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CONSERVED QUANTITIES OF BOX AND BALL SYSTEM

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Abstract. Conserved quantities of box and ball system(BBS) are presented from the hungry Toda molecule equation, an inverse ultra-discrete limit of the BBS.

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Soliton cellular automaton [**1**,**2**], or sometimes referred to box and ball system (BBS), has attracted much interest in physics and mathematics. Recently a method to construct CAs from continuous equations was proposed [**3**]. It is based on a limiting procedure and is called ultra-discretization. This method has been found to be particularly effective for obtaining the CAs in which soliton solutions of original continuous equations are transformed into soliton patterns [**4**–**6**]. The BBS is also extended from the viewpoint of the crystal formulation [**7**–**9**].

In the present paper [**10**], we have given two proofs for solitonical nature of the BBS, utilizing its solutions and conserved quantities. Although these conserved quantities, whose number is equal to that of solitons, are sufficient to prove the solitonical nature, they were not all. The purpose of this paper is to find new conserved quantities of the BBS.

Following reference [**10**], we first review the BBS and its conserved quantities. Let us think of an infinite array of boxes in a line and M ($M \geq 1$) kinds of balls [**2**]. The number of balls is finite. We assume that each of the boxes either contains one ball or is empty. We distinguish these M kinds of balls by integer indices $1, 2, \cdots, M$. The time evolution of the BBS is illustrated in Figure 1. It should be noted that we have changed the time evolution rule, for later convenience' sake, as follows; from time t to $t+1$, move only one kind of balls (index $(t \bmod M)+1$).

In order to find the conserved quantities, let us introduce new dependent variables $Q_1^t, Q_2^t, \cdots, Q_N^t$ and $E_0^t, E_1^t, \cdots, E_N^t$, where N is the number of solitons. Let Q_n^t be the number of balls, which move from t to $t+1$, in the n-th soliton from the left and E_n^t be the number of empty boxes between the n-th and $(n+1)$-th solitons with boundary conditions $E_0^t, E_N^t = \infty$. Then, they satisfy the following set of equations [**10**,**11**]:

$$Q_n^{t+M} = \min\left[\sum_{j=1}^{n} Q_j^t - \sum_{j=1}^{n-1} Q_j^{t+M},\ E_n^t\right], \tag{1}$$

$$E_n^{t+1} = Q_{n+1}^t + E_n^t - Q_n^{t+M}, \tag{2}$$

$$E_0^t, E_N^t = +\infty. \tag{3}$$

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$t=0$ $\cdots$ 0 1 1 1 2 2 3 0 0 0 1 2 2 0 0 2 3 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 $\cdots$
$t=1$ $\cdots$ 0 0 0 0 2 2 3 1 1 1 0 2 2 1 0 2 3 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 $\cdots$
$t=2$ $\cdots$ 0 0 0 0 0 0 3 1 1 1 2 0 0 1 2 0 3 2 2 2 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 $\cdots$
$t=3$ $\cdots$ 0 0 0 0 0 0 0 1 1 1 2 3 0 1 2 0 0 2 2 2 3 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 $\cdots$
$t=4$ $\cdots$ 0 0 0 0 0 0 0 0 0 0 2 3 1 0 2 1 1 2 2 2 3 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 $\cdots$
$t=5$ $\cdots$ 0 0 0 0 0 0 0 0 0 0 0 3 1 2 0 1 1 0 0 0 3 1 2 2 2 2 0 0 0 0 0 0 0 0 0 0 0 0 0 0 $\cdots$
$t=6$ $\cdots$ 0 0 0 0 0 0 0 0 0 0 0 0 1 2 3 1 1 0 0 0 0 1 2 2 2 2 3 0 0 0 0 0 0 0 0 0 0 0 0 0 $\cdots$
$t=7$ $\cdots$ 0 0 0 0 0 0 0 0 0 0 0 0 0 2 3 0 0 1 1 1 0 0 2 2 2 2 3 1 0 0 0 0 0 0 0 0 0 0 0 0 $\cdots$
$t=8$ $\cdots$ 0 0 0 0 0 0 0 0 0 0 0 0 0 0 3 2 0 1 1 1 0 0 0 0 0 0 3 1 2 2 2 2 0 0 0 0 0 0 0 0 $\cdots$
$t=9$ $\cdots$ 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 2 3 1 1 1 0 0 0 0 0 0 0 1 2 2 2 2 3 0 0 0 0 0 0 0 $\cdots$
$t=10$ $\cdots$ 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 2 3 0 0 0 1 1 1 0 0 0 0 0 2 2 2 2 3 1 0 0 0 0 0 0 $\cdots$
$t=11$ $\cdots$ 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 3 2 0 0 1 1 1 0 0 0 0 0 0 0 0 0 3 1 2 2 2 2 0 0 $\cdots$
$t=12$ $\cdots$ 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 2 3 0 1 1 1 0 0 0 0 0 0 0 0 0 0 1 2 2 2 2 3 0 $\cdots$

