation of subobjects and quotients within LNM_X; in any given situation, it suffices to check one of the subobjects or quotients, as the other will follow by dualizing. There is no harm in enlarging K, so we may assume that there exists a section $x : V \to X$ of the projection $\pi_1 : X \to V$; we will write x also to mean the image of x.

We first check that, for $n = 1$ and $V = \operatorname{Max} K$, the property of being constant is stable under taking quotients. Suppose $\mathcal{E} \in \mathrm{LNM}_X$ is constant and $g : \mathcal{E} \to \mathcal{E}'$ is a surjection in LNM_X. Let $\mathcal{F}'$ be the image of $H^0(X, \mathcal{E})$ in $\mathcal{E}'$; then the map $\mathcal{F}' \otimes_K \mathcal{O}_X \to \mathcal{E}'$ is surjective by construction, and injective by Lemma 3.2.19. Thus $\mathcal{E}'$ is constant.

We next check that, for any n and V, the property of being constant is stable under taking subobjects. By induction on n, it suffices to check the case $n = 1$ for arbitrary V. Let $\mathcal{E} \in \mathrm{LNM}_X$ be constant, so that there exists $\mathcal{F} \in \mathrm{LNM}_V$ with $\pi_1^*\mathcal{F} \cong \mathcal{E}$; we can identify $\mathcal{F}$ inside $\mathcal{E}$ as the $\pi_1^{-1}\mathcal{O}_V$-span of the horizontal sections. Let $g : \mathcal{E}' \to \mathcal{E}$ be an injection in LNM_X. Let $\mathcal{F}' \in \mathrm{LNM}_V$ be the image of the restriction of g to x.

We wish to show that $\mathcal{E}' = \pi_1^*\mathcal{F}'$, i.e., that the maps $\mathcal{E}' \to \mathcal{E}/(\pi_1^*\mathcal{F}')$ and $\pi_1^*\mathcal{F}' \to \mathcal{E}/\mathcal{E}'$ are zero. Since this is a property that can be checked pointwise on V, it is certainly enough to check on a polydisc around each point of V. If V is itself a polydisc, we may pass to its generic point and check there. We may thus reduce to the case where V is a point, where we already know that $\mathcal{E}$ is constant, and the equality $\mathcal{E}' = \pi_1^*\mathcal{F}'$ is thus straightforward. Hence $\mathcal{E}' = \pi_1^*\mathcal{F}'$ in general, so $\mathcal{E}'$ is constant.

To conclude, we observe that the property of being unipotent is also stable under subobjects and quotients: we may intersect a unipotent filtration with a subobject or project it onto a quotient, and the successive quotients will be constant by the previous paragraph. Hence ULNM_X is indeed an abelian tensor subcategory. □

Remark 3.2.21. One could in principle consider the Tannakian category consisting of the log-∇-modules over $V \times A_K^n(I)$, and reinterpret the constant/unipotent property for a given log-∇-module in terms of the action of the fundamental group on that module. However, in order to produce a fibre functor by consideration of horizontal sections, it is necessary to restrict to modules with a Frobenius structure and invoke a suitable form of the p-adic local monodromy theorem. We may address this point in a subsequent paper.

3.3 Classification of unipotent log-∇-modules

Our next goal is to give a characterization of unipotent log-∇-modules analogous to the pullback definition of constant log-∇-modules. For this we will need a relative analogue of [ClS99, Proposition 1.1.2], whose proof is straightforward.

Lemma 3.3.1. *Under Hypothesis 3.2.1, let I be an aligned subinterval of $[0, +\infty)$, put $X = V \times A_K^n(I)$, and let $\mathcal{D}_{X/W}$ be the noncommutative ring sheaf of (finite order) $\mathcal{O}_W$-linear log-differential operators on X. Let $\mathcal{E}$ and $\mathcal{E}'$ be (left) $\mathcal{D}_{X/W}$-modules on X, with $\mathcal{E}$ coherent and flat over $\mathcal{O}_X$. Then there is a natural isomorphism*

$$\mathrm{Ext}^i_{\mathcal{D}_{X/W}}(\mathcal{E}, \mathcal{E}') \cong \mathbb{H}^i(X, \mathcal{E}^\vee \otimes \mathcal{E}' \otimes \Omega^{\cdot,\log}_{X/W}),$$

where $\mathbb{H}$ denotes hypercohomology.

Using Lemma 3.3.1, we can show that $\mathcal{U}_I$ commutes with the formation of Yoneda Ext groups.

Lemma 3.3.2. *Under Hypothesis 3.2.1, let I be a quasi-open subinterval of $[0, +\infty)$. Then for any $\mathcal{E}, \mathcal{E}' \in \mathrm{ULNM}_{V \times A_K^n[0,0]/W}$, the natural map*

$$\mathrm{Ext}^i(\mathcal{E}, \mathcal{E}') \to \mathrm{Ext}^i(\mathcal{U}_I(\mathcal{E}), \mathcal{U}_I(\mathcal{E}'))$$

is a bijection.

Proof. If

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$$

is a short exact sequence, then by the long exact sequence for Yoneda Exts and the five lemma (and the fact that the map in question is functorial), we can reduce the question of bijectivity from the case of $\mathcal{E}$ and $\mathcal{E}'$ to the cases of $\mathcal{E}_1$ and $\mathcal{E}'$, and of $\mathcal{E}_2$ and $\mathcal{E}'$. Of course one has an analogous reduction given a short exact sequence with $\mathcal{E}'$ in the middle. We may thus reduce to the case where $\mathcal{E}$ and $\mathcal{E}'$ are constant.

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Next, we observe that it suffices to check the case where W is affinoid, as we may deduce the general case by making an admissible affinoid cover of W and using the spectral sequence provided by the corresponding Čech complex. Similarly, we may reduce to the case where V is affinoid.

We may now formally imitate the construction of the Katz–Oda spectral sequence [KO68, Theorem 3] to produce a spectral sequence with

$$E_2^{pq} = H^p(\Gamma(V, \Omega^{\cdot,\log}_{V/W}) \otimes_{\mathcal{O}(V)} \mathbb{H}^q(\mathcal{E}^\vee \otimes \mathcal{E}' \otimes \Omega^{\cdot,\log}_{X/V})) \implies \mathbb{H}^{p+q}(X, \mathcal{E}^\vee \otimes \mathcal{E}' \otimes \Omega^{q,\log}_{X/W}).$$

Namely, it is the spectral sequence associated to the filtration on the $\Omega^{\cdot,\log}_{X/W}$ with

$$\mathrm{Fil}^i(\Omega^{\cdot,\log}_{X/W}) = \mathrm{im}(\Omega^{\cdot-i,\log}_{X/W} \otimes_{\mathcal{O}_X} f^*(\Omega^{i,\log}_{V/W}) \to \Omega^{\cdot,\log}_{X/W})$$

with respect to the derived functors of $\mathbb{R}^0\Gamma(X, \cdot)$.

Using the Katz–Oda spectral sequence (and Lemma 3.3.1 to translate between Ext groups and cohomology), we may argue that it suffices to prove the desired result in the case $V = W$: if each step in the spectral sequence commutes with the application of $\mathcal{U}_I$, then so does the final result. Again by passing from V to a suitable cover, we may reduce to the case where $\mathcal{E}$ and $\mathcal{E}'$ are actually free over $\mathcal{O}$ (and $V = W$); by arguing again using short exact sequences, we may then reduce to the case $\mathcal{E} = \mathcal{E}' = \mathcal{O}$.

To summarize, we have so far reduced to consider the case where $\mathcal{E} = \mathcal{E}' = \mathcal{O}$ and $V = W$. (Note that since $V = W$, the logarithmic structure on V no longer intervenes in the calculation.) Since V is affinoid, $V \times A^n_K(I)$ is a quasi-Stein space by Corollary 3.1.10, so is acyclic for the cohomology of coherent sheaves by Kiehl's theorem [Kie67, Satz 2.4]. Hence the hypercohomology in Lemma 3.3.1 may be computed directly on global sections. With this in mind, we note that the functoriality map $\mathrm{Ext}^i(\mathcal{O}, \mathcal{O}) \to \mathrm{Ext}^i(\mathcal{U}_I(\mathcal{O}), \mathcal{U}_I(\mathcal{O}))$ translates via Lemma 3.3.1 into the map on cohomologies induced by the map on complexes

$$g : \Gamma(V \times \{0\}/V, \Omega^{\cdot,\log}_{V \times A^n_K[0,0]/V}) \to \Gamma(V \times A^n_K(I)/V, \Omega^{\cdot,\log}_{V \times A^n_K(I)/V})$$

induced by the embedding of $\mathcal{O}(V \times A^n_K[0,0])$ into $\mathcal{O}(V \times A^n_K(I))$.

Let h denote the map on complexes obtained from the 'constant coefficient' map $\mathcal{O}(V \times A^n_K(I)) \to \mathcal{O}(V)$ (that is, expanding a function as a Laurent series in $t_1, \ldots, t_n$ and extracting the constant coefficient). Then once we identify V with $V \times \{0\}$, $h \circ g$ becomes the identity map; we claim that $g \circ h$ is homotopic to the identity map. One such homotopy can be reconstructed from the following description on monomials. Given the k-form

$$t_1^{i_1} \cdots t_n^{i_n} \frac{dt_{j_1}}{t_{j_1}} \wedge \cdots \wedge \frac{dt_{j_k}}{j_k} \quad (i_1, \ldots, i_n \in \mathbb{Z}; 1 \leqslant j_1 < \cdots < j_k \leqslant n),$$

pick out the first integer h such that $i_h \neq 0$, and integrate against dt_h/t_h (obtaining zero if h is not among $j_1, \ldots, j_k$); this gives a well-defined operation on the complex because I is quasi-open, so the convergence condition is not disturbed by the integration.

Since g admits a homotopy inverse, it induces bijections on cohomology, as desired. □

Remark 3.3.3. Note that even though only the cases $i = 0, 1$ of Lemma 3.3.2 are needed in what follows, the higher cases are needed in order to apply the five lemma in the induction within the proof of Lemma 3.3.2.

THEOREM 3.3.4. *Under Hypothesis 3.2.1, for any nonempty quasi-open subinterval I of $[0, +\infty)$, the functor $\mathcal{U}_I : \mathrm{ULNM}_{V \times A^n_K[0,0]/W} \to \mathrm{ULNM}_{V \times A^n_K(I)/W}$ is an equivalence of categories.*

Proof. By Lemma 3.3.2 applied in the cases $i = 0$ and $i = 1$ (as in the proof of [Cre98, Proposition 6.7]), $\mathcal{U}_I$ is an equivalence whenever V and W are both affinoid. In general, faithfulness of $\mathcal{U}_I$

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may be checked locally on V and W, so it follows from the affinoid case. Similarly, given faithfulness, full faithfulness may be checked locally; given full faithfulness, essential surjectivity may be checked locally. $\square$

Corollary 3.3.5. *For I quasi-open and $W = \operatorname{Max} K$, the property of an element of $\mathrm{LNM}_{V\times A^n_K(I)/W}$ being constant/unipotent may be checked locally on V.*

Corollary 3.3.6. *Under Hypothesis 3.2.1 with $W = V$, for $\mathcal{E} \in \mathrm{ULNM}_{V\times A^n_K[0,0]/V}$, and I an aligned subinterval of $[0, +\infty)$ of positive length, there is a natural isomorphism*

$$H^0_V(V \times A^n_K[0,0], \mathcal{E}) \cong H^0_V(V \times A^n_K(I), \mathcal{U}_I(\mathcal{E})).$$

Proof. Note that elements of H^0 can be viewed as homomorphisms from the trivial log-∇-module on $V \times A^n_K[0,0]$ (i.e., the sheaf $\mathcal{O}_V$ equipped with n endomorphisms all equal to zero). Hence if $I \subseteq [0,a) \subseteq [0,+\infty)$, then by Theorem 3.3.4, we have a natural isomorphism

$$H^0_V(V \times A^n_K[0,0], \mathcal{E}) \cong H^0_V(V \times A^n_K[0,a), \mathcal{U}_{[0,a)}(\mathcal{E}))$$

inverting the restriction map. We then have another restriction map

$$H^0_V(V \times A^n_K[0,a), \mathcal{U}_{[0,a)}(\mathcal{E})) \to H^0_V(V \times A^n_K(I), \mathcal{U}_I(\mathcal{E}));$$

by Theorem 3.3.4, the composite map

$$H^0_V(V \times A^n_K[0,0], \mathcal{E}) \to H^0_V(V \times A^n_K(I), \mathcal{U}_I(\mathcal{E}))$$

does not depend on the choice of a. This composite map is clearly injective; to see that it is surjective, compose further with the injection $H^0_V(V \times A^n_K(I), \mathcal{U}_I(\mathcal{E})) \to H^0_V(V \times A^n_K(J), \mathcal{U}_J(\mathcal{E}))$ for any $J \subseteq I$ quasi-open and note that the result is an isomorphism by Theorem 3.3.4. $\square$

Remark 3.3.7. Corollary 3.3.6 depends crucially on the nilpotent residues hypothesis; compare Remark 6.3.3.

As a further consequence of Theorem 3.3.4, we can make an argument that allows us to ignore hereafter 'logarithmic structure on the base'.

Proposition 3.3.8. *Let X be a smooth rigid space, and suppose the zero loci of $t_1, \dots, t_n \in \Gamma(X, \mathcal{O})$ are smooth and meet transversely; let U be the complement of these zero loci. Let $\mathcal{E}$ be a log-∇-module with nilpotent residues on X with respect to $t_1, \dots, t_n$, and let $\mathcal{F}$ be a ∇-submodule of the restriction of $\mathcal{E}$ to U. Then $\mathcal{F}$ extends uniquely to a log-∇-submodule of $\mathcal{E}$ with nilpotent residues.*

Proof. By induction, it suffices to check the following. Let Z be the zero locus of t_n. Suppose that Z is irreducible and that $\mathcal{F}$ is a log-∇-submodule with nilpotent residues of the restriction of $\mathcal{E}$ to $X \setminus Z$. Then $\mathcal{F}$ extends uniquely to a log-∇-submodule of $\mathcal{E}$ with nilpotent residues.

Since this claim is local (because of the uniqueness assertion), we may assume further that $\mathcal{E}$ and $\mathcal{F}$ are free, and that (by imitating the construction of [GK00, Proposition 1.3], as was done already in Lemma 3.2.12) there exists an admissible subspace of X containing Z and isomorphic to $Z \times A^1_K[0,a)$ via a map carrying Z to the zero section. Moreover, by Lemma 3.2.8, we may choose a so that $\mathcal{E}$ is unipotent on $Z \times A^1_K[0,a)$. Note that $Z \times A^1_K[0,a)$ and $X \setminus Z$ form an admissible covering of X; it thus suffices to exhibit a unique extension of $\mathcal{F}$ to $Z \times A^1_K[0,a)$. That is, we may as well assume outright that $X = Z \times A^1_K[0,a)$ at this point.

Write $\mathcal{E} = \mathcal{U}_{[0,a)}(\mathcal{G})$ for some $\mathcal{G} \in \mathrm{LNM}_{Z\times A^1_K[0,0]}$. Then $\mathcal{F}$ is a subobject in $\mathrm{LNM}_{Z\times A^1_K(0,a)}$ of the restriction of $\mathcal{E}$ to $\mathrm{ULNM}_{Z\times A^1_K(0,a)}$; by Proposition 3.2.20, $\mathcal{F}$ is itself unipotent on $Z \times A^1_K(0,a)$. That is, we can write $\mathcal{F} = \mathcal{U}_{(0,a)}(\mathcal{H})$ for some $\mathcal{H} \in \mathrm{LNM}_{Z\times A^1_K[0,0]}$. By Theorem 3.3.4, the inclusion $\mathcal{F} \hookrightarrow \mathcal{E}|_{Z\times A^1_K(0,a)}$ is induced by an inclusion $\mathcal{H} \hookrightarrow \mathcal{G}$, so we may take $\mathcal{U}_{[0,a)}(\mathcal{H})$ as the desired extension of $\mathcal{F}$. To establish uniqueness of the extension, note that any two such extensions are both unipotent on some $Z \times A^1_K[0,a)$ by Lemma 3.2.8, so must be isomorphic by Theorem 3.3.4. $\square$

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3.4 Unipotence and generization

We now adapt a recipe from [Ked06, § 5.3] for iteratively constructing horizontal elements of a differential module; it shows that the property of unipotence is 'generic on the base' in a certain sense.

LEMMA 3.4.1. *Let A be an integral affinoid algebra with $V = \operatorname{Max} A$ smooth over K, and let L be a field containing A which is complete for a norm restricting to the spectral seminorm on A. (Note that the existence of L forces the reduction of A to be integral.) Let I be a quasi-open subinterval of $[0, +\infty)$, take $\mathcal{E} \in \mathrm{LNM}_{V \times A_K^n(I)/V}$, and let $\mathcal{F}$ be the induced element of $\mathrm{LNM}_{A_L^n(I)/\operatorname{Max} L}$. If $\mathcal{F}$ is unipotent, then $H_V^0(V \times A_K^n[b,c], \mathcal{E}) \neq 0$ for any closed aligned subinterval $[b,c]$ of I.*

Proof. By Theorem 3.3.4, we can express $\mathcal{F}$ as $\mathcal{U}_I(W)$ for some finite-dimensional vector space W over L equipped with commuting nilpotent endomorphisms $N_1, \dots, N_n$. Let m be the minimal length of a unipotent filtration of W; we can then choose $i_1, \dots, i_{m-1} \in \{1, \dots, n\}$ such that $N_{i_1} \cdots N_{i_{m-1}} \neq 0$ but $N_{i_1} \cdots N_{i_{m-1}} N_i = 0$ for $i = 1, \dots, n$.

