UNIT ORTHODOX SEMIGROUPS

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Let S be a regular semigroup. Given $x \in S$, we shall say that $a \in S$ is an *associate* of x if $xax = x$. The set of associates of $x \in S$ will be denoted by $A(x)$. Now suppose that S has an identity element 1. Let H_1 denote the group of units of S. Then we say that $u \in S$ is a *unit associate* of x whenever $u \in A(x) \cap H_1$. In what follows we shall write $U(x) = A(x) \cap H_1$ and we shall say that S is *unit regular* [**1**, **3**] if $(\forall\, x \in S)\, U(x) \neq \varnothing$. Examples of unit regular semigroups include the full transformation semigroup on a finite set [**1**] and the semigroup of endomorphisms of a finite-dimensional vector space [**3**]. In this paper we shall be concerned with semigroups that are *unit orthodox* (i.e. unit regular and orthodox), and we shall describe completely the structure of those semigroups that are *uniquely unit orthodox* (i.e. orthodox and uniquely unit regular in the sense that, for every $x \in S$, the set $U(x)$ is a singleton). It is worthy of mention that neither of the examples cited above is of this type.

We begin by examining the sets of unit associates.

Theorem 1. *Let S be a unit orthodox semigroup. If $x \in S$ then*

$$u,\, v,\, w \in U(x) \Rightarrow uv^{-1}w \in U(x).$$

Proof. If $u, v, w \in U(x)$ then it is readily seen that vxw and wxu are inverses of x. Denoting by $\mathcal{Y}$ the finest inverse semigroup congruence on S we therefore have, S being orthodox, that $vxw \,\mathcal{Y}\, wxu$ whence $xw \,\mathcal{Y}\, v^{-1}wxu$. Now since xw is idempotent and $\mathcal{Y}$ is idempotent-determined it follows that $v^{-1}wxu$ is idempotent, so that

$$v^{-1}wxu \,.\, v^{-1}wxu = v^{-1}wxu$$

whence, by cancellation, we obtain $uv^{-1}w \in U(x)$. □

Corollary 1. *For every $x \in S$, $U(x)$ is a coset of some subgroup of H_1.*

Proof. This is immediate from Theorems 22 and 23 of [**2**], which together show that a non-empty subset X of a group G is a coset if and only if $X = XX^{-1}X$. □

Corollary 2. *If $e \in S$ is idempotent then $U(e)$ is a subgroup of H_1.*

Proof. Taking $x = e$ in the theorem we can choose $w = 1$. □

If $u \in U(x)$ then xu is idempotent and so $U(xu)$ is a subgroup of H_1. It is instructive to see how $U(x)$ can be expressed as a coset of this subgroup.

Theorem 2. *Let S be a unit orthodox semigroup. Given $x \in S$, we have that*

$$(\forall\, u \in U(x)) \qquad U(x) = uU(xu).$$

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Proof. Let $u, v \in U(x)$. Then since $xu \,.\, u^{-1}v \,.\, xu = xu$ we have $u^{-1}v \in U(xu)$ whence $v \in uU(xu)$ and consequently $U(x) \subseteq uU(xu)$. But if $w \in U(xu)$ then $xu \,.\, w \,.\, xu = xu$ gives $xuwx = x$ so that $uw \in U(x)$. Thus we have the converse inclusion $uU(xu) \subseteq U(x)$. □

The above observations may be used to obtain a simple necessary and sufficient condition for a unit orthodox semigroup to be uniquely unit orthodox:

THEOREM 3. *A unit orthodox semigroup S is uniquely unit orthodox if and only if, for every idempotent e, the subgroup $U(e)$ is trivial.*

Proof. The condition is clearly necessary. Suppose, conversely, that $U(e) = \{1\}$ for every idempotent e. Given $x \in S$, let $u, v \in U(x)$. By Theorem 2 we have $uU(xu) = U(x) = vU(xv)$ whence, since xu and xv are idempotents, we deduce that $u = v$. □

We recall now that if S and T are semigroups and if $\zeta : T \to \mathrm{Aut}(S)$, described by $x \mapsto \zeta_x$, is a contravariant morphism from T to the group of automorphisms on S then the *semidirect product* (relative to ζ) of S and T is the semigroup $S \times_\zeta T$ consisting of the set $S \times T$ equipped with the law of composition

$$(a, x)(b, y) = (a \,.\, b\zeta_x, xy).$$

[*Note.* The contravariance of ζ is required when writing the mappings ζ_x on the right.]

We now show how the notion of semidirect product can be used to provide a method of constructing uniquely unit orthodox semigroups.

THEOREM 4. *Let E be a band with an identity element and let G be a group. Then for every contravariant morphism $\zeta : G \to \mathrm{Aut}(E)$ the semidirect product $E \times_\zeta G$ is a uniquely unit orthodox semigroup whose band of idempotents is isomorphic to E and whose group of units is isomorphic to G.*

Proof. We shall write, without confusion, the identity elements of G and E as simply 1. Describing ζ by $x \mapsto \zeta_x$ we observe that $\zeta_1 = \mathrm{id}_E$. We also note that, for every $x \in G$ and every $e \in E$,

$$\begin{aligned} e \,.\, 1\zeta_x = e\zeta_1 \,.\, 1\zeta_x &= e\zeta_{xx^{-1}} \,.\, 1\zeta_x = e\zeta_{x^{-1}}\zeta_x \,.\, 1\zeta_x \\ &= (e\zeta_{x^{-1}} \,.\, 1)\zeta_x = e\zeta_{x^{-1}}\zeta_x = e\zeta_1 = e \end{aligned}$$

and similarly $1\zeta_x \,.\, e = e$, whence $1\zeta_x = 1$. It now follows that $(1, 1)$ is the identity element of $E \times_\zeta G$; for

$$(e, x)(1, 1) = (e \,.\, 1\zeta_x, x) = (e, x),$$
$$(1, 1)(e, x) = (1 \,.\, e\zeta_1, x) = (e, x).$$