Figure 1. Time evolution of BBS.

Through the inverse ultra-discretization, or roughly speaking, replacing operators (min, +) by (+, $\times$), we have

$$I_n^{t+M} = I_n^t + V_n^t - V_{n-1}^{t+1}, \tag{4}$$

$$V_n^{t+1} = \frac{I_{n+1}^t V_n^t}{I_n^{t+M}}, \tag{5}$$

$$V_0^t = V_N^t = 0, \tag{6}$$

which we named the hungry Toda molecule equation. This equation is equivalent to the following matrix difference equation;

$$\mathbf{L}(t+1)\mathbf{R}(t+M) = \mathbf{R}(t)\mathbf{L}(t) \tag{7}$$

$$\mathbf{L}(t) = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ V_1^t & 1 & \ddots & & \vdots \\ 0 & V_2^t & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & V_{N-1}^t & 1 \end{bmatrix}, \mathbf{R}(t) = \begin{bmatrix} I_1^t & 1 & 0 & \cdots & 0 \\ 0 & I_2^t & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & I_N^t \end{bmatrix}. \tag{8}$$

By introducing a new matrix $\mathbf{X}(t)$ as

$$\mathbf{X}(t) \equiv \mathbf{L}(t)\mathbf{R}(t+M-1)\cdots\mathbf{R}(t+1)\mathbf{R}(t) \tag{9}$$

equation (7) is equivalent to the following isospectral flow

$$\mathbf{X}(t+1)\mathbf{R}(t) = \mathbf{R}(t)\mathbf{X}(t) \Leftrightarrow \mathbf{X}(t+1) = \mathbf{R}(t)\mathbf{X}(t)\mathbf{R}(t)^{-1}. \tag{10}$$

Therefore, if one expands the determinant as

$$\det[\mathbf{I}_N - \lambda\mathbf{X}(t)] = 1 + \sum_{j=1}^{N} \lambda^j C_j, \quad \mathbf{I}_N : N \times N \text{ identity matrix}, \tag{11}$$

coefficients in λ given by $C_n (n = 1, 2, \cdots, N)$ are conserved quantities of the hungry Toda molecule equation. By taking an ultra-discrete limit of C_i's, we obtain N conserved quantities for the BBS,

$$uC_l = \min_{\substack{j_1, j_2, \cdots, j_l \\ j_1 < j_2 - M < \cdots <= j_l - (l-1)M}} \left(\sum_{m=1}^{M} W^t_{j_1+m} + \sum_{m=1}^{M} W^t_{j_2+m} + \cdots + \sum_{m=1}^{M} W^t_{j_l+m} \right) \tag{12}$$
$$(l = 1, 2, \cdots, N)$$

where $W^t_1, W^t_2, W^t_3, \cdots, W^t_{MN+N-1}$ are equal to

$$Q^t_1, Q^{t+1}_1, \cdots, Q^{t+M-1}_1, E^t_1, Q^t_2, Q^{t+1}_2, \cdots, Q^{t+M-1}_2, E^t_2, Q^t_3, \cdots,$$
$$Q^t_{N-1}, Q^{t+1}_{N-1}, \cdots, Q^{t+M-1}_{N-1}, E^t_{N-1}, Q^t_N, Q^{t+1}_N, \cdots, Q^{t+M-1}_N,$$

respectively. They are also regarded as

$$W^t_{(M+1)(n-1)+m} = \begin{cases} \text{number of balls with index } \{(t+m) \bmod M\}+1 & \\ \quad \text{in the } n\text{-th soliton from the left at time } t & (m \neq M+1), \\ \text{number of boxes between the } n\text{-th} & \\ \quad \text{and } (n+1)\text{-th solitons from the left at time } t & (m = M+1), \end{cases}$$

where $1 \le n \le N, \ 1 \le m \le M+1$ and $1 \le m \le M$ if $n = N$

REMARK 1. In the case of $M = 1$, the above conserved quantities are essentially the same as those in reference [**12**].

