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COMPATIBLE TIGHT RIESZ ORDERS ON THE GROUP OF AUTOMORPHISMS OF AN 0-2-HOMOGENEOUS SET

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Introduction. Davis and Bolz (1974) considered, and to some extent classified, compatible tight Riesz order on the group of all order-preserving permutations of a totally ordered field. Glass (1976) carried out a more general study of compatible tight Riesz orders on ordered permutation groups and, in particular, showed the importance of determining compatible tight Riesz orders on 0-primitive ordered permutation groups. However, the general problems of existence and classification of compatible tight Riesz orders on 0-primitive ordered permutation groups remained open.

In this paper we consider these problems in relation to the group $A(\Omega)$ of all order-preserving permutations of a totally-ordered set Ω with $A(\Omega)$ acting 0-2-transitively on Ω. Such a group has compatible tight Riesz orders (Theorem 7), which answers an implicit question of Glass (1976) and, with a further restriction on Ω, we can describe certain maximal compatible tight Riesz orders on $A(\Omega)$ (Theorem 8). The final section deals with the maximal tangents of the compatible tight Riesz orders we have found.

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For a totally-ordered set Ω we denote by $A^+(\Omega)$ the positive set of $A(\Omega)$ with the usual lattice order. That is $A^+(\Omega) = \{g \in A(\Omega) : xg \geqq x \text{ for all } x \in \Omega\}$. For $x \in \Omega$ the stabilizer of x in $A(\Omega)$ is $A_x(\Omega) = \{g \in A(\Omega) : xg = x\}$, and we write $A_x^+(\Omega)$ for $A_x(\Omega) \cap A^+(\Omega)$.

In the sequel we shall call an order-preserving permutation of Ω an *automorphism* of Ω, and we shall assume always that $A(\Omega)$ is a non-trivial group.

We recall that a subgroup G of $A(\Omega)$ acts 0-2-transitively on Ω if for all $x_1 < x_2$ and $y_1 < y_2$ in Ω there is a $g \in G$ satisfying $x_i g = y_i$ $(i = 1, 2)$. We say that Ω is *homogeneous* (respectively, 0-2-*homogeneous*) if $A(\Omega)$ acts transitively (respectively, 0-2-transitively) on Ω.

For the record we provide a proof of the following piece of folklore (apparently originating with Wielandt), since it is the key to our constructions.

THEOREM 1. *For a totally-ordered set Ω the following are equivalent:*

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(1) *Ω is 0-2-homogeneous.*

(2) *Ω has neither least nor greatest element and all closed intervals of Ω with more than one point have the same order-type.*

Proof ((1) implies (2)). Since Ω is homogeneous it can have neither least nor greatest element. If $x_1 < x_2$ and $y_1 < y_2$ in Ω then $x_i g = y_i$ $(i = 1, 2)$ for some $g \in A(\Omega)$. Clearly the restriction of g to the closed interval $[x_1, x_2]$ is an order-isomorphism onto $[y_1, y_2]$.

((2) implies (1)). By a result of Holland (1965, Theorem 4) we need only show that for all $x < y < z$ in Ω there is a $g \in A_x^+(\Omega)$ satisfying $yg = z$. (Since Ω is without a least element this also shows immediately that Ω is homogeneous). For each *integer n* take $a_n \in \Omega$ satisfying $x < a_n < a_{n+1}$, $y = a_0$ and $z = a_1$. This is possible since Ω is dense in itself and has no greatest element. Now for each n let ϕ_n be an order-isomorphism from $[a_n, a_{n+1}]$ onto $[a_{n+1}, a_{n+2}]$. The map $g : \Omega \rightarrow \Omega$ defined by

$$wg = \begin{cases} w\phi_n & \text{if } w \in [a_n, a_{n+1}) \text{ for some } n \\ w & \text{otherwise} \end{cases}$$

is an element of $A_x^+(\Omega)$ and $yg = z$ (both facts being easy to verify).

By Lemma 9 of Holland (1963) we have the following result, which is important for us:

COROLLARY 2. *If Ω is 0-2-homogeneous then $A(\Omega)$ is divisible.*

Compatible tight Riesz orders. A *compatible tight Riesz order* on $A(\Omega)$ is a subset T of $A(\Omega)$ satisfying the following:

(1) T is a proper dual ideal of $A^+(\Omega)$

(2) T is normal in $A(\Omega)$

(3) $T = TT$

(4) $\inf T = 1$

Our objective in this section is to show that $A(\Omega)$ *has* a compatible tight Riesz order when Ω is 0-2-homogeneous and then, in some cases, to determine maximal compatible tight Riesz orders.

