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A NEW PROOF OF A THEOREM OF MAGNUS

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ABSTRACT. Using naive algebraic geometric methods a new proof of the following celebrated theorem of Magnus is given: Let G be a group with a presentation having n generators and m relations. If G also has a presentation on $n-m$ generators, then G is free of rank $n-m$.

1. **Introduction.** In 1983 Culler and Shalen [CS] employed in the study of finitely generated groups algebraic geometric ideas with origins going back to Poincaré, to Fricke and Klein and to Voght (see [M1]). They took the space of all $\mathrm{SL}_2\,C$ representations of the fundamental group G of a compact 3 manifold. For any finitely generated group this set can be endowed with the structure of a complex affine algebraic variety, $R(G)$, whose coordinate functions are the matrix entries of generators of G. $R(G)$ is up to isomorphism independent of the finitely generated presentation of G chosen. They showed then that the space of characters of representations of G in $\mathrm{SL}_2\,C$ has the structure of an algebraic variety $K(G)$, *the character variety*, and obtained a theorem guaranteeing that the group G would split as an HNN extension or a free product with amalgamation provided the corresponding character variety of G had positive dimension.

Determining if some finitely presented group G is infinite can be a difficult problem in combinatorial group theory. However, in many instances algebraic geometric methods can be used to approach this question. For example, if the dimension of the character variety, $K(G)$, is positive then the group G is infinite. Methods like these were employed by G. Baumslag and Shalen in showing that a class of finitely presented groups were infinite [BS].

We have evidence of the usefulness of algebraic geometric methods in group theory, but can dimension theoretic methods such as the above be used in providing alternative proofs to some classical theorems in combinatorial group theory? In the book *The History of Combinatorial Group Theory* (see [CM]) under the section *The Commutator Calculus and the Lower Central Series* mention is made of the fact that one of the first applications of lower central series methods in combinatorial group theory was given by Magnus in 1939 (see [M2]) in proving the following theorem (we quote from [CM]): "If a group G is presented in terms of $m+r$ generators $a_1,\ldots,a_m,b_1,\ldots,b_r$ and of r defining relations, and if $b_1=,\ldots,b_r=1$ in G, then G is a free group freely generated by $a_1,\ldots,a_m$. This result is not as obvious as it appears to be. A brief proof using entirely different powerful tools which were developed much later is due to Stammbach [1967]".

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It is curious that Magnus' 1939's paper begins with the observation that the lower central series is not needed in proving that finitely generated free groups are Hopfian. In honor of Wilhelm Magnus it is the object of this paper to offer naive algebro-geometric proofs of the above theorem and also of the fact that finitely generated free groups are Hopfian. Let F_n denote the free group of rank n, W a non trivial word freely reduced, and $N(W)$ the normal subgroup generated by W. The proof of the next useful theorem will be given later.

THEOREM 1. $\operatorname{Dim}\Big(R\big(F_n/N(W)\big)\Big) < 3(n)$.

If $V \subset C^m$ and $W \subset C^n$ are varieties, a map from V to W is said to be *regular* if it is the restriction of some map from C^m to C^n which is defined by n polynomials in m variables.

PROPOSITION 1. *Let V and W be irreducible varieties and set $m = \operatorname{Dim}(V)$, $n = Dim(W)$. Let $f\colon V \longrightarrow W$ be an almost surjective regular map; then for each point w in the image of f, each irreducible component of $f^{-1}(w)$ has dimension $\geq m - n$.*

A proof of Proposition 1 is easily deduced from results proven in [MD].

Given a finitely presented non free group G, the set $\operatorname{Hom}(G, \mathrm{SL}_2\, C)$ inherits the structure of a complex affine variety denoted by $R(G)$. To see this isn't difficult; for let $G = \langle x_1, \ldots, x_n;\ w_1 = 1, \ldots, w_m = 1\rangle$, where the words w_i are in the generators $\{x_1, \ldots, x_n\}$. Then the words $w_1, \ldots, w_m$ can be used to define a regular map $\Phi\colon (\mathrm{SL}_2\, C)^n \longrightarrow (\mathrm{SL}_2\, C)^m$, given by $\Phi(M) = \big(w_1(M), \ldots, w_m(M)\big)$, for M any n-tuple of matrices in $(\mathrm{SL}_2\, C)^n$. Notice that, $R(G) = \Phi^{-1}(I, \ldots, I)$, where I is the 2×2 identity matrix. $R(G)$ being the inverse image of a closed set (namely a single point) in the Zariski topology is itself closed, and thus an affine variety. We could assume that G is only finitely generated since by the Hilbert Basis Theorem only a finite number of relations suffice to define $R(G)$. In the case G is a free group of rank n, then $R(G) = (\mathrm{SL}_2\, C)^n$, since any n-tuple of matrices give rise to a representation of G in $(\mathrm{SL}_2\, C)^n$.