REMARK 2. Recently, Fukuda, Okado and Yamada [**9**] derived conserved quantities E_m, making use of the so-called energy function of the BBS. They are equivalent to $uC_1, \cdots, uC_N$ in the following sense: let us construct a tableau whose i-th column from the left is as long as the i-th longest soliton in the BBS for $i = 1, 2, \cdots, N$. (See the left tableau in Figure 2.) Then uC_k corresponds to the size of subset of the tableau which consists of k columns from the right and E_m to one which consists of m rows from the top.

We next derive the other new conserved quantities. Though the conserved quantities given by equation (12) are sufficient to prove the solitonical nature of the

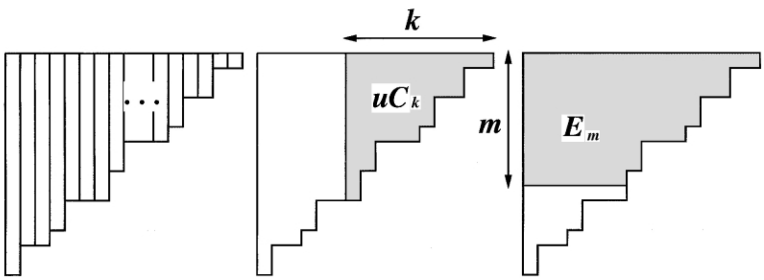

Figure 2. Relation between conserved quantities uC_k and E_m.

BBS, they are not all. As a simple example, the number of balls with a certain index is conserved but is not included in the conserved quantities we have found. Strictly speaking, let us introduce a dependent variable $\tilde{\Xi}^t$, given by

$$\tilde{\Xi}^t = \sum_{n=1}^{N} Q_n^t, \tag{13}$$

that represents the number of balls with index (t mod M) + 1. From the definition of Q_n^t, it is easy to prove the relation,

$$\tilde{\Xi}^{t+M} = \tilde{\Xi}^t. \tag{14}$$

Furthermore, if we define dependent variables Ξ_m as

$$\Xi_m = \min_{0 \le i_1 < i_2 < \cdots < i_m \le M-1} = \left(\tilde{\Xi}^{t+i_1} + \cdots + \tilde{\Xi}^{t+i_m}\right) \quad (m = 1, 2, \cdots, M), \tag{15}$$

they are also conserved quantities ($m = 1, 2, \cdots, M$). However, they are not included in $uC_1, \cdots, uC_N$ except $\Xi_M (= uC_N)$.

In order to investigate the conserved quantities above, we modify the Lax form (7) with one parameter a as follows.

$$\mathbf{L}(t+1)\mathbf{R}_a(t+M) = \mathbf{R}_a(t)\mathbf{L}(t), \tag{16}$$

$$\mathbf{L}(t) = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ V_1^t & 1 & \ddots & & \vdots \\ 0 & V_2^t & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & V_{N-1}^t & 1 \end{bmatrix}, \quad \mathbf{R}_a(t) = \begin{bmatrix} I_1^t & 1 & 0 & \cdots & 0 \\ 0 & I_2^t & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & 1 \\ a & 0 & \cdots & 0 & I_N^t \end{bmatrix}. \tag{17}$$

In fact, equation (16) is also equivalent to the hungry Toda molecule equation, independently of a. Hence it is expected that if one calculates conserved quantities following the same procedures, new conserved quantities are generated from their coefficients in a.