Define the sequence of operators D_l on $\mathcal{E}$ as follows:

$$D_l = \prod_{h=1}^{m-1} \left(t_{i_h} \frac{\partial}{\partial t_{i_h}} \right) \prod_{i=1}^{n} \prod_{j=1}^{l} \left(1 - \frac{t_i}{j} \frac{\partial}{\partial t_i} \right)^m \left(1 + \frac{t_i}{j} \frac{\partial}{\partial t_i} \right)^m .$$

Pick a closed aligned subinterval $[d,e]$ of I with $d \leqslant b$ with strict inequality if $b > 0$, and $c < e$. We claim that for $\mathbf{v} \in \Gamma(V \times A_K^n[d,e], \mathcal{E})$, the sequence $D_l(\mathbf{v})$ converges to an element of $H_V^0(V \times A_K^n[b,c], \mathcal{E})$. It suffices to check this in $\Gamma(A_L^n[b,c], \mathcal{F})$, where we can write $\mathbf{v} = \sum_J \mathbf{v}_J t_1^{j_1} \cdots t_n^{j_n}$ for some $\mathbf{v}_J \in W$. In this representation, we have

$$D_l(\mathbf{v}) = \sum_J t_1^{j_1} \cdots t_n^{j_n} (j_{i_1} + N_{i_1}) \cdots (j_{i_{m-1}} + N_{i_{m-1}}) \prod_{i=1}^{n} \prod_{j=1}^{l} \left(1 - \frac{(j_i + N_i)^2}{j^2} \right)^m \mathbf{v}_J. \tag{3.4.2}$$

We now analyze the situation following [Ked06, Lemmas 5.3.1 and 5.3.2]. We may multiply out the summand in (3.4.2) to get a collection of terms, each of which consists of $t_1^{j_1} \cdots t_n^{j_n}$ times a rational number times at most $m-1$ factors from among $\{N_1, \dots, N_n\}$ (repetitions allowed) times $\mathbf{v}_J$. (Remember that the product of any m of the operators $N_1, \dots, N_n$ vanishes, so we can ignore any such product.) There is a unique term with no N, in which the rational number factor is $j_{i_1} \cdots j_{i_{m-1}}$ times the product of the binomial coefficients $\binom{-j_i-1}{l}\binom{j_i+l}{l}$ for $i = 1, \dots, n$; in particular, this factor is an integer. A term with some number $h \leqslant m-1$ of N as factors will have a rational number factor which can be obtained from the integral product we just described by multiplying by some integer and then dividing by h integers, each of absolute value at most $\max\{|j_1|, \dots, |j_n|\} + l$.

This means that, for $g = \lfloor \log_p(\max\{|j_1|, \dots, |j_n|\} + l) \rfloor$ the norms of the terms of the t-adic expansion of $D_l(\mathbf{v}) - N_{i_1} \cdots N_{i_{m-1}} \mathbf{v}_0$ are dominated by the norms of the terms of the sum

$$\sum_{J : |j_1|, \dots, |j_n| > l} \; \sum_N (p^g)^{-m+1} t_1^{j_1} \cdots t_n^{j_n} N \mathbf{v}_J,$$

where N runs over the number of products of at most $m-1$ of the operators $N_1, \dots, N_n$ with repetitions allowed. For each fixed N, if we were to consider the sequence (as l varies) of summands with the factor $(p^g)^{-m+1}$ removed, then Lemma 3.1.11 would force the sequence to be η-null over $X \times A_K^n[b,c]$ for some $\eta > 1$. Putting the factor back in, we obtain the same conclusion by replacing η by any smaller value, since $|p^g|^{-m+1}$ is dominated by ρ^l for any $\rho > 1$.

We conclude that $\{D_l(\mathbf{v}) - N_{i_1} \cdots N_{i_{m-1}} \mathbf{v}_0\}_{l=0}^{\infty}$ is η-null over $V \times A_K^n[b,c]$ for some $\eta > 1$, and so in particular is convergent to zero. Hence the $D_l(\mathbf{v})$ converge to an element of $H_V^0(V \times A_K^n[b,c], \mathcal{E})$,

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and the limit is nonzero if and only if $N_{i_1} \cdots N_{i_{m-1}} \mathbf{v}_0 \neq 0$ (since the map $\Gamma(V \times A_K^n[b,c], \mathcal{E}) \to \Gamma(A_L^n[b,c], \mathcal{F})$ is injective). Let $\mathbf{v}_1, \dots, \mathbf{v}_k$ be a set of generators of $\Gamma(V \times A_K^n[d,e], \mathcal{E})$. If $0 \in I$, take $S = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$; otherwise, let S be the set consisting of elements of $\Gamma(V \times A_K^n[d,e], \mathcal{E})$ of the form $t^J \mathbf{v}_l$ for $J \in \mathbb{Z}^n$ and $l \in \{1, \dots, k\}$. Then as $\mathbf{v}$ runs over S, the resulting values of $\mathbf{v}_0$ must span W over L; in particular, we can choose $\mathbf{v}$ so that $N_{i_1} \cdots N_{i_{m-1}} \mathbf{v}_0 \neq 0$, and so the limit of the $D_l(\mathbf{v})$ is a nonzero element of $H_V^0(V \times A_K^n[b,c], \mathcal{E})$. □

Proposition 3.4.3. *Let A be an integral affinoid algebra with $V = \operatorname{Max} A$ smooth over K, take $x_1, \dots, x_m \in A$ whose zero loci are smooth and meet transversely, and let L be a field containing A which is complete for a norm restricting to the spectral seminorm on A. Let I be a quasi-open subinterval of $[0,1)$, take $\mathcal{E} \in \mathrm{LNM}_{V \times A_K^n(I)}$, and let $\mathcal{F}$ be the induced element of $\mathrm{LNM}_{A_L^n(I)}$. Then $\mathcal{E}$ is constant (respectively, unipotent) if and only if $\mathcal{F}$ is constant (respectively, unipotent).*

Proof. If $\mathcal{E}$ is constant (respectively, unipotent), then clearly $\mathcal{F}$ is constant (respectively, unipotent). We prove the converse by induction on the rank of $\mathcal{E}$.

Let $[b,c]$ be any closed aligned subinterval of I of positive length. By Lemma 3.4.1, $H_V^0(V \times A_K^n[b,c], \mathcal{E})$ is nonzero; if we let V' be the complement on V of the zero loci of $x_1, \dots, x_m$, then it follows that $H_{V'}^0(V' \times A_K^n[b,c], \mathcal{E})$ is also nonzero. By Proposition 3.3.8, the $\mathcal{O}_{V' \times A_K^n[b,c]}$-span of $H_{V'}^0(V' \times A_K^n[b,c], \mathcal{E})$ extends to a subobject $\mathcal{G}$ of $\mathcal{E}$ in $\mathrm{LNM}_{V \times A_K^n[b,c]}$; we will show that $\mathcal{G}$ is constant.

Let $\mathcal{H} \in \mathrm{LNM}_{A_L^n[b,c]}$ be induced by $\mathcal{G}$. Since $\mathcal{F}$ is unipotent and $\mathcal{H}$ injects into $\mathcal{F}$, $\mathcal{H}$ is unipotent by the proof of Proposition 3.2.20. On the other hand, $\mathcal{H}$ is also generated by global sections, namely those coming from $H_{V'}^0(V' \times A_K^n[b,c], \mathcal{E})$, so $\mathcal{H}$ is constant. In particular, $H^0(A_L^n[b,c], \mathcal{H})$ is a finite-dimensional L-vector space and, writing π_L for the structure map $A_L^n[b,c] \to \operatorname{Max} L$, the natural map $\pi_L^* H^0(A_L^n[b,c], \mathcal{H}) \to \mathcal{H}$ is an isomorphism.

For any finitely generated $\mathcal{O}_V$-submodule M of $H_V^0(V \times A_K^n[b,c], \mathcal{G})$, we have a commuting diagram

$$\begin{tikzcd} \pi_1^* M \arrow[r] \arrow[d] & \mathcal{G} \arrow[d] \\ \pi_L^* H^0(A_L^n[b,c], \mathcal{H}) \arrow[r] & \mathcal{H} \end{tikzcd}$$

in which the vertical arrows are visibly injective. We showed above that the lower horizontal arrow is an isomorphism, so the upper horizontal arrow is also injective. Since $\mathcal{G}$ is a finitely generated module over the noetherian sheaf of rings $\mathcal{O}_{V \times A_K^n[b,c]}$, we can choose some M as above, which we call M_1, such that $\pi_1^* M_1$ is maximal among the $\pi_1^* M$. On the other hand, if M_2 were an $\mathcal{O}_V$-submodule of $H_V^0(V \times A_K^n[b,c], \mathcal{G})$ strictly containing M_1, then $\pi_1^* M_2$ would strictly contain $\pi_1^* M_1$. We conclude that $H_V^0(V \times A_K^n[b,c], \mathcal{G}) = M_1$ is finitely generated over $\mathcal{O}_V$, and that $\pi_1^* H_V^0(V \times A_K^n[b,c], \mathcal{G}) = \pi_1^* M_1 \to \mathcal{G}$ is injective.

We next prove that the map $\pi_1^* H_V^0(V \times A_K^n[b,c], \mathcal{G}) \to \mathcal{G}$ is surjective. With notation as in the proof of Lemma 3.4.1, let $f(\mathbf{v})$ denote the limit of the $D_l(\mathbf{v})$. Then for any $\mathbf{v} \in \Gamma(V \times A_K^n[d,e], \mathcal{G})$, we have $f(\mathbf{v}) \in H_V^0(V \times A_K^n[b,c], \mathcal{G})$.

Suppose that $b \neq 0$. Then

$$\mathbf{v} = \sum_{J \in \mathbb{Z}^n} t_1^{j_1} \cdots t_n^{j_n} f(t_1^{-j_1} \cdots t_n^{-j_n} \mathbf{v})$$

as an equality of sections of $\mathcal{G}$ on $V \times A_K^n[b,c]$; this implies that $\mathbf{v} \in \pi_1^* H_V^0(V \times A_K^n[b,c], \mathcal{G})$. Since $\Gamma(V \times A_K^n[d,e], \mathcal{G})$ is dense in $\Gamma(V \times A_K^n[b,c], \mathcal{G})$, this yields the desired surjectivity.

Suppose now that $b = 0$. Before proceeding further, we verify that, for $\alpha \in \Gamma^*$ and $J \in \mathbb{Z}_{\geqslant 0}^n$, an element $\mathbf{x} \in \Gamma(V \times A_K^n[0,\alpha], \mathcal{G})$ is divisible by a monomial t^J if and only if the restriction $\mathbf{x}_L$ of $\mathbf{x}$

to $\Gamma(A_L^n[0,\alpha],\mathcal{H})$ is divisible by t^J; that is, we can check divisibility by t^J from the expansion of $\mathbf{x}$ as a formal series in $t_1,\dots,t_n$. By induction on the sum of the entries of J, it suffices to check the claim for $t^J = t_1$. Write ι, ι_L for the inclusions

$$V \times A_K^{n-1}[0,\alpha] \to V \times A_K^n[0,\alpha], \quad A_L^{n-1}[0,\alpha] \to A_L^n[0,\alpha]$$

into the locus $t_1 = 0$. Write ι^*, ι_L^* for the induced morphisms

$$\Gamma(V \times A_K^n[0,\alpha],\mathcal{G}) \to \Gamma(V \times A_K^{n-1}[0,\alpha],\mathcal{G}), \quad \Gamma(A_L^n[0,\alpha],\mathcal{H}) \to \Gamma(A_L^{n-1}[0,\alpha],\mathcal{H}).$$

Then $\mathbf{x}$ is divisible by t_1 if and only if $\iota^*(\mathbf{x}) = 0$, which is equivalent to $\iota_L^*(\mathbf{x}_L) = 0$ because the restrictions $\Gamma(V \times A_K^*[0,\alpha],\mathcal{G}) \to \Gamma(A_L^*[0,\alpha],\mathcal{H})$ are injective for $* \in \{n-1,n\}$. The latter is equivalent to $\mathbf{x}_L$ being divisible by t_1, as desired.

For J, J', write $J \leqslant J'$ if J' is componentwise greater than or equal to J. Let $J_0, J_1, \dots$ be a total ordering of $\mathbb{Z}^n_{\geqslant 0}$ refining the partial ordering $\leqslant$; write $J_j = (a_{j,1},\dots,a_{j,n})$. Choose a decreasing sequence of aligned intervals

$$[d,e] = [0,e_0] \supset [0,e_1] \supset \cdots$$

satisfying $\bigcap_j [0,e_j] \supseteq [0,c] = [b,c]$. For α equal to one of the e_j, and $k \in \{1,\dots,n\}$, write

$$\hat{\pi}_k : V \times A_K^n[0,\alpha] \to V \times A_K^{n-1}[0,\alpha]$$

for the projection omitting the kth coordinate of $A_K^n[0,\alpha]$. Let $f_k(\mathbf{v})$ denote the limit of the $D_l(\mathbf{v})$ when computed for the projection $\hat{\pi}_k$; then f_k defines a map

$$\Gamma(V \times A_K^n[0,e_j],\mathcal{G}) \to \Gamma(V \times A_K^n[0,e_{j+1}],\mathcal{G}).$$

For $\mathbf{v} \in \Gamma(V \times A_K^n[0,\alpha],\mathcal{G})$, write $\mathbf{v} = \sum_J t^J \mathbf{v}_J$ for the series expansion of $\mathbf{v}$ over $A_L^n[0,\alpha]$; in terms of these series, f_k acts as

$$\sum_{j \geqslant 0} t^{J_j} \mathbf{v}_j \mapsto \sum_{j \geqslant 0, a_{j,k}=0} t^{J_j} \mathbf{v}_j.$$

We check by induction on j that $\sum_{J_j \leqslant J} t^J \mathbf{v}_J \in \Gamma(V \times A_K^n[0,e_j],\mathcal{G})$. This is given for $j = 0$; if $j > 0$, we can choose $k \in \{1,\dots,n\}$ such that $a_{j,k} > 0$, and there is an index $j' < j$ such that

$$J_{j'} = (a_{j,1},\dots,a_{j,k}-1,\dots,a_{j,n}).$$

By the induction hypothesis,

$$\sum_{J_{j'} \leqslant J} t^J \mathbf{v}_J \in \Gamma(V \times A_K^n[0,e_{j'}],\mathcal{G}) \subseteq \Gamma(V \times A_K^n[0,e_{j-1}],\mathcal{G})$$

and $\sum_{J_{j'} \leqslant J} t^J \mathbf{v}_J$ is divisible by $t_k^{a_{j,k}-1}$, since we showed above that we can check this divisibility on the level of formal series. We now have

$$\sum_{J_j \leqslant J} t^J \mathbf{v}_J = \sum_{J_{j'} \leqslant J} t^J \mathbf{v}_J - t_k^{a_{j,k}-1} f_k\left(t_k^{-a_{j,k}+1} \sum_{J_{j'} \leqslant J} t^J \mathbf{v}_J\right) \in \Gamma(V \times A_K^n[0,e_j],\mathcal{G}),$$

completing the induction.

By the previous paragraph, for each j, $\sum_{J_j \leqslant J} t^J \mathbf{v}_J \in \Gamma(V \times A_K^n[0,e_j],\mathcal{G})$, and $\sum_{J_j \leqslant J} t^J \mathbf{v}_J$ is divisible by t^{J_j} because again we can check this divisibility on the level of formal series. We then have

$$\mathbf{v}_{J_j} = f\left(t^{-J_j} \sum_{J_j \leqslant J} t^J \mathbf{v}_J\right) \in H_V^0(V \times A_K^n[0,c],\mathcal{G}).$$

Because the norm on L is compatible with that on V, the sum $\sum_J t^J \mathbf{v}_J$ converges to $\mathbf{v}$, and so $\mathbf{v} \in \pi_1^* H^0_V(V \times A^n_K[0,c], \mathcal{G})$ as in the case $b \neq 0$. Again because the restriction map

$$H^0_V(V \times A^n_K[0,e], \mathcal{G}) \to H^0_V(V \times A^n_K[0,c], \mathcal{G})$$

has dense image, this yields the desired surjectivity.

In either of the cases $b \neq 0$ or $b = 0$, we now see that the map $\pi_1^* H^0_V(V \times A^n_K[b,c], \mathcal{G}) \to \mathcal{G}$ is surjective; since we already showed injectivity, the map is an isomorphism. At this point, there is no harm in replacing K by a finite extension, as what we are checking is local freeness and nilpotence of residues for $H^0_V(V \times A^n_K[b,c], \mathcal{E})$. In particular, we may assume that $A^n_K[b,c]$ contains a K-rational point x.

Writing i for the injection $V \times \{x\} \to V \times A^n_K[b,c]$, we obtain an isomorphism

$$H^0_V(V \times A^n_K[b,c], \mathcal{G}) = i^* \pi_1^* H^0_V(V \times A^n_K[b,c], \mathcal{G}) \cong i^* \mathcal{G}.$$

Consequently, $H^0_V(V \times A^n_K[b,c], \mathcal{G})$ defines an object in LNM_V, and $\mathcal{G} \cong \pi_1^* H^0_V(V \times A^n_K[b,c], \mathcal{G})$ is constant over V.

