PRIME IDEALS IN MATRIX RINGS

by ARTHUR D. SANDS

(Received 18th October, 1955)

1. *Introduction.* Let R be a ring and let R_n be the complete ring of $n \times n$ matrices with coefficients from R.

If A is any subset of R, we denote by A_n the subset of R_n consisting of the matrices of R_n with coefficients from A.

If R is a ring with a unit element, the ideals† of R_n are the sets A_n corresponding to the ideals A of R. But if R has no unit element, this is not, in general, the case. It is however possible to establish for any ring R, with or without a unit element, results corresponding to the above one for two special types of ideals, namely, prime ideals and prime maximal ideals. Thus in § 2 it is shown that the prime and prime maximal ideals of R_n are the sets A_n corresponding to the prime and prime maximal ideals A of R.

In § 3 it is shown that if M is the M-radical of R, as defined by M. Nagata ((**2**), p. 338), then the M-radical of R_n is M_n.

In § 4 it is shown that those maximal ideals of R_n which are of the form A_n are the sets A_n corresponding to the *prime* maximal ideals A of R, i.e., they are the prime maximal ideals of R_n.

An ideal P in a general ring R is said to be prime if the following condition is satisfied; if A and B are ideals of R such that $AB \subseteq P$, then $A \subseteq P$ or $B \subseteq P$. An ideal of R is said to be semi-prime if it is an intersection of prime ideals of R. In (**1**), theorem 1, N. H. McCoy gives a set of alternative necessary and sufficient conditions that an ideal should be prime; we shall make use of several of these conditions in our proofs. We shall also make use of the fact that any ring R can be embedded in an over-ring $(1, R)$ such that R is an ideal in $(1, R)$, and $(1, R)$ has a unit element. We shall also use a result of M. Nagata ((**2**), remark 2, p. 333) which states that if S is an ideal in the ring R and A is a semi-prime ideal in the ring S, then A is an ideal in R.‡

Throughout the paper we shall denote by $[r]^{i,j}$ the matrix which has r as its (i, j)th coefficient and has all its other coefficients equal to zero.

2. *Prime and prime maximal ideals in R.*

THEOREM 1. *The prime ideals of R_n are the sets A_n corresponding to the prime ideals A of R.*

Proof: We first show that if A is a prime ideal of R, then A_n is a prime ideal of R_n. If A is any ideal of R, it is easily seen that A_n is an ideal of R_n. Let A be a prime ideal of R. Let $[a_{ij}]$ and $[b_{ij}]$ be matrices of R_n such that $[a_{ij}]R_n[b_{ij}] \subseteq A_n$. Suppose that $[a_{ij}] \notin A_n$. Let a_{kl} be a coefficient of $[a_{ij}]$ which is not contained in A. Let r be any element of R and b_{pq} any coefficient from $[b_{ij}]$. Then $[a_{ij}]\,[r]^{l,p}[b_{ij}]$ is a matrix of $[a_{ij}]R_n[b_{ij}]$ which has the element

† Throughout this paper, *ideal* will mean *two-sided ideal.*

‡ For the sake of completeness we give the proof of this result.

Since a semi-prime ideal is an intersection of prime ideals, it is sufficient to prove the corresponding result for prime ideals.

Let S be an ideal in a ring R and A a prime ideal in the ring S. The ideal in R generated by A is $A + RA + AR + RAR$. Since S is an ideal of R containing A, S contains $A + RA + AR + RAR$. This ideal of R is *a fortiori* an ideal of S; hence $S(A + RA + AR + RAR)S \subseteq SAS \subseteq A$. But A is a prime ideal in S. Therefore $A + RA + AR + RAR \subseteq A$. It follows that A is an ideal in R.

This completes the proof.

$a_{kl}rb_{pq}$ as its (k, q)th coefficient. But $[a_{ij}]R_n[b_{ij}] \subseteq A_n$; therefore $a_{kl}rb_{pq} \in A$. This is true for each element r in R; hence $a_{kl}Rb_{pq} \subseteq A$. But A is a prime ideal and $a_{kl} \notin A$; it follows by condition (3) of (**1**), theorem 1, that $b_{pq} \in A$. This is true for each coefficient of $[b_{ij}]$; hence $[b_{ij}] \in A_n$. Thus, if $[a_{ij}]R_n[b_{ij}] \subseteq A_n$ and $[a_{ij}] \notin A_n$, it follows that $[b_{ij}] \in A_n$. Therefore A_n is a prime ideal in R_n.

We now show that every prime ideal of R_n is of this form. Let A^* be a prime ideal in R_n. We denote by A the set of elements of R which are coefficients in matrices of A^*.

R is an ideal in the ring $(1, R)$. Therefore R_n is an ideal in $(1, R)_n$. A^* is a prime ideal in R_n. Hence, by the result of Nagata, A^* is an ideal in $(1, R)_n$. But $(1, R)$ has a unit element. It follows that A is an ideal in $(1, R)$ and that $A^* = A_n$. But the elements of A are contained in R; hence A is an ideal in R.

It remains to show that A is prime in R. Let a and b be elements of R such that $aRb \subseteq A$. Then $[a]^{1,1}R_n[b]^{1,1} \subseteq A_n = A^*$. But A^* is prime in R_n; therefore $[a]^{1,1}$ or $[b]^{1,1}$ is contained in A_n. Hence a or b is contained in A. It follows that A is a prime ideal in R.

This completes the proof.

Since an ideal A in R is different from an ideal B in R if and only if A_n is different from B_n, it follows that the mapping $A \to A_n$ sets up a one-to-one correspondence between the prime ideals of R and of R_n.