To this end, we introduce a matrix $\mathbf{X}_a(t)$ defined by

$$\mathbf{X}_a(t) \equiv \mathbf{L}(t)\mathbf{R}_a(t+M-1)\cdots\mathbf{R}_a(t+1)\mathbf{R}_a(t). \tag{18}$$

Then equation (16) is rewritten as

$$\mathbf{X}_a(t+1) = \mathbf{R}_a(t)\mathbf{X}_a(t)\mathbf{R}_a(t)^{-1}. \tag{19}$$

It is shown by simple calculations that if we collect $\det \mathbf{X}_a(t)$ in a as

$$
\begin{aligned}
\det \mathbf{X}_a(t) &= \det \mathbf{L}(t) \cdot \prod_{j=0}^{M-1} \det \mathbf{R}_a(t+j) \\
&= \prod_{j=0}^{M-1} (a + \prod_{n=1}^{N} I_n^{t+j}) \\
&= a^M + \sum_{k=1}^{M} \sigma_k \left(\prod_{n=1}^{N} I_n^t, \prod_{n=1}^{N} I_n^{t+1}, = \cdots, \prod_{n=1}^{N} I_n^{t+M-1} \right) a^{M-k},
\end{aligned} \tag{20}
$$

where $\sigma_j(x_1, \cdots, x_M)$ is the fundamental symmetric polynomial of j-th order, an ultra-discrete limit of each coefficient $\sigma_k\left(\prod_{n=1}^{N} I_n^t, \prod_{n=1}^{N} I_n^{t+1}, = \cdots, \prod_{n=1}^{N} I_n^{t+M-1}\right)$ gives

$$\Xi_k (k = 1, 2, \cdots, M).$$

We can also construct other conserved quantities. For simplicity, we consider the case $M = 2$. In this case, each element of the matrix $\mathbf{X}_a(t)$ is written as

$$
(\mathbf{X}_a(t))_{i,j} = \begin{cases}
I_{i-1}^t I_{i-1}^{t+1} V_{i-1}^t & i = j+1, \\
I_{i-1}^{t+1} V_{i-1}^t + V_{i-1}^t I_i^t + I_i^t I_i^{t+1} & i = j, \\
V_{i-1}^t + I_i^{t+1} + I_{i+1}^t & i = j-1, \\
1 & i = j-2, \\
a & (i,j) = (N-1, 1), (N, 2), \\
a(I_1^t + V_{N-1}^t + I_N^{t+1}) & (i,j) = (N, 1), \\
0 & \text{otherwise.}
\end{cases} \tag{21}
$$

If one expands a determinant $\det[\mathbf{I}_N - \lambda \mathbf{X}_a(t)]$ as

$$\det[\mathbf{I}_N - \lambda \mathbf{X}_a(t)] = \det[\mathbf{I}_N - \lambda \mathbf{X}_0(t)] + a \sum_{j=1}^{N} = \omega_j \lambda^j + a^2(-\lambda)^N \tag{22}$$

and takes ultra-discrete limits of coefficients $\omega_j (j = 1, \cdots, N)$, which are not identically zero, it is expected that other new conserved quantities are generated. Through straightforward calculations, we obtain the following conserved quantities*:

$$
\left\{
\begin{array}{lcl}
\Omega_1 & = & \min\limits_{1 \le i_1 \le 3N-1} W_{i_1}^t, \\
\Omega_3 & = & \min\limits_{\substack{1 \le i_1 < i_2 < i_3 \le 3N-1 \\ i_{k+1} - i_k \in \{3,5,7,\cdots\}}} = (W_{i_1}^t + W_{i_2}^t + W_{i_3}^t), \\
\Omega_5 & = & \min\limits_{\substack{1 \le i_1 < i_2 < \cdots < i_5 \le 3N-1 \\ i_{k+1} - i_k \in \{3,5,7,\cdots\}}} = (W_{i_1}^t + W_{i_2}^t + \cdots + W_{i_5}^t), \\
 & \vdots & \\
\Omega_N & = & \min\limits_{\substack{1 \le i_1 < i_2 < \cdots < i_N \le 3N-1 \\ i_{k+1} - i_k \in \{3,5,7,\cdots\}}} (W_{i_1}^t + W_{i_2}^t + \cdots + W_{i_N}^t),
\end{array}
\right\} \quad (N \text{ is odd}). \tag{23}
$$

*Each index j in ω_j does not necessarily coincide with one in Ω_j.