By the induction hypothesis, we may deduce that the restriction of $\mathcal{E}$ to $V \times A^n_K(b,c)$ in the case $b > 0$, or $V \times A^n_K[0,c)$ in the case $b = 0$, is unipotent over V. Since $[b,c]$ was an arbitrary closed aligned subinterval of I, we deduce by Theorem 3.3.4 that $\mathcal{E}$ is unipotent over all of $V \times A^n_K(I)$, as desired. (If $\mathcal{F}$ is constant, then $\mathcal{E}$ is constant by comparison of residues.) □

Remark 3.4.4. Proposition 3.4.3 even makes a nontrivial assertion when $V = \operatorname{Max} K$, as we may take L to be any extension of K complete under some extension of $|\cdot|$. The assertion is that unipotence can be tested after making an arbitrary base field extension. (As everything involved is K-linear, this should not be surprising; a special case of this was already proved in [Ked04a, Proposition 6.11] using this linearity.)

Corollary 3.4.5. *Let P be a smooth affine formal scheme of finite type over $\mathfrak{o}_K$, suppose $x_1, \dots, x_m \in \Gamma(P, \mathcal{O})$ have zero loci on P_K which are smooth and meet transversely, and let X be an open dense subscheme of P_k. Given a quasi-open subinterval I of $[0,1)$ and an object $\mathcal{E} \in \mathrm{LNM}_{P_K \times A^n_K(I)}$, suppose that the restriction of $\mathcal{E}$ to $]X[\times A^n_K(I)$ is constant/unipotent. Then $\mathcal{E}$ is constant/unipotent.*

Proof. By shrinking X further, we may reduce to the case where $X = P_k \backslash V(g)$ for some $g \in \Gamma(P, \mathcal{O})$. Then $]X[$ is the affinoid space associated to the affinoid algebra $\Gamma(P_K, \mathcal{O})\langle g^{-1} \rangle$; in particular, $\Gamma(P_K, \mathcal{O})$ and $\Gamma(]X[, \mathcal{O})$ have the same completed fraction field L. We may thus apply Proposition 3.4.3 to deduce that $\mathcal{E}$ induces a constant/unipotent ∇-module over $A^n_L(I)$, and then that $\mathcal{E}$ is constant/unipotent over $P_K \times A^n_K(I)$. □

3.5 Unipotence and overconvergent generization

We will also need a variant of the construction of Proposition 3.4.3 in which we allow an 'overconvergent' base. We start with a Gröbner basis calculation derived from [Ked06, § 2.4], but modified to avoid relying on discreteness of K.

Lemma 3.5.1. *For $\lambda \in [1,\infty) \cap \Gamma^*$, let R_λ be the (affinoid) ring of rigid analytic functions on the subspace*

$$|x_1| \leqslant 1, \dots, |x_{n-1}| \leqslant 1,\ |x_n| \leqslant \lambda$$

of the rigid affine n-space over K, and write $|\cdot|_\lambda$ for the supremum norm on R_λ. Let $\mathfrak{a}$ be an ideal of R_δ for some $\delta \in (1,\infty) \cap \Gamma^*$. Then there exists $\rho_0 \in (1,\delta] \cap \Gamma^*$ such that for any $\rho \in (1,\rho_0] \cap \Gamma^*$ and any $y, z \in R_\delta$ with $y - z \in \mathfrak{a}$, one can find $u \in R_\delta$ with

$$u - z \in \mathfrak{a}, \quad |u|_1 \leqslant |y|_1, \quad |u|_\rho \leqslant |z|_\rho.$$

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Proof. If $y = 0$, we may take $u = 0$, so we assume instead that $y \neq 0$. Choose a total ordering $\leqslant$ on $\mathbb{Z}^n_{\geqslant 0}$ extending the partial order $\preceq$ by termwise comparison and the partial order by comparison only in the last component. The former partial order is well founded, so the total ordering is a well ordering.

For $y = \sum y_I x^I \in R_\delta$ and $\lambda \in [1, \delta] \cap \Gamma^*$, define the λ-*leading term* of y to be the expression $y_I x^I$ for I the largest tuple under $\leqslant$ which maximizes $|y_I x^I|_\lambda = |y_I| \lambda^{i_n}$; such a tuple exists because there only finitely many tuples achieving the maximum.

We claim that, for each $y \in R_\delta$, the 1-leading term of y coincides with the ρ-leading term for each sufficiently small $\rho \in (1, \delta] \cap \Gamma^*$ (depending on y). To see this, let $y_I x^I$ be the 1-leading term of y. For each tuple J, we then have either

(a) $|y_J| < |y_I|$, or

(b) $|y_J| = |y_I|$ and $J \leqslant I$; in this case we have $j_n \leqslant i_n$.

If $|y_J x^J|_\delta \leqslant |y_I x^I|_\delta$, then in case (a), we have $|y_J x^J|_\rho < |y_I x^I|_\rho$ for all $\rho \in [1, \delta) \cap \Gamma^*$; in case (b), we have $|y_J x^J|_\rho \leqslant |y_I x^I|_\rho$ and $J \leqslant I$. So these terms are all acceptable for any ρ; in fact, because $y \in R_\delta$, there are only finitely many tuples $J \neq I$ with $|y_J x^J|_\delta > |y_I x^I|_\delta$. For each such J, we must be in case (a), so $|y_J x^J|_\rho < |y_I x^I|_\rho$ for $\rho \in (1, \delta]$ sufficiently small. This yields the claim.

Define elements $a_1, a_2, \ldots$ of $\mathfrak{a}$ as follows. Given $a_1, \ldots, a_{i-1}$, choose a_i if possible to be an element of $\mathfrak{a}$ whose 1-leading term is not a multiple of the 1-leading term of a_j for any $j < i$; otherwise stop. By the well-foundedness of $\preceq$, this process must eventually stop; at that point, every 1-leading term of every element of $\mathfrak{a}$ is a multiple of the 1-leading term of some a_i. Let A be the finite set consisting of the a_i just constructed.

As shown above, we can choose $\rho_0 \in (1, \delta] \cap \Gamma^*$ such that for $\rho \in [1, \rho_0) \cap \Gamma^*$, the 1-leading term and ρ-leading term of each $a \in A$ coincide. Moreover, we can choose $\epsilon \in (0, 1)$ such that for each $a \in A$, if $y_I x^I$ is the 1-leading term of a, then for each J in case (a) above, we actually have $|y_J| \leqslant \epsilon |y_I|$. (Namely, for any particular ϵ, there are only finitely many J contradicting this inequality; by making ϵ large enough, we can eliminate all of these.)

We construct a sequence $\{c_j\}$ of monomials and a sequence $\{d_j\}$ of elements of A as follows. Given the sequences up to c_j and d_j, put $z_j = z - c_1 d_1 - \cdots - c_j d_j$ (or $z_0 = z$ initially). If $|z_j|_1 \leqslant |y|_1$, then stop. Otherwise, let $e_I x^I$ be the 1-leading term of $z_j - y$. By the construction of A, we can find a monomial c_{j+1} and some $d_{j+1} \in A$ such that $c_{j+1} d_{j+1}$ has 1-leading term, and hence ρ-leading term, equal to $e_I x^I$.

From the construction, we clearly have $|z_j|_\rho \leqslant |z|_\rho$. On the other hand, if the process were never to terminate, we could show that $|z_j|_1 \to 0$ as $j \to \infty$ as follows. It would suffice to show that eventually $|z_j|_1 \leqslant \epsilon |z|_1$, as this argument could then be iterated. Let s_j be the set of monomials of z_j of 1-norm greater than $\epsilon |z|_1$. If s_j is nonempty, then s_{j+1} is obtained from s_j by taking out a term of maximal 1-norm and possibly adding back in some other terms of the same 1-norm which are smaller under $\leqslant$. In particular, the set of all possible 1-norms of elements of the s_j is finite; moreover, since $\leqslant$ is a well-ordering, we must eventually run out of terms of any particular 1-norm. Hence eventually s_j becomes empty, and so $|z_j|_1 \leqslant \epsilon |z|_1$.

Again assuming that the process does not terminate, the previous paragraph would imply that $|z_j|_1 \to 0$ as $j \to \infty$. But since we stop whenever $|z_j|_1 \leqslant |y|_1$, this can only happen if $y = 0$, which contradicts an earlier assumption. Thus the process terminates at some z_j, and we may take $u = z_j$. □

PROPOSITION 3.5.2. *Let X be a reduced affinoid space, and take $f \in \mathcal{O}(X)$ with $|f|_X = \delta > 1$. For $\lambda \in [1, \delta] \cap \Gamma^*$, put $U_\lambda = \{x \in X : |f(x)| \leqslant \lambda\}$. Suppose that $U_1 \neq \emptyset$. Then for each $c \in (0, 1) \cap \mathbb{Q}$,* *there exists* $\lambda \in (1, \delta] \cap \Gamma^*$ *such that, for all* $g \in \mathcal{O}(X)$,

$$|g|_{U_\lambda} \leqslant |g|_{U_1}^c |g|_X^{1-c}.$$

Proof. With notation as in Lemma 3.5.1, we can choose a closed immersion $\phi : X \hookrightarrow \operatorname{Max} R_\delta$ which pulls x_n back to f; then $U_\lambda = \phi^{-1}(\operatorname{Max} R_\lambda)$. We then choose ρ_0 as in Lemma 3.5.1.

For each $\lambda \in [1, \delta] \cap \Gamma^*$, the supremum norm on U_λ is equivalent to the quotient norm induced from R_λ. We can thus choose $\epsilon > 1$ such that, for any $g \in \mathcal{O}(X)$, there exist $y, z \in R_\delta$ with

$$\phi^*(y) = \phi^*(z) = g, \quad |y|_1 \leqslant \epsilon |g|_{U_1}, \quad |z|_{\rho_0} \leqslant \epsilon |g|_{U_{\rho_0}}.$$

By Lemma 3.5.1, we can choose $u \in R_\delta$ with

$$\phi^*(u) = g, \quad |u|_1 \leqslant |y|_1, \quad |u|_{\rho_0} \leqslant |z|_{\rho_0}.$$

Now put $\lambda = \rho_0^{1-c}$; by Lemma 3.1.6, we have

$$\begin{aligned} |g|_{U_\lambda} &\leqslant |u|_\lambda \\ &\leqslant |u|_1^c |u|_{\rho_0}^{1-c} \\ &\leqslant |y|_1^c |z|_{\rho_0}^{1-c} \\ &\leqslant \epsilon |g|_{U_1}^c |g|_{U_{\rho_0}}^{1-c} \\ &\leqslant \epsilon |g|_{U_1}^c |g|_X^{1-c}. \end{aligned}$$

Since supremum norms are multiplicative, applying the same argument to g^n instead of g yields

$$|g|_{U_\lambda} \leqslant \epsilon^{1/n} |g|_{U_1}^c |g|_X^{1-c},$$

and the desired result now follows by taking the limit as $n \to \infty$. $\square$

Proposition 3.5.3. *Let P be an affine formal scheme of finite type over $\mathfrak{o}_K$, and let X be an open dense subscheme of P_k such that P is smooth in a neighborhood of X. Take $x_1, \ldots, x_m \in \Gamma(P, \mathcal{O})$ whose zero loci on P_K are smooth and meet transversely. Let I be a quasi-open subinterval of $[0, 1)$, let V be a strict neighborhood of $]X[$ in P_K, and suppose that $\mathcal{E} \in \mathrm{LNM}_{V \times A_K^n(I)}$ becomes constant/unipotent on $]X[\times A_K^n(I)$. Then for any closed aligned subinterval $[b, c] \subset I$ of positive length, there exists a strict neighborhood V' of $]X[$ in P_K such that $\mathcal{E}$ is constant/unipotent over $V' \times A_K^n[b, c]$.*

Proof. We may assume without loss of generality that V is affinoid. Let $[d, e] \subset I$ be a closed aligned subinterval with $[b, c] \subseteq [d, e)$, and with $d < b$ unless $b = 0$. As in the proof of Lemma 3.4.1, we can choose $\mathbf{v} \in \Gamma(V \times A_K^n[d, e], \mathcal{E})$ such that the sequence $\{D_l(\mathbf{v})\}$ converges to a nonzero element of $H^0_{]X[}(]X[\times A_K^n[b, c], \mathcal{E})$. Moreover, from the construction in Lemma 3.4.1, we see that there exists $\eta > 1$ so that the sequence $\{D_{l+1}(\mathbf{v}) - D_l(\mathbf{v})\}$ is η-null over $]X[\times A_K^n[b, c]$.

Suppose W is a connected affinoid subdomain of $V \times A_K^n[d, e]$ over which $\mathcal{E}$ becomes free. Choose a basis $\mathbf{e}_1, \ldots, \mathbf{e}_r$ of $\Gamma(W, \mathcal{E})$, and for $i = 1, \ldots, r$, let A_i be the matrix via which $t_i\, \partial/\partial t_i$ acts on the basis $\mathbf{e}_1, \ldots, \mathbf{e}_r$. Define a system V_λ of strict neighborhoods of $]X[$ in P_K as in Lemma 2.2.8, and let $g_i(\lambda)$ denote the maximum supremum seminorm of any entry of A_i over $W \cap (V_\lambda \times A_K^n[d, e])$. Then we see directly from the definition of D_l that the sequence $\{D_{l+1}(\mathbf{v}) - D_l(\mathbf{v})\}$ is ρ-null over $W \cap (V_\lambda \times A_K^n[d, e])$ for some $\rho > 0$, e.g.,

$$\rho = (\max\{1, g_1(\lambda)\} \cdots \max\{1, g_n(\lambda)\} |p|^{-1/(p-1)})^{-2m}.$$

If W has nonempty intersection with $]X[\times A_K^n[d, e]$, we may apply Proposition 3.5.2 to deduce that the sequence $\{D_{l+1}(\mathbf{v}) - D_l(\mathbf{v})\}$ is 1-null over $W \cap (V_\lambda \times A_K^n[b, c])$ for some $\lambda \in (0, 1) \cap \Gamma^*$. If on the other hand W has empty intersection with $]X[\times A_K^n[d, e]$, then by the maximum modulus principle, W also has empty intersection with $V_\lambda \times A_K^n[d,e]$ for some $\lambda \in (0,1) \cap \Gamma^*$, so there is nothing to check in this case.

Note that we can cover $V \times A_K^n[d,e]$ with finitely many affinoid subdomains W, over each of which $\mathcal{E}$ becomes free. Hence we can choose $\lambda \in (0,1) \cap \Gamma^*$ such that the limit of the $D_l(\mathbf{v})$ exists over $V_\lambda \times A_K^n[b,c]$. Thus $H^0_{V_\lambda}(V_\lambda \times A_K^n[b,c], \mathcal{E}) \neq 0$ for some λ. As in the proof of Proposition 3.4.3, we may obtain a nonzero constant log-∇-submodule of $\mathcal{E}$, quotient by it, and repeat to obtain the desired result. (The role of L in the proof of Proposition 3.4.3 is played by a complete field containing $\mathcal{O}(]X[)$ whose norm is compatible with the norm on $\mathcal{O}(V_\lambda)$.) □

3.6 Convergence and unipotence

Contrary to what one's intuition from real analysis would suggest, a log-∇-module over $V \times A_K^n[0,1)$ with nilpotent residues need not be unipotent; see Remark 3.6.5 below. What distinguishes unipotent log-∇-modules is η-convergence (see Definition 2.4.2), in the following fashion.

LEMMA 3.6.1. *For any smooth affinoid space X, any $a, b \in (0,1) \cap \Gamma^*$ with $a \leqslant b$, and any $\mathcal{E} \in \mathrm{ULNM}_{X \times A_K^n[a,b]/X}$, $\mathcal{E}$ is η-convergent with respect to $t_1, \dots, t_n$ (relative to X) for any $\eta < a$. Moreover, if $\mathcal{E} \in \mathrm{ULNM}_{X \times A_K^n[a,b]}$ and there exists a point $x \in A_K^n[a,b]$ such that the restriction of $\mathcal{E}$ to $X \times \{x\}$ is η-convergent with respect to some coordinate system $z_1, \dots, z_l$ on X and some $\eta < a$, then $\mathcal{E}$ is η-convergent with respect to $t_1, \dots, t_n, z_1, \dots, z_l$.*

Proof. First note that the question is local on X, so we may reduce to the case where $\mathcal{E}$ admits a filtration whose successive quotients are constant and pulled back from free $\mathcal{O}_X$-modules. By Remark 2.4.5, we may assume that $\mathcal{E}$ itself is constant.

Note that the claim in the first instance holds for $\mathcal{E} = \mathcal{O}$ by direct calculation: for any $x \in \mathcal{O}(X \times A_K^n[a,b])$, any tuple $R = (r_1, \dots, r_n) \in [a,b]^n$, and any tuple $I = (i_1, \dots, i_n)$ of nonnegative integers, one has

$$\left| \frac{1}{I!} \frac{\partial^{i_1}}{\partial t_1^{i_1}} \cdots \frac{\partial^{i_n}}{\partial t_n^{i_n}} x \right|_R \leqslant r_1^{-i_1} \cdots r_n^{-i_n} |x|_R,$$

yielding the η-convergence. In particular, $t_1, \dots, t_n$ form an η-admissible coordinate system on $X \times A_K^n[a,b]$ relative to X.