We equip Ω with the order topology. The collection of all open dense subsets of Ω is denoted by $D(\Omega)$. Clearly $D(\Omega)$ is a filter of the lattice of open subsets of Ω. The *support* of $g \in A(\Omega)$ is the set $\operatorname{supp}(g) = \{x \in \Omega : xg \neq x\}$. Each support set is open for the order topology.

The collection $\Sigma(\Omega) = \{\operatorname{supp}(g) : g \in A(\Omega)\}$, ordered by inclusion, is a sublattice of the lattice of open sets of Ω and is called the *support lattice* of Ω. Thus $\Sigma(\Omega)$ is a distributive lattice with least element $\square = \operatorname{supp}(1)$, but in general without a greatest element. We denote the annihilator of Δ in $\Sigma(\Omega)$ by Δ^*. Thus $\Delta^* = \{\Delta' \in \Sigma(\Omega) : \Delta \cap \Delta' = \square\}$, and we denote $\Sigma(\Omega) \cap D(\Omega)$ by $\delta(\Omega)$.

We say that a closed interval $[a, b]$ in Ω supports a non-identity automorphism if $A([a, b]) \neq \langle 1 \rangle$.

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LEMMA 3. *If each closed interval of* Ω *with more than one point supports a non-identity automorphism then* $\delta(\Omega) = \{\Delta \in \Sigma(\Omega) : \Delta^* = \{\square\}\}$.

Proof. Let Ω satisfy the hypothesis of the lemma and take any Δ in $\delta(\Omega)$. If $\Delta' \cap \Delta = \square$, with Δ' in $\Sigma(\Omega)$, then $\Delta' = \square$ (otherwise Δ', being open, meets the open dense set Δ). Thus $\Delta^* = \{\square\}$. Conversely, suppose that $\Delta \in \Sigma(\Omega)$ and that the closure $\bar{\Delta}$ of Δ is not Ω. Then $[y, z] \subseteq \Omega\backslash\bar{\Delta}$ for some $y < z$ in Ω, so if we let h be a non-identity automorphism of $[y, z]$ and define $g : \Omega \rightarrow \Omega$ by

$$xg = \begin{cases} xh & \text{if } x \in [y, z] \\ x & \text{otherwise} \end{cases}$$

then $g \in A(\Omega)$. Since $\text{supp}(g) \neq \square$ and $\Delta \cap \text{supp}(g) = \square$ we have $\Delta^* \neq \{\square\}$. Thus for Δ in $\Sigma(\Omega)$, $\Delta^* = \{\square\}$ implies $\Delta \in D(\Omega)$.

COROLLARY 4. *If* Ω *is* 0-2-*homogeneous then* $\delta(\Omega) = \{\Delta \in \Sigma(\Omega) : \Delta^* = \{\square\}\}$.

Proof. Let $[x, y]$, with $x < y$, be a proper closed interval of Ω. A non-identity automorphism of $[x, y]$ can be constructed as in Theorem 1.

Now we define a candidate for a compatible tight Riesz order on $A(\Omega)$:

$$T_\delta = \{g \in A^+(\Omega) : \text{supp}(g) \text{ is dense in } \Omega\}.$$

LEMMA 5. T_δ *is either empty or a proper normal dual ideal of* $A^+(\Omega)$.

Proof. Suppose $T_\delta \neq \square$. Take $f, g \in T_\delta$ and any $h \in A(\Omega)$. Recall that $D(\Omega)$ is a filter of the lattice of open subsets of Ω. Since $f \leqq h$ implies $\text{supp}(f) \subseteq \text{supp}(h)$, and since $\text{supp}(f \wedge g) = \text{supp}(f) \cap \text{supp}(g)$, it follows that T_δ is a dual ideal of $A^+(\Omega)$. Also $\text{supp}(h^{-1}fh) = \text{supp}(f)h$, and h is a homeomorphism of Ω, so that T_δ is normal in $A(\Omega)$. Clearly $1 \notin T_\delta$ so T_δ is either empty or a proper normal dual ideal of $A(\Omega)$.

