Canad. Math. Bull. Vol. **49** (1), 1997 pp. 74–99

CONSTRAINED APPROXIMATION IN SOBOLEV SPACES

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ABSTRACT. Positive, copositive, onesided and intertwining (co-onesided) polynomial and spline approximations of functions $f \in \mathbf{W}_p^k[-1,1]$ are considered. Both uniform and pointwise estimates, which are exact in some sense, are obtained.

1. **Introduction and main results.** We start by recalling some of the notations and definitions used throughout this paper. Let $\mathbf{C}[a,b]$ and $\mathbf{C}^k[a,b]$ be, respectively, the sets of all continuous and k-times continuously differentiable functions on $[a,b]$, and let $\mathbf{L}_p[a,b]$, $0 < p < \infty$, be the set of measurable functions on $[a,b]$ such that $\|f\|_{\mathbf{L}_p[a,b]} < \infty$, where

$$\|f\|_{\mathbf{L}_p[a,b]} := \left\{ \int_a^b |f(x)|^p \, dx \right\}^{\frac{1}{p}}.$$

Throughout this paper $\mathbf{L}_\infty[a,b]$ is understood as $\mathbf{C}[a,b]$ with the usual uniform norm, to simplify the notation. We also denote by $\mathbf{W}_p^k[a,b]$, $p \geq 1$, the set of all functions f on $[a,b]$ such that $f^{(k-1)}$ are absolutely continuous and $f^{(k)} \in \mathbf{L}_p$, and by $\mathbf{P}_n$ the set of all polynomials of degree $\leq n$. The m-th symmetric difference of f is given by

$$\Delta_h^m(f,x,[a,b]) := \begin{cases} \sum_{i=0}^m \binom{m}{i}(-1)^{m-i} f(x - \frac{mh}{2} + ih), & \text{if } x \pm \frac{mh}{2} \in [a,b], \\ 0, & \text{otherwise.} \end{cases}$$

Then the m-th (usual) modulus of smoothness of $f \in \mathbf{L}_p[a,b]$ is defined by

$$\omega^m(f,t,[a,b])_p := \sup_{0<h\leq t} \|\Delta_h^m(f,\cdot,[a,b])\|_{\mathbf{L}_p[a,b]}.$$

We will also use the so-called τ-modulus, an averaged modulus of smoothness, defined for all bounded measurable functions on $[a,b]$ by

$$\tau^m(f,t,[a,b])_p := \|\omega^m(f,\cdot,t)\|_{\mathbf{L}_p[a,b]},$$

where

$$\omega^m(f,x,t) := \sup\{|\Delta_h^m(f,y)| : y \pm mh/2 \in [x - mt/2, x + mt/2] \cap [a,b]\}$$

is the m-th local modulus of smoothness of f. (We set $\tau^m(f,t,[a,b])_p := \infty$ if the function f is unbounded.) From the definition one can easily see that

$$\tau^m(f,t,[a,b])_\infty = \omega^m(f,t,[a,b])_\infty. \qquad (1.1)$$

Received by the editors February 22, 1996.
AMS subject classification: 41A10, 41A15, 41A25, 41A29.
Key words and phrases: Constrained approximation, polynomials, splines, degree of approximation, L_p space, Sobolev space.

https://doi.org/10.4153/CJM-1997-004-2 Published online by Cambridge University Press

The following relationship between the ω- and τ-moduli holds for any $f \in \mathbf{W}_p^1[a,b]$ and $1 \le p \le \infty$ (Sendov and Popov, [29, Theorem 1.5])

$$\tau^m(f,t,[a,b])_p \le C_m t \omega^{m-1}(f',t,[a,b])_p, \quad t \ge 0. \tag{1.2}$$

If the interval $[-1,1]$ is used in any of the above notations, it will be omitted for the sake of simplicity, for example,

$$\|f\|_p := \|f\|_{\mathbf{L}_p[-1,1]}, \quad \omega^m(f,t)_p := \omega^m(f,t,[-1,1])_p.$$