In the second instance, $\mathcal{E}$ is obtained by pullback from a log-∇-module $\mathcal{F}$ on X, which by the given hypothesis is η-convergent with respect to $z_1, \dots, z_l$. The η-convergence of $\mathcal{E}$ follows by the same calculation as in the previous paragraph. □

LEMMA 3.6.2. *Let X be a smooth affinoid space, and take $\mathcal{E} \in \mathrm{LNM}_{X \times A_K^n[0,b]}$ for some $b \in (0,1) \cap \Gamma^*$. Suppose that the restriction of $\mathcal{E}$ to $X \times A_K^n[a,b]$ is η-convergent with respect to $t_1, \dots, t_n$ (relative to X) for some $a \in (0,b) \cap \Gamma^*$ and some $\eta \in (0,a) \cap \Gamma^*$. Then $\mathcal{E}$ is unipotent on $X \times A_K^n[0,\eta)$. Moreover, if all of the residues are zero, then $\mathcal{E}$ is constant on $X \times A_K^n[0,\eta)$.*

Proof. We proceed by induction on n. Write $Y = (X \times A_K^{n-1}[0,b]) \times A_K^1[0,\eta]$. Suppose $\mathcal{F} \in \mathrm{ULNM}_Y$ is a (possibly zero) proper subobject of the restriction of $\mathcal{E}$. Let d be the length of the shortest unipotent filtration of the restriction of the residue of $\mathcal{E}/\mathcal{F}$ along $t_n = 0$. Let $P_j(x)$ denote the jth binomial polynomial, i.e.,

$$P_j(x) = \frac{x(x-1)\cdots(x-j+1)}{j!} \quad (j = 0, 1, \dots).$$

Then an exercise in elementary number theory shows that the $\mathbb{Z}$-module of polynomials with rational coefficients carrying $\mathbb{Z}$ into itself is freely generated by the $P_n(x)$. Moreover, if Q is a polynomial carrying $\mathbb{Z}$ into itself and $Q(0) = \cdots = Q(j-1) = 0$, then Q is an integer linear combination of $P_j, P_{j+1}, \dots, P_{\deg Q}$. (Evaluating at 0 shows that the coefficient of P_0 vanishes; then evaluating at 1

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shows that the coefficient of P_1 vanishes, and so on.) In particular, if we set

$$Q_j(x) = x^{d-1}\left(\frac{(1-x)\cdots(j-x)}{j!}\right)^d,$$

then $Q_{j+1}(x) - Q_j(x)$ is an integer linear combination of $P_{j+1}(x), \dots, P_{dj+d-1}(x)$.

By computing on formal power series in t_n (with which we can formally construct a basis of sections killed by $(t_n\,\partial/\partial t_n)^d$) or invoking Lemma 3.2.8, we see that

$$(Q_{j+1} - Q_j)\left(t_n\frac{\partial}{\partial t_n}\right)$$

carries any element of $\Gamma(Y, \mathcal{E}/\mathcal{F})$ to a multiple of t_n^{j+1} in the same module. That is,

$$\frac{1}{t_n^{j+1}}(Q_{j+1} - Q_j)\left(t_n\frac{\partial}{\partial t_n}\right)$$

is a well-defined operator on $\mathcal{E}/\mathcal{F}$. As we saw above,

$$\frac{1}{t_n^{j+1}}(Q_{j+1} - Q_j)\left(t_n\frac{\partial}{\partial t_n}\right)$$

is a $\mathbb{Z}$-linear combination of

$$\frac{1}{t_n^{j+1}}P_l\left(t_n\frac{\partial}{\partial t_n}\right) \quad (l = j+1, \dots, dj+d-1),$$

and hence is a $\Gamma(Y, \mathfrak{o})$-linear combination of the

$$\frac{1}{t_n^{l}}P_l\left(t_n\frac{\partial}{\partial t_n}\right) \quad \text{for } l = j+1, \dots, dj+d-1.$$

However, we have

$$\frac{1}{t_n^{l}}P_l\left(t_n\frac{\partial}{\partial t_n}\right) = \frac{1}{l!}\frac{\partial^l}{\partial t_n^l}.$$

By the η-convergence condition, for any $\mathbf{w} \in \Gamma(A_K^n[a,b], \mathcal{E})$, the sequence

$$\left\{\frac{1}{j!}\frac{\partial^j}{\partial t_n^j}\mathbf{w}\right\}_{j=1}^{\infty}$$

is η-null on $A_K^n[a,b]$. If we choose $\mathbf{w} \in \Gamma(A_K^n[0,b], \mathcal{E})$, then Lemma 3.1.6 implies that

$$\left\{\frac{1}{j!}\frac{\partial^j}{\partial t_n^j}\mathbf{w}\right\}_{j=1}^{\infty}$$

is also η-null on $A_K^n[0,b]$, so in particular on Y. In particular, if $\mathbf{v}$ denotes the image of $\mathbf{w}$ in $\Gamma(Y, \mathcal{E}/\mathcal{F})$, then the sequence

$$\left\{t_n^{-j-1}(Q_{j+1} - Q_j)\left(t_n\frac{\partial}{\partial t_n}\right)\mathbf{v}\right\}_{j=1}^{\infty} \tag{3.6.3}$$

is η-null on Y; that means that the sequence

$$\left\{(Q_{j+1} - Q_j)\left(t_n\frac{\partial}{\partial t_n}\right)\mathbf{v}\right\}$$

is 1-null on Y. That is, the limit

$$f(\mathbf{v}) = \lim_{j\to\infty} Q_j\left(t_n\frac{\partial}{\partial t_n}\right)\mathbf{v}$$

exists in $\Gamma(Y, \mathcal{E}/\mathcal{F})$.

Again from the formal power series computation, we see that $f(\mathbf{v})$ is killed by $\partial/\partial t_n$; that is, the kernel of $\partial/\partial t_n$ is nonempty. We may now repeat the proof of Proposition 3.4.3, using this last result to replace Lemma 3.4.1 (and inspecting its proof similarly) to produce a nonzero constant subobject $\mathcal{G}$ of $\mathcal{E}/\mathcal{F}$. (The role of L in the proof of Proposition 3.4.3 is played by the completed fraction field of $\mathcal{O}(X \times A_K^{n-1}[0,b])$.) Repeating the argument with $\mathcal{F}$ replaced by the preimage of $\mathcal{G}$ in $\mathcal{E}$, we eventually deduce that $\mathcal{E} \in \mathrm{ULNM}_Y$.

To summarize, we have shown that $\mathcal{E}$ is unipotent on $Y = (X \times A_K^{n-1}[0,b]) \times A_K^1[0,\eta)$ relative to $X \times A_K^{n-1}[0,b]$. Since the restriction of $\mathcal{E}$ to $X \times A_K^{n-1}[0,b] \times \{0\}$ again satisfies the convergence hypothesis (by Lemma 3.1.6 again), we may invoke the induction hypothesis to obtain the desired result. ∎

Remark 3.6.4. The subtlety in the above proof is that the application of Lemma 3.1.6 must be to a sequence without poles; this is why we must apply it to (3.6.3) rather than to the sequence

$$\left\{\frac{1}{j!}\frac{\partial^j}{\partial t_n^j}\mathbf{v}\right\}$$

directly.

Remark 3.6.5. We have already seen (in Lemma 3.2.12 for $X = K$; apply Proposition 3.4.3 to deduce the general case) that without the convergence hypothesis one can only prove that $\mathcal{E}$ is unipotent over $X \times A_K^n[0,a)$ for some $a \in [0,1]$. Indeed, simple examples show that the stronger conclusion of unipotence over $X \times A_K^n[0,1)$ cannot be achieved; for instance, the log-∇-module of rank 1 on $A_K^1[0,1)$ with generator $\mathbf{v}$ satisfying

$$\frac{\partial}{\partial t}\mathbf{v} = \mathbf{v}$$

is only unipotent on $A_K^1[0,|p|^{1/(p-1)})$. (Its horizontal sections are the scalar multiples of $\exp(-t)\mathbf{v}$, and the exponential only converges on the smaller disk.)

Definition 3.6.6. Let X be a smooth rigid space, and let $\mathcal{E}$ be a log-∇-module on $X \times A_K^n[a,1)$ or $X \times A_K^n(a,1)$ for some $a \in [0,1) \cap \Gamma^*$. We say that $\mathcal{E}$ is *convergent* if, for any $\eta \in (0,1)$, there exists $b \in (a,1) \cap \Gamma^*$ such that, for all $c \in [b,1) \cap \Gamma^*$, $\mathcal{E}$ is η-convergent with respect to $t_1,\dots,t_n$ on $X \times A_K^n[b,c]$ (relative to X).

Example 3.6.7. If $\mathcal{E}$ is constant, then it is convergent by Lemma 3.6.1. It follows (from the fact that η-convergence is stable under formation of extensions) that any unipotent log-∇-module is also convergent. It also follows that $t_1,\dots,t_n$ is an η-convergent coordinate system on $X \times A_K^n[b,c]$ (relative to X), so we may check η-convergence of $\mathcal{E}$ on X by just checking η-convergence at a set of generators.

Remark 3.6.8. If $\mathcal{E}$ is the ∇-module over $A_K^1[a,1)$ associated to a finite free module M over $\Gamma(A_K^1[a,1),\mathcal{O})$, then $\mathcal{E}$ is convergent if and only if M is 'soluble at 1' in the terminology of [CM00, 4.1-1]. (See also [CD94, § 2.3], where the notion of 'generic radius of convergence' used in [CM00] is introduced.)

Putting Lemma 3.6.2 together with Theorem 3.3.4 gives us the following characterization of constant/unipotent ∇-modules.

Proposition 3.6.9. *Under Hypothesis 3.2.1 with $W = \operatorname{Max} K$, take $a \in [0,1) \cap \Gamma^*$ and suppose that $\mathcal{E} \in \mathrm{LNM}_{V \times A_K^n(a,1)}$ is convergent. Then $\mathcal{E}$ is unipotent if and only if $\mathcal{E}$ extends to a log-∇-module with nilpotent residues on $V \times A_K^n[0,1)$. Moreover, this extension is unique if it exists, and $\mathcal{E}$ is constant if and only if the residues of $\partial/\partial t_i$ are all zero.*

Proof. If $\mathcal{E}$ is unipotent, then the desired extension exists and is unique thanks to Theorem 3.3.4. Conversely, if $\mathcal{E}$ extends, then the extension is unipotent by Lemma 3.6.2. ☐

Remark 3.6.10. If $\mathcal{E}$ is already known to be isomorphic as an $\mathcal{O}$-module to the pullback of a coherent locally free $\mathcal{O}$-module on V, one may invoke [BC93, Corollary 6.5.2] to give an alternative derivation of Proposition 3.6.9.

4. Monodromy of isocrystals

In this section, we explain what it means for an isocrystal on a smooth variety to have 'constant/unipotent monodromy' along a divisor, and show that one can 'fill in' an overconvergent isocrystal along a divisor of constant monodromy.

4.1 Partial compactifications

Definition 4.1.1. Let X be a k-variety. By a *partial compactification* of X, we will mean a pair (Y, j), where Y is a k-variety and $j : X \hookrightarrow Y$ is an immersion. We do not require that j have dense image, though we will see soon (Remark 4.1.4) that this permissiveness is not so critical. If X is closed in Y (e.g., if $Y = X$ and $j = \mathrm{id}_X$), we say that (Y, j) is a *trivial compactification.* If the closure of X in Y is proper over k (e.g., if Y is proper over k), we say that (Y, j) is a *full compactification.*

Definition 4.1.2. Given a k-variety X and two partial compactifications (Y_i, j_i) of X $(j = 1, 2)$, put $Y_3 = Y_1 \times_k Y_2$; then j_1 and j_2 induce an immersion $j : X \hookrightarrow Y_3$. Let $\overline{X}_i$ denote the Zariski closure of X within Y_i for $i = 1, 2, 3$. We write $(Y_1, j_1) \geqslant (Y_2, j_2)$ if the map $\overline{X}_3 \to \overline{X}_2$ is proper; clearly this relation is a reflexive partial ordering. In particular, we say that (Y_1, j_1) and (Y_2, j_2) are *equivalent* if they are mutually comparable under $\geqslant$. Note that this does indeed give an equivalence relation; moreover, a compactification is trivial/full if and only if it is minimal/maximal under $\geqslant$.

In practice, instead of checking the definition of equivalence directly, we use the following result.

Lemma 4.1.3. *With notation as in Definition 4.1.2, suppose that there exists a proper map $\phi : Y_1 \to Y_2$ such that $j_2 = \phi \circ j_1$. Then (Y_1, j_1) and (Y_2, j_2) are equivalent.*

Proof. The map $\mathrm{id}_{Y_1} \times \phi : Y_1 \to Y_3$ is proper and sections the projection $\pi_1 : Y_3 \to Y_1$; we thus have regular maps $\overline{X_3} \to \overline{X_1}$ and $\overline{X_1} \to \overline{X_3}$, induced by π_1 and $\mathrm{id}_{Y_1} \times \phi$, respectively, which compose both ways to give maps which restrict to the identity map on X. Since X is dense in both $\overline{X_1}$ and $\overline{X_3}$, the compositions really are the identity maps; that is, the induced maps $\overline{X_3} \to \overline{X_1}$ and $\overline{X_1} \to \overline{X_3}$ are isomorphisms.

In particular, $\pi_1 : \overline{X_3} \to \overline{X_1}$ is proper; since $\pi_2 : \overline{X_3} \to \overline{X_2}$ factors as $\phi \circ \pi_1$, it is also proper. This yields the desired equivalence. ☐

Remark 4.1.4. In particular, if (Y, j) is a partial compactification and $\overline{X}$ is the Zariski closure of X within Y, then (Y, j) and $(\overline{X}, j)$ are equivalent, because a closed immersion is proper.

Remark 4.1.5. We have observed previously (Definition 2.6.7) that if (Y_1, j_1) and (Y_2, j_2) are equivalent partial compactifications, and $Y_3 = Y_1 \times_k Y_2$, then the inverse image functors $\mathrm{Isoc}^\dagger(X, Y_1/K) \to \mathrm{Isoc}^\dagger(X, Y_3/K)$ and $\mathrm{Isoc}^\dagger(X, Y_2/K) \to \mathrm{Isoc}^\dagger(X, Y_3/K)$ are equivalences of categories. In other words, the category of isocrystals on X overconvergent along $Y \setminus X$ depends only on the equivalence class of the partial compactification (Y, j).

Since any variety can be covered by open subvarieties which are affine and hence quasi-projective, it will be helpful to know something similar for partial compactifications; the following lemma is a step in this direction.

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Lemma 4.1.6. *Let X be a quasi-projective k-variety. Then for any partial compactification (Y, j) of X, there exists a partial compactification (Y', j') with Y' quasi-projective and a proper map $\phi : Y' \to Y$ such that $j = \phi \circ j'$. In particular, the two partial compactifications (Y, j) and (Y', j') are equivalent.*

Proof. This is precisely the statement (restricted from algebraic spaces to varieties) of the quantitative Chow's lemma of Gruson and Raynaud [GR71, Corollaire 5.7.14]. □

4.2 Smooth varieties and small frames

We now focus attention on isocrystals on smooth varieties; it will be convenient to handle them using a special sort of frame.

Definition 4.2.1. A *small frame* is a frame (X, Y, P, i, j) in which $Y = P_k$, the map i is the identity, and $Y \setminus X$ is the zero locus of some regular function on Y. We will drop Y and i from the notation for a small frame, denoting it by (X, P, j). Note that in any small frame, X must be smooth, since X is open in P_k and P is smooth in a neighborhood of X.

In order to make much use of small frames, we need the following lemma.

Lemma 4.2.2. *Let $j : X \hookrightarrow Y$ be an open immersion of k-varieties, with X dense in Y. Then there exists a blowup $Y' \to Y$ centered in $Y \setminus X$, an open cover $U_1, \ldots, U_n$ of Y', and for $i = 1, \ldots, n$, a partial compactification (Y_i, j_i) of $X \cap U_i$, enclosed by a small frame, such that Y_i admits a proper morphism ϕ_i to $Y' \cap U_i$ with $j = \phi_i \circ j_i$ on $X \cap U_i$. In particular, (Y_i, j_i) is equivalent to $(Y' \cap U_i, j)$.*

Proof. By blowing up in $Y \setminus X$, we may reduce to the case where all components of $Y \setminus X$ have codimension 1 in Y. By then passing to open affine covers, we may reduce to the case where X and Y are affine (and $Y \setminus X$ is still a divisor). By a theorem of Arabia [Ara01, Théorème 1.3.1] (generalizing a theorem of Elkik [Elk73] in the case of K discretely valued), there exists a smooth affine scheme $\tilde{X}$ over $\mathfrak{o}_K$ with $\tilde{X} \times_{\mathfrak{o}_K} k \cong X$. Choose an embedding of $\tilde{X}$ into a projective space $\mathbb{P}^n_{\mathfrak{o}_K}$ and let P be the formal completion along the projective closure of $\tilde{X}$ in $\mathbb{P}^n_{\mathfrak{o}_K}$.