In fact, when Ω is 0-2-homogeneous T_δ is not empty. The next lemma describes the elements of $\delta(\Omega)$ in this case.

We shall say that a pairwise disjoint collection $\{K_i : i \in I\}$ of subsets of Ω is a *topological partition* of Ω if $\bigcup\{K_i : i \in I\}$ is dense in Ω (for the order topology). If $K \subseteq \Omega$ we say that $S \subseteq K$ is *terminal* in K if for all $x \in K$ there are $a, b \in S$ such that $a \leqq x \leqq b$.

LEMMA 6. *If* Ω *is dense in itself then there is a topological partition* $\{K_i : i \in I\}$ *of* Ω *for which each* K_i *is a convex set with a countable terminal subset.*

Proof. Let X denote the set of all collections $\{K_i : i \in I\}$ where each K_i is a convex subset of Ω with a countable terminal subset, and $K_i \cap K_j = \square$ if $i \neq j$. Then X, ordered by inclusion, is an inductive set, so let $\{K_i : i \in I\}$ be a maximal element of X. If $\Delta = \bigcup\{K_i : i \in I\}$ is not dense in Ω then there is a non-empty open interval (x, y) contained in $\Omega\backslash\bar{\Delta}$. Since Ω is order-dense the interval (x, y) contains a convex set K with a countable terminal subset, and for this K we have $K \cap K_i = \Box$ for all $i \in I$. However this contradicts the maximality of $\{K_i : i \in I\}$ in X.

THEOREM 7. *If Ω is 0-2-homogeneous then T_δ is a compatible tight Riesz order on $A(\Omega)$.*

Proof. Suppose that Ω is 0-2-homogeneous. To show that both $T_\delta \neq \Box$ and $\inf T = 1$, it is sufficient to take any $w \in \Omega$ and then find $g \in T_\delta \cap A_w(\Omega)$. So take $w \in \Omega$ and let $\Omega_1 = \{x \in \Omega : x < w\}$ and $\Omega_2 = \{x \in \Omega : x > w\}$. By Lemma 6 we can write $\Omega_1 = \overline{\dot{\bigcup}\{K_i : i \in I\}}$, where $K_i = \bigcup\{[x_{i(n)}, x_{i(n+1)}] : n \in \mathbf{Z}\}$ with $x_{i(n)} < x_{i(n+1)}$ for all $n \in \mathbf{Z}$. For each $i \in I$ and $n \in \mathbf{Z}$ let $\phi_{i(n)}$ be an order-isomorphism from $[x_{i(n)}, x_{i(n+1)}]$ onto $[x_{i(n+1)}, x_{i(n+2)}]$. Then $g_1 : \Omega_1 \to \Omega_1$ defined by

$$xg_1 = \begin{cases} x\phi_{i(n)} & \text{if } x \in [x_{i(n)}, x_{i(n+1)}) \text{ for some } i(n) \\ x & \text{otherwise} \end{cases}$$

is an element of $A^+(\Omega_1)$, and $\operatorname{supp}(g_1) = \bigcup\{K_i : i \in I\}$ is dense in Ω_1. Similarly we can find $g_2 \in A^+(\Omega_2)$ with $\operatorname{supp}(g_2)$ dense in Ω_2. Then $g : \Omega \to \Omega$ defined by

$$xg = \begin{cases} xg_1 & \text{if } x \in \Omega_1 \\ x & \text{if } x = w \\ xg_2 & \text{if } x \in \Omega_2 \end{cases}$$

is an element of $T_\delta \cap A_w(\Omega)$. By Lemma 5 it remains to show that $T_\delta = T_\delta T_\delta$. Since T_δ is a dual ideal of $A^+(\Omega)$ it is also a subsemigroup, and since $A(\Omega)$ is divisible (Corollary 2) and $\operatorname{supp}(g) = \operatorname{supp}(g^2)$ for all $g \in A(\Omega)$ we have $T_\delta \subseteq T_\delta T_\delta$.

There are two obvious compatible tight Riesz orders larger than T_δ. Namely $T_\rho = \{g \in A^+(\Omega) : \operatorname{supp}(g) \cap [x, \infty) \text{ is dense in } [x, \infty) \text{ for some } x \in \Omega\}$, and its dual T_λ. (Here $[x, \infty) = \{y \in \Omega : y \geqq x\}$). When are these compatible tight Riesz orders maximal? Not always, we suspect. The following theorem gives a partial answer.