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$$\left\{\begin{array}{lcl} \Omega_2 & = & \displaystyle\min_{\substack{1\le i_1<i_2\le 3N-1= \\ i_2-i_1\in\{3,5,7,\cdots\}}} (W^t_{i_1}+W^t_{i_2}), \\ \Omega_4 & = & \displaystyle\min_{\substack{1\le i_1<i_2<i_3<i_4\le 3N-1 \\ i_{k+1}-i_k\in\{3,5,7,\cdots\}}} = (W^t_{i_1}+W^t_{i_2}+W^t_{i_3}+W^t_{i_4}), \\ \Omega_6 & = & \displaystyle\min_{\substack{1\le i_1<i_2<\cdots<i_6\le 3N-1 \\ i_{k+1}-i_k\in\{3,5,7,\cdots\}}} = (W^t_{i_1}+W^t_{i_2}+\cdots+W^t_{i_6}), \\ & \vdots & \\ \Omega_N & = & \displaystyle\min_{\substack{1\le i_1<i_2<\cdots<i_N\le 3N-1 \\ i_{k+1}-i_k\in\{3,5,7,\cdots\}}} (W^t_{i_1}+W^t_{i_2}+\cdots+W^t_{i_N}), \end{array}\right\} \quad (N \text{ is even}). \tag{24}$$

We remark that the conserved quantity Ω_N is equal to Ξ_1.

The result above shows that conserved quantities of even or odd degree appear from the modified Lax form (16), depending on whether the number of solitons N is even or odd, respectively. In fact, however, if N is even, one can observe from numerical experiment that quantities of odd degree $(\Omega_1, \Omega_3, \Omega_5, \cdots, \Omega_{N-1})$ are also conserved. This is also valid in the case in which N is odd.

To overcome such a problem, we consider an additional soliton with length $Q^0_{N+1}+Q^1_{N+1}$ to the right of the N-th soliton at intervals of E^0_N (Figure 3). If N is odd (even), a degree of each conserved quantity calculated according to the procedures above is even (odd) because this BBS consists of $N+1$ solitons. Taking the limits $E^0_N, Q^0_{N+1}, Q^1_{N+1} \to +\infty$, we obtain conserved quantities of an N-soliton system, degrees of which are even (odd), because quantities $E^0_N, Q^0_{N+1}, Q^1_{N+1}$ do not contribute in taking the minimum.

Using similar procedures, we can calculate new conserved quantities in the case $M \ge 3$. They are given by

$$\Omega_l = \min_{\substack{1\le i_1, i_2, \cdots, i_l \le MN+N-1 \\ i_{k+1}-i_k\in\{-1, M-1, 2M-1, \cdots\} \\ i_{k+M-1}-i_k \ne -M+1}} (W^t_{i_1}+W^t_{i_2}+\cdots+W^t_{i_l}), \tag{25}$$

where $l = 1, 2, \cdots, MN-N$.

We have obtained three kinds of conserved quantities given by $\{uC_1, uC_2, \cdots, uC_N\}$, $\{\Xi_1, \Xi_2, \cdots, \Xi_M\}$ and $\{\Omega_1, \Omega_2, \cdots, \Omega_{MN-N}\}$. Since the relations $uC_N = \Xi_M$ and $\Omega_{MN-N} = \Xi_{M-1}$ hold, we have

$$N+M-1+MN-N-1 = MN+M-2 \ (M \ge 2)$$

conserved quantities in all. However, their independence is confirmed only in the case M is small and still remains unresolved otherwise. Conserved quantities for the BBS with larger size of boxes are not obtained because of the difficulty in the definition of dependent variables Q^t_n, E^t_n. In such cases, it is necessary to employ the ultra-discrete Lotka-Volterra equation instead of the Toda molecule equation. It is

$$t=0: \cdots 001112200012200012\overbrace{0\cdots0}^{E^0_N}\overbrace{1\cdots1}^{Q^0_{N+1}}\overbrace{2\cdots2}^{Q^1_{N+1}}0\cdots$$

Figure 3. The BBS with an additional soliton.

also interesting to note that the Toda molecule equation serves as numerical algorithms [**13**,**14**] and therefore the hungry Toda molecule equation also does as an extension of such algorithms.

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