Choose a closed immersion $Y \hookrightarrow \mathbb{A}^l_k$, where the latter has coordinates $x_1, \ldots, x_l$. Then along the rational map $P_k \dashrightarrow Y \hookrightarrow \mathbb{A}^l_k$ induced by the isomorphism between the two copies of X, each of $x_1, \ldots, x_l$ pulls back to a rational function $f_1, \ldots, f_l$ on P_k. For some $m > 0$, these functions can be written as quotients of homogeneous polynomials of degree m (i.e., sections of $\mathcal{O}(m)$); lift these polynomials to homogeneous polynomials of degree m over $\mathfrak{o}_K$. The resulting rational functions define a rational map $P \dashrightarrow \widehat{\mathbb{A}^l_{\mathfrak{o}_K}}$; let P' denote the closure of the graph of this rational map. Then P'_k is a partial compactification of X admitting a proper map to Y, and the complement $P'_k \setminus X$ is the zero locus of a regular function; we can cover P' with affines to obtain the desired small frames. □

Remark 4.2.3. Lemma 4.2.2 may be interpreted as saying that any isocrystal can be described entirely using small frames. However, this does not assert by itself that one can reconstruct the whole theory of isocrystals using only small frames, since functoriality is defined by passing to a restriction from a product frame, which is not small. One could get around this using sophisticated 'lifting lemmas' of the sort given in [Ara01]; this would amount to giving a development of isocrystals from the point of view of Monsky and Washnitzer's 'formal cohomology' (see [MW68] for the construction, and [Brt96, § 2.5] for its relationship to Berthelot's construction). We will not give such a development here.

4.3 Monodromy: a restricted definition

Lemma 4.3.1. *Let A be a noetherian ring, such that A is complete with respect to the x-adic topology for some $x \in A$ not a zero divisor, and let R be a subring of A. Suppose that $B = A/xA$ is* *formally smooth over R. Then there is an isomorphism $A \cong B[\![x]\!]$ sending x to x, whose composition with the quotient $B[\![x]\!] \to B[\![x]\!]/xB[\![x]\!] \cong B$ gives the quotient map $A \to A/xA \cong B$.*

Proof. The proof is as in [Har75, Lemma II.1.2], except that there R is taken to be a field (but the argument does not change). See also [Gro71, Exposé III, 5.6]. □

Hypothesis 4.3.2. Let $X \hookrightarrow Y$ be an open immersion of smooth affine k-varieties, with X dense in Y and $Z = Y \setminus X$ also smooth. Suppose that there exists a small frame (X, P, j) enclosing Y, and that there exists $f \in \Gamma(P, \mathcal{O}_P)$ which cuts out Z within Y, such that df generates a direct summand of Ω^1 in a neighborhood of Z. Let Q be the zero locus of f on P.

Lemma 4.3.3. *Under Hypothesis 4.3.2, there exists an isomorphism $\phi :]Z[_Q \times A^1_K[0,1) \to]Z[_P$.*

Proof. Apply Lemma 4.3.1 to produce an isomorphism $\Gamma(]Z[_Q, \mathfrak{o})[\![t_1]\!] \cong \Gamma(]Z[_P, \mathfrak{o})$. This yields the desired map. (This can also be proved using the strong fibration theorem; compare the proof of Lemma 5.1.1.) □

Definition 4.3.4. Under Hypothesis 4.3.2, let $\mathcal{E}$ be an isocrystal on X overconvergent along $Y \setminus X$. We confound $\mathcal{E}$ with its realization on the small frame $F = (X, P, j)$; the latter is a ∇-module on a strict neighborhood V of $]X[_P$ in $]Y[_P$. Since $]Y[_P = P_K$ is an affinoid space, by Lemma 2.2.8, $V \cap]Z[_P$ contains a subspace of the form

$$\{y \in P_K : |f(y)| \geqslant \lambda\}$$

for some $\lambda \in (0,1) \cap \Gamma^*$. Under ϕ^{-1}, such a space maps to $]Z[_Q \times A^1_K[\lambda, 1)$, so $\mathcal{E}$ restricts to a ∇-module on $]Z[_Q \times A^1_K[\lambda, 1)$, which is convergent thanks to Proposition 2.5.6 (applied with $g = f$). We say that $\mathcal{E}$ has *constant/unipotent monodromy along* Z (with respect to f, ϕ) if $\mathcal{E}$ is constant/unipotent over $]Z[_Q \times A^1_K[\lambda, 1)$ for some $\lambda \in (0,1) \cap \Gamma^*$.

So far, the definition of the phrase '$\mathcal{E}$ has constant/unipotent monodromy along Z' depends on the choices of the frame (X, P, j), the map ϕ, and the function f. To eliminate these dependences, we make the usual argument of passing to a product frame, but since the latter is not a small frame, some care is required.

Proposition 4.3.5. *Under Hypothesis 4.3.2, let (X, P', j') be another small frame satisfying the same hypotheses (with corresponding objects denoted by primes). Let $\mathcal{E}'$ be the realization of $\mathcal{E}$ on (X, P', j'). Then $\mathcal{E}$ has constant/unipotent monodromy along Z if and only if $\mathcal{E}'$ has constant/unipotent monodromy along Z.*

Proof. We first note that by Proposition 3.6.9, $\mathcal{E}$ has constant monodromy along Z if and only if $\mathcal{E}$ extends from some $]Z[_Q \times A^1_K[\lambda, 1)$ to a ∇-module on $]Z[_Q \times A^1_K[0,1)$. Similarly, $\mathcal{E}$ has unipotent monodromy along Z if and only if $\mathcal{E}$ admits a filtration $0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_l = \mathcal{E}$ whose successive quotients extend to ∇-modules on $]Z[_Q \times A^1_K[0,1) =]Z[_P$.

Suppose now that $\mathcal{E}$ has unipotent monodromy along Z. By passing to an affine cover, we may assume that there exist $x_1, \dots, x_m \in \Gamma(P, \mathcal{O}_P)$ and $x'_1, \dots, x'_m \in \Gamma(P', \mathcal{O}_{P'})$ whose differentials generate Ω^1 on P and P', respectively, such that $x_i \equiv x'_i$ as elements of $\Gamma(Y, \mathcal{O}) = \Gamma(P_k, \mathcal{O}) = \Gamma(P'_k, \mathcal{O})$. Put $P'' = P \times P'$, put $j'' = j \times j'$, put $t_i = x_i - x'_i \in \Gamma(P'', \mathcal{O}_{P''})$, and let $\mathcal{E}''$ be the realization of $\mathcal{E}$ on (Y, P'', j''). On the one hand, $\mathcal{E}''$ is isomorphic to the pullback $\pi_1^* \mathcal{E}$ along the projection $P'' \to P$; so on the intersection of $]Z[_{P''}$ with some strict neighborhood of $]X[_{P''}$ in $]Y[_{P''}$, $\mathcal{E}''$ admits a filtration $0 = \mathcal{E}''_0 \subset \mathcal{E}''_1 \subset \cdots \subset \mathcal{E}''_l = \mathcal{E}''$ whose successive quotients extend to ∇-modules on $]Z[_{P''}$. On the other hand, $\mathcal{E}''$ is also isomorphic to the pullback $\pi_2^* \mathcal{E}'$, and in fact we can recover $\mathcal{E}'$ from $\mathcal{E}''$ by restricting to a component of the subspace $t_1 = \cdots = t_m = 0$ of P''. In particular, we obtain a filtration $0 = \mathcal{E}'_0 \subset \mathcal{E}'_1 \subset \cdots \subset \mathcal{E}'_l = \mathcal{E}'$ whose successive quotients

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extend to ∇-modules on $]Z[_{P'} =]Z[_{Q'} \times A^1_K[0,1)$. Hence $\mathcal{E}'$ also has unipotent monodromy along Z. Moreover, if $\mathcal{E}$ actually has constant monodromy along Z, then we can take the filtration of $\mathcal{E}$ to be the trivial one $0 = \mathcal{E}_0 \subset \mathcal{E}_1 = \mathcal{E}$, move it through the above argument, and deduce that $\mathcal{E}'$ has constant monodromy along Z. □

Remark 4.3.6. If $\mathcal{E}$ extends to a convergent isocrystal on Y, then $\mathcal{E}$ has constant monodromy along Z by Proposition 3.6.9. We will prove a converse of this observation; see Theorem 5.2.1.

Remark 4.3.7. As noted in Remark 3.2.21, one could in principle construct a local monodromy representation (along $Y \setminus X$) for an isocrystal on X overconvergent along $Y \setminus X$. We will defer doing so to a subsequent paper.

4.4 Monodromy: a general definition

We now wish to extend the definition of constant/unipotent monodromy; first we make some comments about the existing definition.

Remark 4.4.1. Under Hypothesis 4.3.2, let $\mathcal{E}$ be the realization, on a fixed small frame F, of an isocrystal on X overconvergent along $Z = Y \setminus X$. Then the following are true.

(i) Let $U_1, \dots, U_n$ be an open cover of Y. Then $\mathcal{E}$ has constant/unipotent monodromy along Z if and only if for $i = 1, \dots, n$, the restriction of $\mathcal{E}$ to $U_i \cap X$ has constant/unipotent monodromy along $U_i \cap Z$; this follows from Corollary 3.3.5 applied to the admissible cover $\{]U_i \cap Z[\}$ of $]Z[$.

(ii) Let K' be a field containing K which is complete under an extension of $|\cdot|$. Then $\mathcal{E}$ has constant/unipotent monodromy along Z if and only if this is true after changing the base field to K'; this follows from Proposition 3.4.3.

(iii) Let U be an open subscheme of Y such that $U \cap Z$ is dense in Z. Then $\mathcal{E}$ has constant/unipotent monodromy along Z if and only if the restriction of $\mathcal{E}$, to an isocrystal on $U \cap X$ overconvergent along $U \cap Z$, has constant/unipotent monodromy along $U \cap Z$; this also follows from Proposition 3.4.3, or more precisely from Corollary 3.4.5.

(iv) If $\mathcal{E}$ extends to a convergent isocrystal on Y, then $\mathcal{E}$ has constant monodromy along Z, by Proposition 3.6.9.

Definition 4.4.2. Let $X \hookrightarrow Y$ be an open immersion of smooth k-varieties, and let $\mathcal{E}$ be an isocrystal on X overconvergent along $Z = Y \setminus X$. We say that $\mathcal{E}$ has *constant/unipotent monodromy along* Z if for any extension field k' of k, any field K' containing K which is complete under an extension of $|\cdot|$ with residue field k', and any small frame (U, P, j) over K enclosing an open subset $V = P_k$ of Y (with $U = V \cap X$) which satisfies Hypothesis 4.3.2 (i.e., $V \setminus U$ is smooth and is the zero locus of some $f \in \Gamma(P, \mathcal{O}_P)$), the realization of $\mathcal{E}$ on (U, P, j) has constant/unipotent monodromy along $V \setminus U$. By virtue of Remark 4.4.1, this agrees with Definition 4.3.4 when they both apply; also, the analogue of Remark 4.4.1 holds for this expanded definition.

Remark 4.4.3. The checking over extension fields is only necessary when k is imperfect: when k is perfect, Z (being reduced, thanks to our running hypothesis that all k-varieties are reduced) is generically smooth, so we may sample on a suitable open subset of Y without enlarging k. However, if k is imperfect, then Z may fail to be geometrically reduced, and one must extend k in order to guarantee that the underlying reduced subscheme is generically smooth. This will require us to do a bit of work in the case of k imperfect in order to complete the proof of the extension theorem (Theorem 5.2.1).

An important property of the definition of constant/unipotent monodromy is its 'codimension 1 nature'.

Proposition 4.4.4. *Let $U \hookrightarrow X \hookrightarrow Y$ be open immersions of smooth k-varieties, such that $Y \setminus X$ has codimension at least 2 in Y. Let $\mathcal{E}$ be an isocrystal on U overconvergent along $Y \setminus U$. Then $\mathcal{E}$ has constant/unipotent monodromy along $Y \setminus U$ if and only if $\mathcal{E}$ has constant/unipotent monodromy along $X \setminus U$.*

Proof. There is no harm in shrinking U so that $Y \setminus U$ becomes purely of codimension 1, as $\mathcal{E}$ automatically has constant monodromy along any added component. In this case, $X \setminus U$ is dense in $Y \setminus U$, so we obtain the desired equivalence as in Remark 4.4.1. □

5. Monodromy and extensions

In this section, we clarify the relationship between extendability of an isocrystal and the property of having constant monodromy along some boundary variety.

5.1 An extension lemma

We now prove a lemma about extending ∇-modules in a key geometric setting. To avoid having to repeat effort, we set up the lemma so that it also handles log-∇-modules with nilpotent residues; hence the somewhat complicated statement.

Lemma 5.1.1. *Let $V \hookrightarrow U \hookrightarrow X \hookrightarrow Y$ be open immersions of k-varieties such that X is smooth, V is dense in Y, $X \setminus V$ is a strict normal crossings divisor on X, and $X \setminus U$ is a single component of $X \setminus V$. Suppose further that there exist:*

(i) *a small frame $F = (X, P, j)$ enclosing (X, Y);*

(ii) *functions $f_1, \ldots, f_r \in \Gamma(P, \mathcal{O}_P)$ whose zero loci cut out the components of the closure of $X \setminus V$ in Y, with f_1 cutting out $X \setminus U$;*

(iii) *functions $f_{r+1}, \ldots, f_n \in \Gamma(P, \mathcal{O}_P)$ such that $df_1, \ldots, df_n$ freely generate Ω^1 in a neighborhood of X;*

(iv) *a function $g \in \Gamma(P, \mathcal{O}_P)$ whose zero locus cuts out $Y \setminus X$ within Y.*

Then the following results hold.

(a) *Let $\mathcal{E}$ be a ∇-module on a strict neighborhood of $]U[_P$ in $]Y[_P = P_K$ representing an isocrystal on U overconvergent along $Y \setminus U$. Then $\mathcal{E}$ has constant monodromy along $X \setminus U$ if and only if $\mathcal{E}$ extends to an isocrystal on X overconvergent along $Y \setminus X$.*

(b) *Let $\mathcal{E}$ be a log-∇-module with nilpotent residues on a strict neighborhood of $]U[_P$ in P_K with respect to $f_1, \ldots, f_r$, whose restriction to a strict neighborhood of $]V[_P$ in P_K represents an isocrystal on V overconvergent along $Y \setminus V$. Then $\mathcal{E}$ has unipotent monodromy along $X \setminus U$ if and only if $\mathcal{E}$ extends to a log-∇-module with nilpotent residues on a strict neighborhood of $]X[_P$ in P_K with respect to $f_1, \ldots, f_r$.*

(c) *In both cases (a) and (b), the implied restriction functor is fully faithful: that is, morphisms between $\mathcal{E}$ and $\mathcal{E}'$ always uniquely induce morphisms on their extensions.*

Proof. Let P' be the zero locus of f_1 on P. Let F' be the frame $(X \setminus U, P', j')$, and let $f'_2, \ldots, f'_n$ be the restrictions of $f_2, \ldots, f_n$ to P'. Put $Z = Y \setminus U$. By the strong fibration theorem (Proposition 2.2.9), there exists a strict neighborhood of $]X \setminus U[_{P \times P'}$ in $]Z[_{P \times P'}$ isomorphic on the one hand to a strict neighborhood V_1 of $]X \setminus U[_{\widehat{P \times \mathbb{A}^{n-1}}} \cong]X \setminus U[_P \times A_K^{n-1}[0,1)$ in $]Z[_{\widehat{P \times \mathbb{A}^{n-1}}} =]Z[_P \times A_K^{n-1}[0,1)$ via the functions $f_2 - f'_2, \ldots, f_n - f'_n$, and on the other hand to a strict neighborhood V_2 of $]X \setminus U[_{P' \times \widehat{\mathbb{A}^n}} =]X \setminus U[_{P'} \times A_K^1[0,1) \times A_K^{n-1}[0,1)$ in $]Z[_{P' \times \widehat{\mathbb{A}^n}} =]Z[_P \times A_K^1[0,1) \times A_K^{n-1}[0,1)$ via the functions $f_1, f_2 - f'_2, \ldots, f_n - f'_n$. If we restrict the resulting isomorphism $V_1 \to V_2$ to the inverse image

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of $0 \in A_K^{n-1}[0,1)$ in both factors, we get an isomorphism between a strict neighborhood of $]X \setminus U[_P$ in $]Z[_P$ with a strict neighborhood of $]X \setminus U[_{P'} \times A_K^1[0,1)$ in $]Z[_{P'} \times A_K^1[0,1)$, whose composition with the projection $]Z[_{P'} \times A_K^1[0,1) \to A_K^1[0,1)$ is precisely f_1.

By assumption, $\mathcal{E}$ is defined on some subset of P_K of the form

$$V_\lambda = \{x \in P_K : |f_1(x)| \geqslant \lambda, |g(x)| \geqslant \lambda\}$$

with $\lambda \in (0,1) \cap \Gamma^*$, and its restriction to $V_\lambda \cap]X \setminus U[_P$ is in case (a) a constant ∇-module and in case (b) a unipotent log-∇-module. Now pass $\mathcal{E}$ over to a strict neighborhood of $]X \setminus U[_{P'} \times A_K^1[0,1)$ in $]Z[_{P'} \times A_K^1[0,1)$; then for each closed subinterval $[a,b]$ of $(\lambda,1)$, $\mathcal{E}$ is defined on $V_0 \times A_K^1[a,b]$ for some strict neighborhood V_0 of $]X \setminus U[_{P'}$ in $]Z[_{P'}$. By Proposition 3.5.3, there exists another strict neighborhood V_1 of $]X \setminus U[_{P'}$ in $]Z[_{P'}$ such that $\mathcal{E}$ becomes constant/unipotent on $V_1 \times A_K^1[a,b]$. By Theorem 3.3.4, this restriction of $\mathcal{E}$ extends in case (a) to a ∇-module, or in case (b) to a log-∇-module with nilpotent residues, on $V_1 \times A_K^1[0,b]$, which we may glue with the original $\mathcal{E}$ to extend it to a strict neighborhood of $]X[_P$ in $]Y[_P$. The assertion of case (c) follows from Corollary 3.3.6.