Now if (e, x) is a unit in $E \times_\zeta G$ then it follows that e is a unit in the band E and so $e = 1$; and conversely, every element of the form $(1, x)$ is a unit of $E \times_\zeta G$ with $(1, x)^{-1} = (1, x^{-1})$. Thus the set of units of $E \times_\zeta G$ is $\{(1, x); x \in G\}$ and this is clearly isomorphic to the group G under the assignment $x \mapsto (1, x)$.

If now (e, x) is an idempotent then obviously $x=1$; and conversely $(e, 1)(e, 1)=(e \,.\, e\zeta_1, 1)=(e^2, 1)=(e, 1)$. Since $(e, 1)(f, 1)=(ef, 1)$ it is now clear that the mapping described by $e \mapsto (e, 1)$ is an isomorphism from E to the band of idempotents of $E \times_\zeta G$.

Finally, the fact that

$$(e, x)(1, x^{-1})(e, x)=(e \,.\, 1\zeta_x, 1)(e, x)=(e, 1)(e, x)=(e, x)$$

shows that $E \times_\zeta G$ is unit orthodox; and

$$(e, x)(1, y)(e, x)=(e, x) \Leftrightarrow y=x^{-1}$$

so that $E \times_\zeta G$ is uniquely unit orthodox. □

That every uniquely unit orthodox semigroup can be obtained in the above way is established in the following result.

THEOREM 5. *Let S be a uniquely unit orthodox semigroup with band of idempotents E. For every $a \in S$ let $U(a)=\{u_a\}$; and for every $u \in H_1$ let $\zeta_u : E \to E$ be described by $e\zeta_u = ueu^{-1}$. Then $u \mapsto \zeta_u$ describes a contravariant morphism $\zeta : H_1 \to \mathrm{Aut}(E)$ and $S \simeq E \times_\zeta H_1$ under the assignment $a \mapsto (au_a, u_a^{-1})$.*

Proof. We note first that, for every $u \in H_1$, the mapping ζ_u is clearly injective and a morphism. It is also surjective since for every $f \in E$ we have $u^{-1}fu \in E$ with $(u^{-1}fu)\zeta_u = f$. Thus $\zeta_u \in \mathrm{Aut}(E)$ for every $u \in H_1$. That $\zeta : H_1 \to \mathrm{Aut}(E)$ described by $u \mapsto \zeta_u$ is a contravariant morphism is readily seen. We can therefore construct the semidirect product $E \times_\zeta H_1$. Now since for every $a \in S$ we have $U(a)=\{u_a\}$, it follows that $au_a \in E$ and we can consider the mapping $\theta : S \to E \times_\zeta H_1$ described by $a\theta = (au_a, u_a^{-1})$. Since

$$(au_a, u_a^{-1})=(bu_b, u_b^{-1}) \Rightarrow a = au_a \,.\, u_a^{-1} = bu_b \,.\, u_b^{-1} = b$$

we have that θ is injective. To see that θ is also surjective, let $(e, x) \in E \times_\zeta H_1$. Then we deduce from $ex \,.\, x^{-1} \,.\, ex = ex$ that $u_{ex} = x^{-1}$, so

$$(ex)\theta = (ex \,.\, u_{ex}, u_{ex}^{-1}) = (ex \,.\, x^{-1}, x) = (e, x).$$

Finally, we show that θ is a morphism. For this purpose, we note that $a = au_a a$ implies that $u_a a u_a$ is an inverse of a. Since S is orthodox, $u_b b u_b \,.\, u_a a u_a$ is then an inverse of ab. Thus

$$ab = ab \,.\, u_b b u_b u_a a u_a \,.\, ab = a \,.\, bu_b b \,.\, u_b u_a \,.\, au_a a \,.\, b = ab \,.\, u_b u_a \,.\, ab$$

and hence $u_b u_a \in U(ab)$. But $U(ab)=\{u_{ab}\}$ so we have that $u_b u_a = u_{ab}$. Using this we now see that

$$\begin{aligned} a\theta \,.\, b\theta &= (au_a, u_a^{-1})(bu_b, u_b^{-1}) \\ &= (au_a \,.\, (bu_b)\zeta_{u_a^{-1}}, u_a^{-1}u_b^{-1}) \\ &= (au_a u_a^{-1} bu_b u_a, u_a^{-1}u_b^{-1}) \\ &= (abu_b u_a, (u_b u_a)^{-1}) \\ &= (abu_{ab}, u_{ab}^{-1}) \\ &= (ab)\theta. \quad \square \end{aligned}$$

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REMARK. In Theorem 5 we proved that if S is uniquely unit orthodox then for all $a, b \in S$ we have $u_{ab} = u_b u_a$. Conversely, this property implies that S is orthodox; for if e, f are idempotents then

$$ef = efu_{ef}ef = efu_f u_e ef = ef \,.\, 1 \,.\, 1 \,.\, ef = (ef)^2.$$

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