Notice that an immediate consequence of this is that if G is free of rank n, then $\operatorname{Dim}\big(R(G)\big) = 3n$ and $R(G)$ is irreducible since it is the product of n-copies of $(\mathrm{SL}_2\, C)$, an irreducible variety of dimension 3.

LEMMA 1. *Let G have presentation $G = \langle x_1, \ldots, x_n;\ w_1 = 1, \ldots, w_m = 1\rangle$, where the W_i's are freely reduced. Then $\operatorname{Dim}\big(R(G)\big) \geq 3(\operatorname{Def} G)$, where Def stands for the deficiency of the presentation.*

PROOF. This is a direct result of Proposition 1 and the fact that the presentation of G can be used to obtain a regular map $\Phi\colon (\mathrm{SL}_2\, C)^n \longrightarrow (\mathrm{SL}_2\, C)^m$. Now by Proposition 1, and using the fact that $\operatorname{Dim}(\mathrm{SL}_2\, C)^n = 3n$, and $\operatorname{Dim}\big((\mathrm{SL}_2\, C)^m\big) = 3m$, we obtain that $\operatorname{Dim}\big(R(G)\big) \geq 3(n-m) = 3(\operatorname{Def} G)$.

THEOREM 1. $\operatorname{Dim}\Big(R\big(F_n/N(W)\big)\Big) < 3(n)$.

PROOF. A well known fact is that the free group of rank n has many faithful representations in $\mathrm{SL}_2 C$ (see [SN]). Let $(m_1, \ldots, m_n)$ be the n-tuple of matrices assigned to the n generators by a faithful representation ρ of F_n in $\mathrm{SL}_2 C$. Then, since W is a freely reduced word in F_n, $\rho(W) \neq 1$. Consequently, $R\big(F_n/N(W)\big) \neq R(F_n)$. However $R\big(F_n/N(W)\big) \subset R(F_n)$. It follows then (since any proper subvariety of an irreducible variety has to be of dimension strictly smaller) that $\mathrm{Dim}\Big(R\big(F_n/N(W)\big)\Big) < 3(n)$.

COROLLARY 1. *Let $G = \langle x_1, \ldots, x_n;\ w = 1\rangle$, where $w \neq 1$ is a freely reduced word. Then* $\mathrm{Dim}\, R(G) < 3n$.

COROLLARY 2 (NIELSEN). *Any set of n elements which generate a free group of rank n are a set of free generators.*

COMMENT. A very interesting proof of Corollary 2 was given by J. Birman in 1973 as an application of her inverse function theorem for free groups (see [BJ]).

THEOREM (NIELSEN 1921, HOPF 1931). *Finitely generated free groups are Hopfian.*

PROOF. This is a direct result of Theorem 1 and the fact that if N is a normal subgroup of G, then $\mathrm{Dim}\big(R(G)\big) \geq \mathrm{Dim}\big(R(G/N)\big)$. The minor details are left to the reader.

THEOREM (MAGNUS, 1939). *Let G have presentation $G = \langle x_1, \ldots, x_n;\ r_1, \ldots, r_m\rangle$. If G can be generated by $n - m$ elements, then G is free of rank $n - m$.*

PROOF. If G can be generated by $n - m$ elements then $\mathrm{Dim}\big(R(G)\big) \leq 3(n - m)$, and if G is not free $\mathrm{Dim}\big(R(G)\big) < 3(n - m)$ by Theorem 1. Assume that G is not free of rank $(n-m)$, and a contradiction will arise. Since G has a presentation with deficiency $(n-m)$, by Lemma 1 $\mathrm{Dim}\big(R(G)\big) \geq 3(n - m)$. So $\mathrm{Dim}\big(R(G)\big) < 3(n - m)$ and $\mathrm{Dim}\big(R(G)\big) \geq 3(n - m)$. This is a contradiction. We must then conclude that G is free of rank $(n - m)$.

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