Finally, we check the overconvergence of the extension in case (a), by verifying the condition of Proposition 2.5.6; that is, we claim that our extension is η-convergent with respect to $f_1, f_2, \dots, f_n$ on some affinoid strict neighborhood of $]X[_P$ in $]Y[_P$ (which may depend on η). We need only verify the η-convergence condition for each of a set of generating sections; by Proposition 2.5.6, we already know this on some V_λ. Now run the aforementioned construction for a choice of $[a,b]$ with $\eta < a$. Then the fact that $\mathcal{E}$ is constant on $V_1 \times A_K^1[a,b]$ means (by Lemma 3.6.1) that the extension of $\mathcal{E}$ to $V_1 \times A_K^1[0,b]$ is η-convergent. This yields η-convergence of the extension of $\mathcal{E}$ to a strict neighborhood of $]X[_P$ in $]Y[_P$, as desired. □

5.2 Extension of overconvergent isocrystals

With Lemma 5.1.1 in hand, we can now prove a definitive theorem about extending overconvergent isocrystals.

THEOREM 5.2.1. *Let $U \hookrightarrow X \hookrightarrow Y$ be open immersions of k-varieties, such that X is smooth and U is dense in Y. Let $\mathcal{E}$ be an isocrystal on U overconvergent along $Y \setminus U$. Then $\mathcal{E}$ has constant monodromy along $X \setminus U$ if and only if $\mathcal{E}$ extends to an isocrystal on X overconvergent along $Y \setminus X$. Moreover, the functor $\mathrm{Isoc}^\dagger(X, Y/K) \to \mathrm{Isoc}^\dagger(U, Y/K)$ is fully faithful, so the extension is unique if it exists.*

Proof. As in Remark 4.3.6, if $\mathcal{E}$ extends, it must have constant monodromy along $X \setminus U$. We will prove the converse and the full faithfulness under several sets of hypotheses, culminating in the unrestricted form.

To begin with, suppose that $X \setminus U$ is a smooth divisor on X. By applying Lemma 4.2.2 (allowing Y to be replaced by a blowup centered in $Y \setminus X$), then passing to an open cover of Y and replacing each open subset of Y by an equivalent partial compactification (of the subset of X it contains), we may reduce the desired assertion to a collection of instances of Lemma 5.1.1, in which we fill in one component of $X \setminus U$ at a time. (Note that part (c) of the lemma ensures that the extensions produced can be glued back together.)

Next, suppose that k is perfect but U, X, Y are not further restricted. If $X \setminus U$ is nonempty, we can find a smooth closed point x on (the reduced subscheme underlying) $X \setminus U$, since the latter is also geometrically reduced. Let Z be the unique component of $X \setminus U$ passing through x, and let D be an irreducible divisor of X containing Z which is smooth in a neighborhood V of x. (For instance, choose functions $t_1, \dots, t_r$ cutting out Z within X whose differentials form part of a basis of Ω^1 in a neighborhood of x, then take D to be the component of the zero locus of t_1 passing through x.) Then either $D = Z$, or $D \setminus Z$ is dense in D. In either case, the restriction of $\mathcal{E}$ to $X \setminus D$

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has constant monodromy along D: in the former case this is by hypothesis, whereas in the latter case this is automatic.

Let Z' be the union of the components of $X \setminus U$ other than Z, together with the nonsmooth locus of D. By the previously treated case, $\mathcal{E}$ extends to an isocrystal on $X \setminus Z'$ overconvergent along $Y \setminus (X \setminus Z')$, and the corresponding restriction functor is fully faithful. Since $x \in X \setminus Z'$, we may glue to obtain an extension of $\mathcal{E}$ to an open subset of X which is strictly larger than U. By noetherian induction, repeating this process eventually yields an extension of $\mathcal{E}$ to X and the full faithfulness of the restriction functor.

Finally, suppose that k is arbitrary. In this case, we can still run the previous argument at the expense of replacing k by a finite radicial extension. It thus suffices to show the following: suppose that $K' = K(y^{1/p})$ for some $y \in \mathfrak{o}_K$ whose image in k is not a pth power, and that the assertion of the theorem holds for U, X, Y over K'. Then it also holds for U, X, Y over K. (Namely, with this result in hand, we can enlarge the residue field from k to any desired finite radicial extension by a sequence of such extensions of K, then back down the tower to deduce the theorem.)

Since everything under consideration is local, we may assume thanks to Lemma 4.2.2 that (X, Y) is enclosed by a small frame (X, P, j). Take $\mathcal{E} \in \mathrm{Isoc}^\dagger(U, Y/K)$ with constant monodromy along $X \setminus U$. For V an affinoid strict neighborhood of $]U[$ in $]Y[$, put $A_V = \Gamma(V, \mathcal{O})$ and $M_V = \Gamma(V, \mathcal{E})$. For W an affinoid strict neighborhood of $]X[$ in $]Y[$, put $B_W = \Gamma(W, \mathcal{O})$. For everything in sight, insert a prime to denote tensoring with K' over K. We have (by applying the theorem over K') that, for some affinoid strict neighborhood V of $]U[$ in $]Y[$, there exists an affinoid strict neighborhood W of $]X[$ in $]Y[$ containing V and a finitely generated B'_W-submodule N'_W of M'_V, stable under ∇ and satisfying $N'_W \otimes_{B'_W} A'_V = M'_V$. By the full faithfulness of restriction from X to U over K', N'_W is uniquely determined by these conditions.

Put $N_W = N'_W \cap M_V$; then N_W is a B_W-submodule of M_V which is stable under ∇. We will show that N_W is finitely generated and that $N_W \otimes_{B_W} A_V = M_V$. It suffices to check this after enlarging K and K' to contain a primitive pth root of unity ζ_p (since $K(\zeta_p)$ and K' are linearly disjoint over K, by the hypothesis on y). In this case, K' becomes Galois with group $G = \mathrm{Gal}(K'/K)$, which we identify with $\mathbb{Z}/p\mathbb{Z}$ by declaring that $e \in \mathbb{Z}/p\mathbb{Z}$ carries $y^{1/p}$ to $\zeta_p^e y^{1/p}$.

Thanks to Proposition 2.6.1 and the fact that G acts trivially modulo $\mathfrak{m}_K$, we obtain a canonical action of G on M'_V with invariants M_V (at least after shrinking V, which is harmless). By the uniqueness of N'_W, N'_W also carries an action of G. For $i = 0, \ldots, p-1$ and $\mathbf{v} \in M'_V$, set

$$f_i(\mathbf{v}) = (y^{1/p})^{-i} \sum_{e \in \mathbb{Z}/p\mathbb{Z}} \zeta_p^{-ei} \mathbf{v}^e.$$

Then each f_i carries M'_V into M_V, and so carries N'_W into N_W.

It is clear that the natural map $N_W \otimes_K K' \to N'_W$ is injective. On the other hand, for $\mathbf{v} = \sum_{l=0}^{p-1} (y^{1/p})^l \mathbf{v}_l \in N'_W$, with each $\mathbf{v}_l \in M_V$, we have $\mathbf{v}_l = p^{-1} f_l(\mathbf{v}) \in N_W$ as in the previous paragraph. Hence $N_W \otimes_K K' \to N'_W$ is also surjective, so

$$(N_W \otimes_{B_W} A_V) \otimes_K K' = N'_W \otimes_{B'_W} A'_V = M'_V = M_V \otimes_K K'$$

and so $N_W \otimes_{B_W} A_V = M_V$ by Galois descent.

Moreover, if $\mathbf{w} \in M_V$ and $\mathbf{v}_j \in M'_V$ satisfy $\sum b_j \mathbf{v}_j = \mathbf{w}$ for some $b_j \in B'_W$, write $b_j = \sum_{l=0}^{p-1} b_{j,l} (y^{1/p})^{-l}$ with $b_{j,l} \in A_V$ (respectively, $b_{j,l} \in B_W$); we then have

$$\begin{aligned} p\mathbf{w} &= f_0(\mathbf{w}) \\ &= \sum_j \sum_{e \in \mathbb{Z}/p\mathbb{Z}} b_j^e \mathbf{v}_j^e \end{aligned}$$

$$= \sum_j \sum_{e \in \mathbb{Z}/p\mathbb{Z}} \sum_{l=0}^{p-1} b_{j,l} \zeta_p^{-el} (y^{1/p})^{-l} \mathbf{v}_j^e$$
$$= \sum_j \sum_{l=0}^{p-1} b_{j,l} f_l(\mathbf{v}_j).$$

That is, $\mathbf{w}$ is also a B_W-linear combination of the $f_l(\mathbf{v}_j)$. Consequently, given any finite set of generators of N'_W over B'_W which also generate M'_V over A'_V, their images under all of the f_i generate N_W over B_W.

Since N_W is finitely generated and $N_W \otimes_{B_W} A_W = M_V$, we can extend $\mathcal{E}$ to a ∇-module on W; its overconvergence can be checked after tensoring with K'. Thus $\mathcal{E}$ extends to an element of $\mathrm{Isoc}^\dagger(X, Y/K)$.

To obtain the extension of horizontal sections, suppose $\mathbf{v} \in M_V$ is horizontal. Then on the one hand $\mathbf{v} \in N'_W$ by the assertion of the theorem over K'; on the other hand, $\mathbf{v}$ is G-invariant. Hence $\mathbf{v} \in N_W$, i.e., $\mathbf{v}$ extends to X as desired. □

Remark 5.2.2. The full faithfulness of restriction to an open subscheme generalizes a result of Étesse [Ete02, Théorème 4], by eliminating the restrictions that K be discretely valued and that the isocrystals carry Frobenius structures. On the other hand, the extension criterion seems to be new in essentially all cases except perhaps on curves (where it is straightforward).

5.3 Consequences of overconvergent extension

Before proceeding to the logarithmic situation, we pause to record some consequences of Theorem 5.2.1. Some of these may be of independent interest.

We first give a result about extending subisocrystals.

PROPOSITION 5.3.1. *Let $U \hookrightarrow X \hookrightarrow Y$ be open immersions of k-varieties, such that X is smooth and U is dense in Y. Let $\mathcal{E}$ be an isocrystal on X overconvergent along $Y \setminus X$, and let $\mathcal{F}$ be a subisocrystal of $\mathcal{E}$ over U overconvergent along $Y \setminus U$. Then $\mathcal{F}$ is the restriction to U of a subisocrystal of $\mathcal{E}$ over X overconvergent along $Y \setminus X$.*

Proof. By Theorem 5.2.1, $\mathcal{E}$ has constant monodromy along $X \setminus U$, as then does $\mathcal{F}$ by Proposition 3.2.20, so $\mathcal{F}$ extends to an isocrystal $\mathcal{G}$ on X overconvergent along $Y \setminus X$. By the full faithfulness component of Theorem 5.2.1, the inclusion $\mathcal{G} \hookrightarrow \mathcal{E}$ extends from U to X. This yields the desired result. □

Remark 5.3.2. This situation should be contrasted with the situation that arises when proving that the forgetful functor from overconvergent to convergent F-isocrystals (isocrystals with Frobenius structures; see Definition 7.1.1) is fully faithful, as in [Ked04b]. There one does not have an analogue of Proposition 5.3.1, as an overconvergent F-isocrystal can have nonconstant convergent subcrystals that do not descend to the overconvergent category. For instance, if $f : X \to B$ is the Legendre family of elliptic curves minus the supersingular fibres, then $R^1 f_* \mathcal{O}_X$ is a rank 2 overconvergent F-isocrystal on B which has a unit-root subobject in the convergent category, but not in the overconvergent category. (If it had a unit-root subobject in the overconvergent category, then by Proposition 5.3.1, it would also have a unit-root subobject even if the supersingular fibres were not excluded, which is absurd.)

We next observe that isocrystals extend across holes of codimension at least 2.

PROPOSITION 5.3.3. *Let $U \hookrightarrow X \hookrightarrow Y$ be open immersions of k-varieties, such that X is smooth, U is dense in Y, and $X \setminus U$ has codimension at least 2 in X. Then the restriction functor $\mathrm{Isoc}^\dagger(X, Y/K) \to \mathrm{Isoc}^\dagger(U, Y/K)$ is an equivalence of categories.*

Proof. The restriction functor is fully faithful by Theorem 5.2.1, so we must show that it is essentially surjective. Let $\mathcal{E}$ be an isocrystal on U overconvergent along $Y \setminus U$. Then applying Proposition 4.4.4 shows that $\mathcal{E}$ has constant monodromy along $X \setminus U$ if and only if it has constant monodromy along the empty scheme. The latter is vacuously true, so $\mathcal{E}$ extends to X. This yields the desired essential surjectivity. □

Remark 5.3.4. The restriction that X be smooth is critical, just as the regularity restriction is critical in the Zariski–Nagata purity theorem; one can construct counterexamples in the nonsmooth case much as in the algebraic de Rham setting, e.g., by taking the rank 1 ∇-module defined by $\nabla(\mathbf{v}) = \mathbf{v} \otimes dx/2x$ on the surface $z^2 = xy$ away from $x = y = z = 0$. On the other hand, Grothendieck [Gro71, Exposé X, Théorème 3.1] gives another form of the purity theorem which we are unable to analogize using our techniques; we leave it as a question.

Question 5.3.5. Let $U \hookrightarrow X \hookrightarrow Y$ be open immersions of k-varieties, such that X is a local complete intersection, U is dense in Y, and $X \setminus U$ has codimension at least 3 in X. Is the restriction functor $\mathrm{Isoc}^\dagger(X, Y/K) \to \mathrm{Isoc}^\dagger(U, Y/K)$ an equivalence of categories? This has been verified explicitly in some special cases by Tsuzuki (private communication).

Using Proposition 5.3.3, we can analogize the invariance of the algebraic fundamental group under a blowup.

Proposition 5.3.6. *Let $f : Y \to X$ be a proper birational morphism of smooth k-varieties, and let $\mathcal{E}$ be an overconvergent isocrystal on Y. Then there exists an overconvergent isocrystal $\mathcal{F}$ on X such that $\mathcal{E} \cong f^*\mathcal{F}$.*

Proof. Since f is birational, there is an open subset U of X, whose complement has codimension at least 2 in X, on which f is an isomorphism. The restriction of $\mathcal{E}$ to U extends to an overconvergent isocrystal $\mathcal{F}$ on X by Proposition 5.3.3; the isomorphism $\mathcal{E} \cong f^*\mathcal{F}$ over U extends to X by the full faithfulness aspect of Theorem 5.2.1. □

Finally, we give a result to the effect that 'overconvergence is contagious'.

Proposition 5.3.7. *Let $U \hookrightarrow X \hookrightarrow Y$ be open immersions of k-varieties, such that X is smooth and U is dense in Y. Let $\mathcal{E}$ be a convergent isocrystal on X whose restriction to $\mathrm{Isoc}^\dagger(U, X/K)$ is isomorphic to the restriction of an isocrystal on U overconvergent along $Y \setminus U$. Then $\mathcal{E}$ itself is the restriction to $\mathrm{Isoc}(X/K)$ of an isocrystal on X overconvergent along $Y \setminus X$.*

Proof. Let $\mathcal{F}$ be an isocrystal on U overconvergent along $Y \setminus U$ whose restriction to $\mathrm{Isoc}^\dagger(U, X/K)$ is isomorphic to the restriction of $\mathcal{E}$. Then $\mathcal{F}$ has constant monodromy along $X \setminus U$, so by Theorem 5.2.1 it extends to an isocrystal $\mathcal{G}$ on X overconvergent along $Y \setminus X$. If we compare $\mathcal{E}$ and the restriction of $\mathcal{G}$ to $\mathrm{Isoc}(X/K)$, we see that they become isomorphic in $\mathrm{Isoc}^\dagger(U, X/K)$; by the full faithfulness aspect of Theorem 5.2.1, they are isomorphic in $\mathrm{Isoc}(X/K)$. This yields the desired result. □

Remark 5.3.8. Proposition 5.3.7 seems tantalizingly close to, but distinct from, a result of Matsuda and Trihan [MT04, Theorem 1]. The latter says (with more restrictive hypotheses, namely discreteness of K and presence of a Frobenius structure) that, on a curve, whether a convergent isocrystal is overconvergent can be checked locally. It would be interesting to give a higher-dimensional analogue of the result of Matsuda and Trihan; our methodology is unsuited for this, as we must have some sort of global overconvergence in order to make any monodromy constructions.

Remark 5.3.9. If one knew that restriction from $\mathrm{Isoc}^\dagger(U, X/K)$ to $\mathrm{Isoc}(U/K)$ were fully faithful, one could perform the comparison in Proposition 5.3.7 in $\mathrm{Isoc}(U/K)$ instead. By [Ked04b, Theorem 1.1] this full faithfulness is known under some additional restrictions: K must be discretely valued, X

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must be proper (so that $\mathrm{Isoc}^\dagger(U, X/K) = \mathrm{Isoc}^\dagger(U/K)$), and one must consider isocrystals with Frobenius structures. (Strictly speaking, that work [Ked04b, Theorem 1.1] only extends morphisms which commute with the Frobenius structures, but it is not difficult to remove that restriction.)