Corollary. *The semi-prime ideals of R_n are the sets A_n corresponding to the semi-prime ideals A of R.*

Theorem 2. *The prime maximal ideals of R_n are the sets A_n corresponding to the prime maximal ideals A of R.*

Proof: We first show that the prime maximal ideals of R_n are of this form. Let A^* be a prime maximal ideal of R_n and let A be defined as in theorem 1. Then, since A^* is prime, it follows from theorem 1 that A is a prime ideal in R and that $A^* = A_n$. Let B be an ideal of R which strictly contains A. Then B_n strictly contains A_n. Hence, by the maximality of A_n, $B_n = R_n$. Therefore $B = R$. It follows that A is a maximal ideal of R.

It remains to show that every ideal of this form in R_n is a prime maximal ideal. Let A be a prime maximal ideal of R. Then, by theorem 1, A_n is a prime ideal of R_n. Let B^* be an ideal of R_n which strictly contains A_n. Then the set B strictly contains the ideal A. Let b be an element of B which is not an element of A. Then, since A is prime, RbR is not contained in A; for $RbR \subseteq A$ implies that $bRbR \subseteq A$. By condition (4) of (**1**), theorem 1, it follows that $bR \subseteq A$ and hence that $bRb \subseteq A$. But from this it follows that $b \in A$. Thus RbR is not contained in A. But RbR is an ideal and A is a maximal ideal. Hence $A + RbR = R$. Thus if r is any element of R, there exist an element a of A and elements s_k and r_k of R such that $r = a + \sum_k s_k b r_k$. Let $[r_{ij}]$ be any matrix of R_n. Then for each pair of integers i, j there exist an element a_{ij} of A and elements s_{ijk} and r_{ijk} of R such that $r_{ij} = a_{ij} + \sum_k s_{ijk} b r_{ijk}$. Thus

$$[r_{ij}] = [a_{ij} + \sum_k s_{ijk} b r_{ijk}] = [a_{ij}] + \sum_{i,j} [\sum_k s_{ijk} b r_{ijk}]^{i,j}.$$

Let b^* be a matrix of B^* with b as a coefficient, say in the (p, q)th position. Then $[r_{ij}] = [a_{ij}] + \sum_{i,j} \sum_k [s_{ijk}]^{i,p} b^* [r_{ijk}]^{q,j}$. But B^* is an ideal of R_n; therefore $[s_{ijk}]^{i,p} b^* [r_{ijk}]^{q,j}$ is an element of B^*. Now $[a_{ij}]$ is an element of A_n and so of B^*. It follows that $[r_{ij}]$ is an element of B^*. But this is true for each matrix $[r_{ij}]$ of R_n. Therefore $B^* = R_n$. Hence A_n is a maximal ideal and so a prime maximal ideal of R_n.

This completes the proof.

3. *The M-radical of a matrix ring* R_n.

M. Nagata defined the M-radical of a ring R to be the intersection of all prime ideals A of R such that R/A is a simple ring. It is easily seen that this is just the intersection of all prime maximal ideals of R.† We now use theorem 2 to show that the usual relationship between the radical of a ring R and the radical of R_n holds for the M-radical.

THEOREM 3. *If M is the M-radical of a ring R, then the M-radical $M(R_n)$ of R_n is equal to M_n.*

Proof: Let A_α be the complete set of prime maximal ideals of R. Then, by theorem 2, $(A_\alpha)_n$ is the complete set of prime maximal ideals of R_n. Hence

$$M(R_n) = \cap (A_\alpha)_n = (\cap A_\alpha)_n = M_n.$$

4. *Maximal ideals in* R_n.

LEMMA. *Let A be a maximal ideal in a ring R. Then A is a prime ideal if and only if R^2 is not contained in A.*

Proof: Let A be prime. Then $R^2 \subseteq A$ implies that $R \subseteq A$. This is not so, since A is maximal. Hence R^2 is not contained in A.

Conversely, suppose that R^2 is not contained in A. Let B and C be ideals such that $BC \subseteq A$. If neither B nor C is contained in A, it follows from the maximality of A that $B + A = C + A = R$. In this case $R^2 = (B + A)(C + A) \subseteq BC + A \subseteq A$. But this contradicts the hypothesis that R^2 is not contained in A. Hence either B or C is contained in A. Therefore A is prime.

THEOREM 4. *Let A be a maximal ideal in a ring R. Then A_n is a maximal ideal in R_n if and only if A is a prime ideal in R.*

Proof: Let A be a prime ideal in R; then, by theorem 2, A_n is a prime maximal ideal in R_n.

Conversely, suppose that A is not a prime ideal. Then, by the lemma, $R^2 \subseteq A$. Therefore $R_n^2 \subseteq A_n$. Let b be an element of R which is not an element of A. Consider the set B^* of R_n. consisting of all matrices of the form $[a_{ij}] + [kb]^{1,1}$, where $[a_{ij}]$ is a matrix of A_n and k is an integer. Then B^* is strictly contained in R_n and strictly contains A_n. Clearly the difference of two matrices of B^* is a matrix of B^*. Also

$$R_nB^* \subseteq R_n^2 \subseteq A_n \subseteq B^* \quad \text{and} \quad B^*R_n \subseteq R_n^2 \subseteq A_n \subseteq B^*.$$

Therefore B^* is an ideal in R_n. Hence A_n is not a maximal ideal in R_n.

This completes the proof.

REFERENCES

(1) N. H. McCoy, *Prime Ideals in General Rings*, Amer. J. Math., **71**, 1949, 823–833.

(2) M. Nagata, *On the Theory of Radicals In a Ring*, J. Math. Soc. Jap., **3**, 1951, 330–344.

THE UNIVERSITY,
GLASGOW

† We adopt the convention that the intersection of an empty set of ideals is the whole ring R.