THEOREM 8. *If Ω is 0-2-homogeneous and has a countable cofinal (coinitial) subset then $T_\rho(T_\lambda)$ is a maximal compatible tight Riesz order on $A(\Omega)$.*

Proof. Suppose Ω is 0-2-homogeneous with countable cofinal subset $z_1 < z_2 < \ldots$ (that is, for each $x \in \Omega$ there is an n for which $x \leqq z_n$). Assume that T is a compatible tight Riesz order properly containing T_ρ. Then there is a $g \in T$ with fixed intervals $[x_n, y_n]$ such that $z_n \leqq x_n < y_n \leqq x_{n+1}$ for all natural numbers n. We choose arbitrary elements x_n, y_n $(n = 0, -1, -2, \ldots)$ in Ω satisfying $y_{n-1} < x_n < y_n < z_1$. Then (as in our previous constructions) there is an $h \in A(\Omega)$ satisfying $x_nh = y_n$ and $y_nh = x_{n+1}$ for all integers n. We see that the support of $g \wedge h^{-1}gh$ is bounded above. If $x \geqq x_1$, then $x \in [x_n, y_n]$ for some integer n, in which case $xg = x$, or $x \in [y_n, x_{n+1}]$ for some integer n, in which case $x(h^{-1}gh) = x$. Since these are the only possibilities for $x \geqq x_1$ it follows that $\operatorname{supp}(g \wedge h^{-1}gh)$ is bounded above by x_1. We can then find $k \in T_\rho$ with $xk = x$ for $x \leqq x_1$, and therefore $1 = k \wedge g \wedge h^{-1}gh \in T$—a contradiction.

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Maximal tangents. If F is a filter of the distributive lattice $A^+(\Omega)$ of positive elements of $A(\Omega)$ then any subset of $A^+(\Omega)$ maximal with respect to being a lattice ideal not meeting F is a prime ideal (this is a specialization of a well-known theorem of M. H. Stone). When T is a compatible tight Riesz order on $A(\Omega)$ the subsets of $A(\Omega)$ that are maximal with respect to being convex sublattice subgroups not meeting T, are called the *maximal tangents* of T. Since convex sublattice subgroups of $A(\Omega)$ are generated by their intersection with $A^+(\Omega)$ as lattice ideals it follows that the maximal tangents of a compatible tight Riesz order are *prime subgroups* of $A(\Omega)$ (i.e. convex sublattice subgroups M of $A(\Omega)$ for which $A^+(\Omega)\backslash M$ is a dual ideal).

We shall denote the set of maximal tangents for a compatible tight Riesz order T by $\mathrm{Max}(T)$. A fundamental theorem due to Norman Reilly (1973) asserts that, always, $T = A^+(\Omega)\backslash \bigcup \mathrm{Max}(T)$.

Our objective in this section is to determine the maximal tangents of T_δ, and this turns out to be a piece of lattice theory.

We recall that a distributive lattice $\mathscr{L}$ with least element 0 is *quasi-pseudo-complemented* (or a distributive *-lattice) if for each $x \in \mathscr{L}$ there is a $y \in \mathscr{L}$ such $x \wedge y = 0$ and $(x \vee y)^* = (0)$ where, for $z \in \mathscr{L}$, $z^* = \{z' \in \mathscr{L} : z \wedge z' = 0\}$.

If we denote by R the congruence on $\mathscr{L}$ defined by xRy if $x^* = y^*$, and by D the set $\{z \in \mathscr{L} : z^* = (0)\}$ of dense elements of $\mathscr{L}$, then the following conditions, amongst others, are known to be equivalent (see, for instance, T. P. Speed (1969)):

(1) $\mathscr{L}$ is quasi-pseudo-complemented

(2) $\mathscr{L}/R$ is Boolean

(3) for any $x \in \mathscr{L}$ there is a $y \in \mathscr{L}$ satisfying $x^{**} = y^*$

(4) for any ideal I of $\mathscr{L}$ with $I \cap D = \square$ there is a minimal prime ideal $\supseteq I$.

Since a quasi-pseudo-complemented lattice $\mathscr{L}$ *has* dense elements the set D of dense elements of $\mathscr{L}$ is a filter and the prime ideals of $\mathscr{L}$ not meeting D are precisely the minimal prime ideals (Grätzer (1971), p. 169).