6. Logarithmic extensions

We now turn to the problem of extending isocrystals into log-isocrystals. The context in which we will do this is the work of Shiho [Shi00, Shi02], which constructs categories of 'convergent log-isocrystals' analogous to the convergent isocrystals of Berthelot and Ogus; indeed, the bulk of this section will be spent reviewing foundational aspects of logarithmic structures on schemes, then making explicit one of Shiho's constructions for a smooth pair (a smooth variety equipped with a strict normal crossings divisor).

In principle, our methods can also be used to construct 'overconvergent log-isocrystals'; the trouble is that there is no analogue of Shiho's work to use as the foundation. Since building such a foundation is somewhat orthogonal to our present purposes, we will not do so here; see Remark 6.4.3 for further discussion.

CONVENTION 6.0.1. We continue to assume that the field K has characteristic 0 and residue field k. However, throughout this section, we also assume that K is complete with respect to a *discrete* valuation; this is in order to invoke Shiho's results. Also, 'locally' on a scheme or formal scheme (e.g., in the notion of a sheaf) will always mean locally for the Zariski topology; note that though some of the constructions can be made using the étale topology (as in [Kat89] or [Shi00]), the relevant constructions in [Shi02] require working Zariski locally. Finally, all monoids to which we refer will be commutative, and for M a monoid, M^{gp} will denote the group generated by M.

CONVENTION 6.0.2. For definitions and notation regarding log-schemes, see [Kat89] and [Shi00, § 2.1]. We follow the following convention from [Shi00]: if $(X, \mathcal{M})$ is a p-adic log formal scheme, we get an ordinary log scheme by reduction modulo p^n; we call the result $(X_n, \mathcal{M}_n)$. We do likewise for morphisms between p-adic log formal schemes.

Remark 6.0.3. It was explained to us by Shiho that the results of this section can be extended to the case of nondiscrete K. We omit this verification here, since it requires repeating a fair bit of [Shi00] in restricted generality, it being not completely clear whether one can redo [Shi00] at full strength for nondiscrete K.

6.1 Convergent log-isocrystals

In the process of introducing Fontaine–Illusie logarithmic structures, Kato constructed the category of crystals on a log-scheme and checked some of its basic properties. The analogue of the Berthelot–Ogus constructions of convergent isocrystals in the logarithmic setting is the work of Shiho [Shi00, Shi02]. We will not recall Shiho's original definition here; rather, we will use the alternative description in the case of interest provided by [Shi02, Proposition 2.2.7].

HYPOTHESIS 6.1.1. Let $(X, \mathcal{M})$ be a fine log scheme over k, and let $i : (X, \mathcal{M}) \hookrightarrow (P, \mathcal{L})$ be a closed immersion of $(X, \mathcal{M})$ into a noetherian fine log formal scheme $(P, \mathcal{L})$ over $\mathrm{Spf}\, \mathfrak{o}_K$ whose underlying scheme is of finite type over k. Assume also that there exists a factorization of i of the form

$$(X, \mathcal{M}) \xrightarrow{i'} (P', \mathcal{L}') \xrightarrow{f'} (P, \mathcal{L}), \tag{6.1.2}$$

in which i' is an exact closed immersion and f' is a formally log étale morphism.

By [Shi02, Lemma 2.2.2], one has the following lemma.

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Lemma 6.1.3. *Under Hypothesis 6.1.1, let $\widehat{P}'$ be the completion of P' along X. Then the rigid analytic space $\widehat{P}'_K$ is independent of the choice of the factorization, up to canonical isomorphism.*

Definition 6.1.4. Under Hypothesis 6.1.1, we write $](X,\mathcal{M})[_{(P,\mathcal{L})}$ for the space $\widehat{P}'_K$ defined in Lemma 6.1.3; for brevity, we also denote it by $]X[^{\log}_P$ if the sheaves of monoids are to be understood. Define the specialization map $\mathrm{sp} :]X[^{\log}_P \to X$ as the composite of the ordinary specialization map $\mathrm{sp} : \widehat{P'}_K \to \widehat{P}'_k$ with the map $\hat{f}'_k$.

Remark 6.1.5. It is shown in [Shi02, (2.2.1)] that Hypothesis 6.1.1 and Definition 6.1.4 admit a natural sheafification for the Zariski topology, but it is not clear whether this is true for the étale topology. Shiho handles this by hypothesizing that $(X,\mathcal{M})$ and $(P,\mathcal{L})$ are of 'Zariski type', i.e., is Zariski locally associated to a finitely generated monoid; given Convention 6.0.1, this is automatic for us.

Hypothesis 6.1.6. Suppose that

$$\begin{tikzcd} (X,\mathcal{M}) \arrow[r, "i"] \arrow[d, "f"'] & (P,\mathcal{L}) \arrow[d, "g"] \\ \operatorname{Spec} k \arrow[r, "\iota"] & \operatorname{Spf} \mathfrak{o}_K \end{tikzcd}$$

is a commuting diagram, where the top row satisfies Hypothesis 6.1.1, the log structures on the bottom row are trivial, and g is formally log smooth. For $j \in \mathbb{N}$, let $(P(j),\mathcal{L}(j))$ denote the $(j+1)$th fibre product of $(P,\mathcal{L}(j))$ over $\operatorname{Spf}\mathfrak{o}_K$, and let $i(j) : (X,\mathcal{M}) \to (P(j),\mathcal{L}(j))$ be the locally closed immersion induced by i (and the diagonal $X \mapsto X(j)$). It can be shown [Shi02, Proposition 2.2.4] that each of the $i(j)$ also satisfies Hypothesis 6.1.1 Zariski locally.

Definition 6.1.7. Under Hypothesis 6.1.6, define a *convergent log-isocrystal* on $(X,\mathcal{M})$ (with respect to i) to be a pair $(\mathcal{E},\epsilon)$, where $\mathcal{E}$ is a coherent $\mathcal{O}_{]X[^{\log}_P}$-module and $\epsilon : \pi_2^*(\mathcal{E}) \stackrel{\sim}{\to} \pi_1^*(\mathcal{E})$ is an isomorphism of $\mathcal{O}_{]X[^{\log}_{P(1)}}$-modules such that $\Delta^*(\epsilon) = \mathrm{id}_{\mathcal{E}}$, where $\Delta : (P,\mathcal{L}) \to (P(1),\mathcal{L}(1))$ is the diagonal, and the cocycle condition $\pi_{12}^*(\epsilon) \circ \pi_{23}^*(\epsilon) = \pi_{13}^*(\epsilon)$ holds on $]X[^{\log}_{P(2)}$. Then by [Shi02, Proposition 2.2.7], the category of convergent log-isocrystals on $(X,\mathcal{M})$ in this sense is equivalent to the category of convergent log-isocrystals on $(X,\mathcal{M})$ in Shiho's sense; in particular, the former is canonically independent of the choice of i.

Remark 6.1.8. The specific analogue of [Shi02, Proposition 2.2.7] in the nonlogarithmic case is the combination of Ogus's description of convergent isocrystals in terms of a canonical sequence of enlargements [Ogu84, Proposition 2.11] and Berthelot's reinterpretation of Ogus's description in terms of rigid analytic geometry [Brt96, (2.3.4)].

6.2 Log-∇-modules and Shiho's construction

We now clarify how to construct a convergent log-isocrystal, in the sense of Definition 6.1.7, from a log-∇-module arising as an extension of an overconvergent isocrystal.

Hypothesis 6.2.1. Let $F = (X,P,j)$ be a small frame with $X = P_k$, and suppose that the differentials of $t_1,\dots,t_n \in \Gamma(P,\mathcal{O}_P)$ freely generate Ω^1. Choose $m \leqslant n$, let Q denote the zero locus of $t_1 \cdots t_m$ on P, and put $Z = Q_k$ and $U = X \setminus Z$. Since Z and Q are (relative) strict normal crossings divisors on X and P, respectively, we obtain log structures $(X,\mathcal{M})$ and $(P,\mathcal{N})$ and a morphism $i : (X,\mathcal{M}) \to (P,\mathcal{N})$ satisfying Hypothesis 6.1.6. Define $X(j)$ and $P(j)$ accordingly, and put

$$Z(j) = \pi_1^{-1}(Z) \cup \cdots \cup \pi_j^{-1}(Z) \subset X(j),$$
$$Q(j) = \pi_1^{-1}(Q) \cup \cdots \cup \pi_j^{-1}(Q) \subset P(j).$$

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In order to apply Definition 6.1.7, we need to identify explicitly the spaces $]X[^{\log}_{P(j)}$ for $j = 1, 2$.

DEFINITION 6.2.2. Under Hypothesis 6.2.1, for $i = 1, \dots, m$ and $l = 1, \dots, j$, put $t_i^{(l)} = \pi_l^*(t_i)$. Let $\widehat{\mathbb{A}^{mj^2}_{\mathfrak{o}_K}}$ be the completion of the affine space with coordinates $u_i^{(l,l')}$ for $i = 1, \dots, m$ and $l, l' = 1, \dots, j$. Let $P'(j)$ be the closure in $P(j) \times \widehat{\mathbb{A}^{mj^2}_{m,\mathfrak{o}_K}}$ of the graph of the map $P(j)^{\mathrm{triv}} \to P(j) \times \widehat{\mathbb{A}^{mj^2}_{\mathfrak{o}_K}}$ induced by the functions $t_i^{(l)}/t_i^{(l')}$ for $i = 1, \dots, m$ and $l, l' = 1, \dots, j$. Let $i'(j) : X \to P'(j)$ be the map induced by composing $i(j) : X \to P(j)$ with the rational map $P(j) \dashrightarrow P'(j)$; note that $i'(j)$ is a *regular* map, not just a rational map. Let $f'(j) : P'(j) \to P(j)$ be the map obtained by composing the injection $P'(j) \hookrightarrow P(j) \times \widehat{\mathbb{A}^{mj^2}_{\mathfrak{o}_K}}$ with the first projection from $P(j) \times \widehat{\mathbb{A}^{mj^2}_{\mathfrak{o}_K}}$; then $i(j) = f'(j) \circ i'(j)$.

LEMMA 6.2.3. *Let $X = \operatorname{Spec} A \to S = \operatorname{Spec} B$ be a morphism of integral affine schemes, and suppose that, for some $n \geqslant 2$, the differentials of $t_1, \dots, t_n \in A$ freely generate $\Omega^1_{X/S}$. Put $A' = A[t_1/t_2, t_2/t_1]$ and $X' = \operatorname{Spec} A'$. Then $\Omega^1_{X'/S}$ is freely generated by the differentials of $t_1/t_2, t_2, \dots, t_n$.*

Proof. Given $f \in A[t_1/t_2]$, we can write $f = (t_1/t_2)^l a$ with $a \in A$ for some $l \in \mathbb{N}$. Then $df = la(t_1/t_2)^{l-1}\, d(t_1/t_2) + (t_1/t_2)^l\, da$ is a linear combination of $d(t_1/t_2), dt_2, \dots, dt_n$, since $dt_1 = t_2\, d(t_1/t_2) + (t_1/t_2)\, dt_2$ can be reexpressed in terms of $d(t_1/t_2)$ and dt_2. The same is true if $f \in A[t_2/t_1]$. Finally, any element of $A[t_1/t_2, t_2/t_1]$ can be written as the sum of an element of $A[t_1/t_2]$ and an element of $A[t_2/t_1]$, so $\Omega^1_{X'/S}$ is indeed generated by $d(t_1/t_2), dt_2, \dots, dt_n$.

On the other hand, suppose that $f\, d(t_1/t_2) + e_2\, dt_2 + \cdots + e_n\, dt_n = 0$ in $\Omega^1_{X'/S}$ for some $e_2, \dots, e_n, f \in A[t_1/t_2, t_2/t_1]$. By multiplying through by a power of $t_1 t_2$, we may reduce to the case where $e_2, \dots, e_n \in A$ and $f \in t_2^2 A$. Then

$$0 = (f/t_2)\, dt_1 + (e_2 - t_1 f/t_2^2)\, dt_2 + e_3\, dt_3 + \cdots + e_n\, dt_n$$

(using the fact that X is integral, so the division f/t_2^2 makes sense), so we must have $e_3 = \cdots = e_n = f/t_2 = 0$, so that $f = 0$, and then $e_2 - t_1 f/t_2^2 = 0$, so that $e_2 = 0$. Thus $d(t_1/t_2), dt_2, \dots, dt_n$ freely generate $\Omega^1_{X'/S}$, as desired. $\square$

Remark 6.2.4. All that Lemma 6.2.3 is doing is blowing up the smooth S-scheme X along the smooth S-subscheme $t_1 = t_2 = 0$.

COROLLARY 6.2.5. *The sheaf $\Omega^1_{P'(j)/\mathfrak{o}_K}$ is freely generated by the differentials of the regular functions*

$$t_i^{(1)}, t_i^{(2)}/t_i^{(1)}, \dots, t_i^{(j)}/t_i^{(j-1)} \ (i = 1, \dots, m), \quad t_i^{(1)}, t_i^{(2)}, \dots, t_i^{(j)} \ (i = m+1, \dots, n).$$

In particular, the divisor $f'(j)^{-1}(Q(j))$ is a relative strict normal crossings divisor on $P'(j)$ (relative to $\mathfrak{o}_K$).

Remark 6.2.6. In fact, $f'(j)^{-1}(Q(j))$ is quite simple: for $i = 1, \dots, m$, the zero locus of $t_i^{(1)}$ on $P'(j)$ is isomorphic to the zero locus of t_i on P via the first projection from $P(j)$, and the union of these loci is all of $f'(j)^{-1}(Q(j))$ since the functions $t_i^{(2)}/t_i^{(1)}, \dots, t_i^{(j)}/t_i^{(j-1)}$ are all invertible on $P'(j)$.

DEFINITION 6.2.7. Let $\mathcal{L}'(j)$ be the canonical log-structure on $P'(j)$ associated to $f'(j)^{-1}(Q(j))$, which is a relative strict normal crossings divisor by Corollary 6.2.5. Then $f'(j)$ gives rise to a natural morphism $(P'(j), \mathcal{L}'(j)) \to (P(j), \mathcal{L}(j))$. On the other hand, since $i'(j)^{-1}(f'(j)^{-1}(Q(j))) = Z$, $i'(j)$ extends to a morphism $(X, \mathcal{M}) \to (P'(j), \mathcal{L}'(j))$ of log formal schemes, and the composition $f'(j) \circ i'(j)$ coincides with $i(j)$ as a map of log formal schemes.

Remark 6.2.8. Suppose that (X, Z) and (X', Z') are (formal) smooth pairs, and $i : X \to X'$ is a closed immersion such that $i^{-1}(Z') \subseteq Z$ as (formal) schemes. Then i induces a morphism between

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the canonical log schemes $\mathcal{M}$ and $\mathcal{M}'$ corresponding to (X, Z) and (X', Z'). On an open subset U of X, $(i^*\mathcal{M}')/\mathcal{O}_X^* \cong i^{-1}(\mathcal{M}'/\mathcal{O}_X^*)$ is generated by the components of Z' meeting $i(U)$, whereas $\mathcal{M}/\mathcal{O}_X^*$ is generated by the components of Z meeting U. Hence a sufficient (but not necessary) condition for the map $i^*(\mathcal{M}') \to \mathcal{M}$ to be an isomorphism is that each component of Z' that meets X does so in a single component of Z, and no two components of Z' meet X along the same component of Z.

Lemma 6.2.9. *The factorization*

$$(X, \mathcal{M}) \xrightarrow{i'(j)} (P'(j), \mathcal{L}'(j)) \xrightarrow{f'(j)} (P(j), \mathcal{L}(j))$$

of $i(j)$ satisfies Hypothesis 6.1.1.

Proof. We first check that the map $i'(j)$ is an exact closed immersion using the criterion from Remark 6.2.8. Namely, each component of $f'(j)^{-1}(Q(j))$ is a component of the zero locus of $t_i^{(1)}$ for some $i \in \{1, \ldots, m\}$, which meets X in the corresponding component of the zero locus of t_i. In particular, each component $f'(j)^{-1}(Q(j))$ meeting X does so in a single component and no two of these intersections coincide. Hence the map $i'(j)^*\mathcal{L}'(j) \to \mathcal{M}$ is an isomorphism, and $i'(j)$ is an exact closed immersion.

We next check that the map $f'(j)$ is formally log étale. The structural map $(P'(j), \mathcal{L}'(j)) \to \operatorname{Spf} \mathfrak{o}_K$ is formally log smooth; by the formal analogue of [Kat89, Proposition 3.12], it then suffices to show that the map $f'(j)^*(\Omega^1_{P(j)/K}) \to \Omega^1_{P'(j)/K}$ is an isomorphism. But this is a straightforward consequence of the fact that

$$d \log(t_i^{(l)}/t_i^{(l')}) = d \log(t_i^{(l)}) - d \log(t_i^{(l')}).$$

Namely, as we adjoin each fraction $t_i^{(l)}/t_i^{(l')}$, we do not change Ω^1. □

We now have the tools with which to construct convergent log-isocrystals on the log schemes associated to strict normal crossings divisors on smooth k-varieties. Before doing so, we must collect a bit of information about log-∇-modules.

6.3 Log-∇-modules and unipotent monodromy

Definition 6.3.1. Under Hypothesis 6.2.1, let $\mathcal{E}$ be a log-∇-module on $]X[$ with respect to $t_1, \ldots, t_n$. We say that $\mathcal{E}$ is *convergent* if the restriction of $\mathcal{E}$ to a strict neighborhood of $]U[$ in $]X[$ is overconvergent along Z.

We now have the following limited logarithmic analogue of Theorem 5.2.1. (Note however that the work has been done already in the proof of Lemma 5.1.1.)

Proposition 6.3.2. *Under Hypothesis 6.2.1, let $\mathcal{E}$ be a ∇-module on a strict neighborhood on $]U[$ in $]X[$ which is overconvergent along Z. Then $\mathcal{E}$ has unipotent monodromy along Z if and only if $\mathcal{E}$ extends to a convergent log-∇-module on $]X[$ with nilpotent residues. Moreover, the restriction functor, from convergent log-∇-modules with nilpotent residues on $]X[$ to isocrystals on U overconvergent along Z, is fully faithful.*

Proof. By covering X with affines, we may reduce to the case where we may repeatedly apply Lemma 5.1.1 to obtain the desired result. □

Remark 6.3.3. The full faithfulness assertion in Proposition 6.3.2 depends crucially on the nilpotent residues hypothesis. This is analogous to the situation in [Del70, II.5], where logarithmic extensions with nilpotent residue are 'canonical' and logarithmic extensions with arbitrary residue are not; indeed, one of the simplest examples in that setting is relevant here also. Namely, put $P = \operatorname{Spf} K\langle t \rangle$, $X = P_k = \mathbb{A}^1_k$, and $U = \mathbb{A}^1_k \setminus \{0\}$, let n be a positive integer, and let $\mathcal{E}$ be the ∇-module on P_K

generated by a single element $\mathbf{v}$ such that $\nabla \mathbf{v} = n\mathbf{v} \otimes dt/t$ for some $n \in \mathbb{N}$. Then one easily verifies that ∇ is overconvergent along $X \setminus U$, and the kernel of ∇ on $]X[$ is trivial, but the kernel of ∇ on any strict neighborhood of $]U[$ in $]X[$ not containing the point $t = 0$ includes the section $t^{-n}\mathbf{v}$. (A similar point arises in [LST01], which is concerned with the passage from a log-F-crystal to an isocrystal on the log-trivial subscheme overconvergent along the complement.)

LEMMA 6.3.4. *Under Hypothesis 6.2.1, let $\mathcal{E}$ be a convergent log-∇-module on P_K. Then for any $\mathbf{v} \in \Gamma(]X[, \mathcal{E})$ and any $\eta \in (0,1)$, the multisequence*

$$\frac{1}{i_1! \cdots i_n!}\left(\prod_{j=1}^{n}\prod_{l=0}^{i_j-1}\left(t_j\frac{\partial}{\partial t_j} - l\right)\right)\mathbf{v}$$

is η-null.

Proof. Since $\mathcal{E}$ restricts to a convergent isocrystal on U, the multisequence

$$\frac{1}{i_1! \cdots i_n!}\left(\prod_{j=1}^{n}\frac{\partial^{i_j}}{\partial t_j^{i_j}}\right)\mathbf{v}$$

is η-null on $]U[$ by the definition of η-convergence plus Proposition 2.5.6. Since $|t_i| \leqslant 1$ for each i, the multisequence

$$\frac{t_1^{i_1} \cdots t_n^{i_n}}{i_1! \cdots i_n!}\left(\prod_{j=1}^{n}\frac{\partial^{i_j}}{\partial t_j^{i_j}}\right)\mathbf{v}$$

is also η-null on $]U[$. However, this is precisely the desired multisequence, and the fact that it is η-null on $]U[$ implies the fact that it is η-null on $]X[$. Namely, this follows from the fact that the spectral seminorm on $\mathcal{O}(]U[)$ restricts to the spectral seminorm on $\mathcal{O}(]X[)$, which is true because U is open dense in X. □

6.4 Convergent log-isocrystals and log-∇-modules

With the constructions of the previous subsection in hand, we can now explicitly describe convergent log-isocrystals, in the case of the log structure associated to a smooth pair, in terms of log-∇-modules.

THEOREM 6.4.1. *Under Hypothesis 6.2.1, there is an equivalence between the category of convergent log-isocrystals on (X, Z) and the category of convergent log-∇-modules on P_K.*

Proof. Suppose $\mathcal{E}$ is a convergent log-isocrystal on (X, Z) in the sense of Definition 6.1.7. Then $\mathcal{E}$ restricts to an isocrystal on X overconvergent along Z, and hence to an overconvergent ∇-module on some strict neighborhood V of $]X[_P$. Moreover, by [Shi02, Proposition 1.2.7], the isomorphism $\epsilon : \pi_2^*(\mathcal{E}) \to \pi_1^*(\mathcal{E})$ on the second infinitesimal neighborhood of X in $P'(1)$ defines a log-connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^{1,\log}_{P_K/K}$ extending the connection on V. This yields the data of a convergent log-∇-module on P_K.

Conversely, suppose that $\mathcal{F}$ is a convergent log-∇-module on P_K. Write u_i for the function $t_i^{(2)}/t_i^{(1)}$ on $P'(1)$. Following [Kat89, (6.7.1)], we observe that the isomorphism $\epsilon : \pi_2^*(\mathcal{F}) \xrightarrow{\sim} \pi_1^*(\mathcal{F})$ over a suitable strict neighborhood of $]U[_{P'(1)}$ in $]X[_{P'(1)}$ induced by ∇ can be written in the form

$$1 \otimes \mathbf{v} \mapsto \sum_{i_1,\dots,i_n=0}^{\infty}\left(\prod_{j=1}^{n}\frac{(u_j-1)^{i_j}}{i_j!}\right) \otimes \left(\prod_{j=1}^{n}\prod_{l=0}^{i_j-1}\left(t_i\frac{\partial}{\partial t_i} - l\right)(\mathbf{v})\right).$$

By Lemma 6.3.4, this series converges uniformly on any affinoid subspace of $]X[_{P'(1)}$ of the form $\max_j\{|u_j - 1|\} \leqslant \lambda$ for $\lambda \in (0,1) \cap \Gamma^*$. Hence ϵ is defined on all of $]X[_{P'(1)} =]X[^{\log}_{P(1)}$.

We now have an isomorphism $\epsilon : \pi_2^*(\mathcal{F}) \xrightarrow{\sim} \pi_1^*(\mathcal{F})$ on $]X[^{\log}_{P(1)}$ satisfying $\Delta^*(\epsilon) = \mathrm{id}$. It is straightforward to check that the cocycle condition $\pi_{12}^*(\epsilon) \circ \pi_{23}^*(\epsilon) = \pi_{13}^*(\epsilon)$ holds on $]X[^{\log}_{P(2)}$ from the formula, but it is easier to deduce it by restricting to a strict neighborhood of $]U[_{P'(2)}$, where it holds because of the equivalence of categories between ordinary overconvergent isocrystals and overconvergent ∇-modules.

We conclude that every convergent log-∇-module on P_K does indeed give rise to a convergent log-isocrystal. This establishes the desired equivalence. □

Remark 6.4.2. Note that the equivalence in Theorem 6.4.1 is compatible with restriction to an open subscheme, so in principle its statement can be 'sheafified'.

Remark 6.4.3. While Lemma 5.1.1 can also be applied with $Y \neq X$ to construct 'overconvergent log-∇-modules', their interpretation in the Grothendieckian sense (i.e., as isomorphisms between two pullbacks to the diagonal) seems subtle. Probably the right thing to do is to globally replace tubes with strict neighborhoods throughout the proof of Theorem 6.4.1; however, in the absence of a 'reference category' of overconvergent log-isocrystals, one then has to check all the relevant compatibilities by hand. The main problem is that we do not presently have an 'overconvergent topos' analogizing [Ogu90]; however, the ongoing work of le Stum mentioned earlier [LS04, LS06] seems to be heading in the right direction, and it is possible it will ultimately be adapted to include logarithmic structures. In the meantime, however, we will stick to convergent log-isocrystals.

Definition 6.4.4. Under Hypothesis 6.2.1, we say that a convergent log-isocrystal on (X, Z) *has nilpotent residues* if its image under the functor of Theorem 6.4.1 is a log-∇-module with nilpotent residues. More generally, if X is a smooth k-variety and Z is a strict normal crossings divisor on X, we say that a convergent log-isocrystal $\mathcal{E}$ on (X, Z) has nilpotent residues if there is an open cover $U_1, \dots, U_n$ of X such that each pair $(U_i \cap X, U_i \cap Z)$ satisfies Hypothesis 6.2.1, and the restriction of $\mathcal{E}$ to $U_i \cap X$ has nilpotent residues. The same is then true on any open cover.

From Theorem 6.4.1, we obtain the following theorem.

Theorem 6.4.5. *Let $U \hookrightarrow X$ be an open immersion of smooth k-varieties such that $Z = X \setminus U$ is a strict normal crossings divisor on X. Let $\mathcal{E}$ be an isocrystal on U overconvergent along Z. Then $\mathcal{E}$ has unipotent monodromy along Z if and only if $\mathcal{E}$ extends to a convergent log-isocrystal with nilpotent residues on (X, Z). Moreover, the restriction functor, from convergent log-isocrystals with nilpotent residues on (X, Z) to isocrystals on U overconvergent along Z, is fully faithful.*

Proof. Everything being asserted is Zariski local, so we may reduce to the case where Hypothesis 6.2.1 holds. In this case, Proposition 6.3.2 and Theorem 6.4.1 together yield the claim. □

Remark 6.4.6. The word 'strict' is probably not necessary in Theorem 6.4.5; removing it would require performing an appropriate étale descent (but beware of some technical problems, as in Remark 6.1.5). However, in the desired application to semistable reduction, we can always get to the strict normal crossings situation using an alteration, in the manner of de Jong [DJ96].

Remark 6.4.7. It should be possible to improve the full faithfulness conclusion of Theorem 6.4.5 to allow restriction all the way to the category of convergent isocrystals on U. In fact, this is possible under additional hypotheses; see Remark 5.3.9.

Remark 6.4.8. In some cases, one may want to apply Theorem 6.4.5 to construct logarithmic extensions of crystals in coherent $\mathcal{O}$-modules, rather than isocrystals. This should be a straightforward consequence of the fact that isocrystals can be viewed as elements of the isogeny category of crystals (as in [Ogu84]), but we have not checked any details.

6.5 Extension classes of log-isocrystals

In the logarithmic setting, one can show that restriction to the log-trivial subscheme preserves extension classes.

PROPOSITION 6.5.1. *Let (X, Z) be a smooth pair, and let $\mathcal{E}_1$ and $\mathcal{E}_2$ be convergent log-isocrystals with nilpotent residues on (X, Z). Then $\mathrm{Ext}^1(\mathcal{E}_1, \mathcal{E}_2)$ is the same whether computed in the category of convergent log-isocrystals on (X, Z) or in the category of isocrystals on $U = X \setminus Z$ overconvergent along Z.*

Proof. Recall that the Yoneda Ext group $\mathrm{Ext}^1(\mathcal{E}_1, \mathcal{E}_2)$ classifies short exact sequences

$$0 \to \mathcal{E}_1 \to \mathcal{F} \to \mathcal{E}_2 \to 0.$$

Let Ext_X and Ext_U denote the group $\mathrm{Ext}^1(\mathcal{E}_1, \mathcal{E}_2)$ computed in the category of convergent log-isocrystals on (X, Z) and in the category of isocrystals on U overconvergent along Z, respectively; then restriction gives a map $\mathrm{Ext}_X \to \mathrm{Ext}_U$. Note that this map is injective thanks to full faithfulness of restriction (Theorem 6.4.5): any isomorphism over U between two short exact sequences over X extends to X.

To see that $\mathrm{Ext}_X \to \mathrm{Ext}_U$ is surjective, note that, if $\mathcal{F}$ fits into a sequence over U, then $\mathcal{F}$ has unipotent monodromy along Z, because $\mathcal{E}_1$ and $\mathcal{E}_2$ both do. Hence $\mathcal{F}$ extends to a convergent log-isocrystal on (X, Z), as do the maps $\mathcal{E}_1 \to \mathcal{F}$ and $\mathcal{F} \to \mathcal{E}_2$ by Theorem 6.4.5. Hence $\mathrm{Ext}_X \to \mathrm{Ext}_U$ is surjective, and thus is a bijection as desired. □

7. Conclusion: a look ahead

We conclude by cataloguing some of the questions we will be discussing later in this series of papers, in the terminology we have established. Note that this section is intended as a 'pre-introduction' to the subsequent papers, and so statements here have not been made in a precise fashion; they will be articulated properly in due course.

7.1 Semistable reduction: Shiho's conjecture

We give the statement of Shiho's conjecture [Shi02, Conjecture 3.1.8], or, in our terminology, the 'semistable reduction problem'. First, we must recall the notion of a Frobenius structure on an isocrystal.

DEFINITION 7.1.1. Suppose that $\sigma_K : \mathfrak{o}_K \to \mathfrak{o}_K$ is an endomorphism lifting the p^a-power Frobenius map on k, for some positive integer a. Let $X \hookrightarrow Y$ be an open immersion of k-varieties. A *Frobenius structure (of order a)* on an isocrystal $\mathcal{E}$ on X overconvergent along $Y \setminus X$ is an isomorphism $F_X^* \sigma_K^* \mathcal{E} \xrightarrow{\sim} \mathcal{E}$, where F_X is the relative p^a-power Frobenius. An isocrystal equipped with a Frobenius structure of order a is called an *F^a-isocrystal*.

CONJECTURE 7.1.2 (Shiho). Assume that the field k is perfect. Let X be a smooth k-variety and let $\mathcal{E}$ be an overconvergent F^a-isocrystal on X. Then there exists a proper, surjective, generically étale morphism $f : X_1 \to X$, and an open immersion $j : X_1 \hookrightarrow \overline{X_1}$ of X_1 into a smooth projective k-variety in which the complement $D = \overline{X_1} \setminus X_1$ is a strict normal crossings divisor, such that $f^*\mathcal{E}$ extends to a convergent F^a-log-isocrystal $\mathcal{F}$ on $(\overline{X_1}, D)$.

Remark 7.1.3. Absent the isocrystal, the existence of the maps f and j is the content of de Jong's alterations theorem [DJ96, Theorem 4.1]; indeed, the map f is precisely an alteration in de Jong's sense.

Remark 7.1.4. Note that it is actually enough to show that $f^*\mathcal{E}$ extends as a convergent log-isocrystal; then the Frobenius structure will extend from X_1 to $\overline{X_1}$ thanks to the full faithfulness

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aspect of Theorem 6.4.5. Note also that a convergent log-F-isocrystal necessarily has nilpotent residues.

Remark 7.1.5. Shiho's conjecture is a higher-dimensional version of de Jong's formulation of Crew's conjecture [DJ98b]; the case where X is a curve is known to follow from the p-adic local monodromy theorem [Ked03]. As noted in the introduction, its resolution is expected to have various consequences for the theory of rigid cohomology, especially in the relative setting, and perhaps for the theory of arithmetic $\mathcal{D}$-modules, which are to the isocrystals considered here as constructible sheaves are to lisse sheaves in étale cohomology.

7.2 Monodromy of exceptional components

The p-adic local monodromy theorem of André [And02], Mebkhout [Meb02], and the present author [Ked04a] implies a strong statement in the direction of Conjecture 7.1.2. (We will describe the exact statement of the p-adic local monodromy theorem and the nature of its application here more thoroughly later in the series.) Namely, if one starts with a compactification $X \hookrightarrow \overline{X}$ such that $(\overline{X}, \overline{X} \setminus X)$ is a smooth pair (which one may do without loss of generality by pulling back along an alteration, thanks to de Jong's theorem), one can construct the maps f and j so that f extends to a map $\overline{X_1} \to \overline{X}$, and $\mathcal{E}$ has unipotent monodromy along each component of $\overline{X_1} \setminus X_1$ which dominates a component of $\overline{X} \setminus X$.

Unfortunately, this statement together with Theorem 6.4.5 do not suffice to imply Conjecture 7.1.2, because there may be components of $\overline{X_1} \setminus X_1$ which do not dominate any component of $\overline{X} \setminus X$. In order to deduce Conjecture 7.1.2 along these lines, one must somehow gain control of the monodromy of these 'exceptional' divisors. Otherwise, one is forced to alter again, possibly introduce more exceptional divisors, and perhaps repeat *ad infinitum* without reaching the desired conclusion.

The control of exceptional divisors will be accomplished by considering monodromy also along certain 'fake annuli', corresponding to irrational valuations on the function field $K(X)$. These form a compact space (an example of a Gelfand spectrum, as in Berkovich's foundations of rigid analytic geometry [Brk98]), so one can prove a global quasi-unipotence theorem 'topologically', by verifying it on an open neighborhood of each valuation.

It must be stressed that the presence of the exceptional divisors is not an artifact of the use of de Jong's theorem in lieu of the as-yet-unknown resolution of singularities in positive characteristic. That is because the underlying finite cover given by the p-adic local monodromy theorem is typically unavoidably singular, due to wild ramification; contrast this situation to what happens in the complex analytic setting, where one can locally avoid introducing any singularities by making the right toroidal cover.

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