THEOREM 9. *If Ω is 0-2-homogeneous then $\Sigma(\Omega)$ is quasi-pseudo-complemented.*

Proof. Take any $g \in A^+(\Omega)$. Then $\Omega\backslash\mathrm{supp}(g)$ is closed for the order topology and can be written as a disjoint union of maximal closed intervals (whose end-points may be in $\bar{\Omega}$, the Dedekind completion of Ω). For each such interval $[x, y]$ we can find an automorphism of Ω whose support set is contained in and dense in $[x, y]$ by Theorem 7. If g' is the join of these automorphisms of Ω then $g \wedge g' = 1$ so that $\mathrm{supp}(g) \cap \mathrm{supp}(g') = \square$ and $\mathrm{supp}(g) \cup \mathrm{supp}(g') = \mathrm{supp}(g \vee g')$ is dense in Ω. By Corollary 5 we then have $(\mathrm{supp}(g) \cup \mathrm{supp}(g'))^* = \{\square\}$.

We recall that a prime subgroup M of a lattice-ordered group G is minimal prime if and only if for all $m \in M \cap G^+$ there is a $g \in G^+\backslash M$ such that $m \wedge g = 1$.

THEOREM 10. *If Ω is 0-2-homogeneous then the maximal tangents of T_δ are precisely the minimal prime subgroups of $A(\Omega)$.*

Proof. Let M be a maximal tangent. Then M is a prime subgroup. If $m \in M^+$ then, since $m \notin T_\delta$ and $\Sigma(\Omega)$ is quasi-pseudo-complemented, there exists an $m^* \in A^+(\Omega)$ with $m \wedge m^* = 1$ and $m \vee m^* \in T_\delta$. Thus $m^* \notin M$ and M is a minimal prime.

Conversely, let M be a minimal prime and $m \in M^+$. Then $m \wedge m^* = 1$, for some $m^* \in A^+(\Omega)\backslash M$. Therefore $m \notin T_\delta$ and $M \cap T_\delta = \emptyset$. Hence M is contained in a maximal tangent which, by the first part of the proof, is a minimal prime and therefore equal to M.

COROLLARY 11. *If $\{M_\lambda : \lambda \in \Lambda\}$ is a non-empty collection of minimal prime subgroups of $A(\Omega)$ left invariant by conjugation then $T = A^+(\Omega)\backslash \bigcup\{M_\lambda : \lambda \in \Lambda\}$ is a compatible tight Riesz order on $A(\Omega)$.*

We denote by A the normal convex sublattice subgroup of $A(\Omega)$ consisting of all $g \in A(\Omega)$ for which $\operatorname{supp}(g) \subseteq \Omega\backslash[x, \infty)$ for some $x \in \Omega$, and by B the dual normal convex sublattice subgroup of $A(\Omega)$.

COROLLARY 12. *The maximal tangent of the compatible tight Riesz order $T_\rho(T_\lambda)$ are precisely the minimal prime subgroups of $A(\Omega)$ lying above $A(B)$.*

Proof. Let M be a maximal tangent of T_λ. Since $T_\lambda \supseteq T_\delta$, M is a prime subgroup of $A(\Omega)$ not meeting T_δ and therefore M is contained in a maximal tangent of T_δ. That is, M is a minimal prime subgroup. Suppose that there is a $g \geqq 1$ in $A\backslash M$, so that, for some $x \in \Omega$, $[x, \infty) \subseteq \operatorname{fix}(g) = \Omega\backslash\operatorname{supp}(g)$. We can then find $h \in T_\rho$ satisfying $g \wedge h = 1$, so that either $g \in M$ or $h \in M$—both contradictory.

Suppose on the other hand that M is a minimal prime subgroup lying above A and that $M \cap T_\rho \neq \square$. Then there is a $g > 1$, $g \in M$ such that $[x, \infty) \cap \operatorname{supp}(g)$ is dense in $[x, \infty)$ for some $x \in \Omega$. Since M is a minimal prime subgroup there is an $h \geqq 1$ satisfying $g \wedge h = 1$ and $h \notin M$. Then we have $zh = z$ on $[x, \infty) \cap \operatorname{supp}(g)$—a dense subset of $[x, \infty)$—so $[x, \infty) \subseteq \operatorname{fix}(h) = \Omega\backslash\operatorname{supp}(h)$. That is, $h \in A \subseteq M$—a contradiction.

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