The moduli ω and τ measure the smoothness of f over the interval uniformly. It is well known that polynomials approximate better near the endpoints of the interval than in the middle, and this leads to either pointwise estimates (if $p = \infty$), or the introduction of "non-uniform" moduli of smoothness. The pointwise estimates for constrained approximation that we obtain in this paper are given in terms of $\omega^m\big(f,\Delta_n(x)\big)_\infty$, where $\Delta_n(x) := n^{-1}\sqrt{1-x^2}+n^{-2}$. The "non-uniform" modulus that we use is the m-th Ditzian-Totik modulus of smoothness, defined for $f \in \mathbf{L}_p[-1,1]$ by

$$\omega_\varphi^m(f,t)_p := \sup_{0<h\le t} \|\Delta_{h\varphi(\cdot)}^m(f,\cdot,[-1,1])\|_p,$$

with $\varphi(x) := \sqrt{1-x^2}$. We have

$$\omega_\varphi^m(f,t)_p \le \omega^m(f,t)_p \le \tau^m(f,t)_p \le 2^{\frac{1}{p}}\omega^m(f,t)_\infty, \quad 1 \le p \le \infty$$

and

$$\omega_\varphi^m(f,t)_p \le \omega^m(f,t)_p \le 2^{\frac{1}{p}}\omega^m(f,t)_\infty, \quad 0 < p < 1.$$

Let $Y_s := \{y_1,\dots,y_s \mid y_0 := -1 < y_1 < y_2 < \cdots < y_s < 1 =: y_{s+1}\}$, $s \ge 0$. We denote by $\Delta^0(Y_s)$ the set of all functions f such that $(-1)^{s-k}f(x) \ge 0$ for $x \in [y_k,y_{k+1}]$, $k = 0,\dots,s$, *i.e.*, those that have $0 \le s < \infty$ sign changes at the points in Y_s and are nonnegative near 1. In particular, $\Delta^0 := \Delta^0(Y_0)$ denotes the set of all nonnegative functions on $[-1,1]$. Functions f and g which belong to the same class $\Delta^0(Y_s)$ are said to be *copositive*.

Copositive approximation is the approximation of functions f from $\Delta^0(Y_s)$ class by polynomials and splines that are copositive with f. For $f \in \mathbf{L}_p[-1,1]$ let

$$E_n(f)_p := \inf_{P_n \in \mathbf{P}_n} \|f-P_n\|_p$$

denote the degree of unconstrained approximation, and let

$$E_n^{(0)}(f,Y_s)_p := \inf_{P_n \in \mathbf{P}_n \cap \Delta^0(Y_s)} \|f-P_n\|_p$$

be the degree of copositive polynomial approximation of f. (In particular, $E_n^{(0)}(f)_p := E_n^{(0)}(f,Y_0)_p := \inf_{P_n\in\mathbf{P}_n\cap\Delta^0} \|f-P_n\|_p$ is the degree of positive approximation.)

The best *onesided* approximation of f by means of algebraic polynomials $P_n \in \mathbf{P}_n$ in $\mathbf{L}_p$-metric is given by

$$\tilde{E}_n(f)_p := \inf\{\|P-Q\|_p; P, Q \in \mathbf{P}_n \text{ and } P(x) \geq f(x) \geq Q(x), -1 \leq x \leq 1\}.$$

A natural extension of (co)positive and onesided approximations is the concept of so-called *intertwining* (or co-onesided) approximation.

DEFINITION. For the set $Y_s := \{y_1, \ldots, y_s \mid y_0 := -1 < y_1 < y_2 < \cdots < y_s < 1 =: y_{s+1}\}$ the best intertwining polynomial approximation of a function $f \in \mathbf{L}_p[-1,1]$ is given by

$$\tilde{E}_n(f, Y_s)_p := \inf\{\|P-Q\|_p; P, Q \in \mathbf{P}_n, P - f \in \Delta^0(Y_s) \text{ and } f - Q \in \Delta^0(Y_s)\}.$$

We call $\{P, Q\}$ an intertwining pair of polynomials for f with respect to Y_s if $P-f, f-Q \in \Delta^0(Y_s)$.

Clearly, in the case $s = 0$ the above definition becomes the definition of the best onesided polynomial approximation: $\tilde{E}_n(f, Y_0)_p = \tilde{E}_n(f)_p$.

We have the following relationships among the above quantities:

- If $f(x) \geq 0$, $x \in [-1,1]$, then $E_n^{(0)}(f)_p \leq \tilde{E}_n(f)_p$.
- If $f \in \Delta^0(Y_s)$, then $E_n^{(0)}(f, Y_s)_p \leq \tilde{E}_n(f, Y_s)_p$.

1.1. *Positive and onesided approximations.* First of all, if $p = \infty$ (*i.e.*, in the uniform metric) the uniform estimates for positive and onesided approximations are not of much interest since

$$E_n(f)_\infty \leq \tilde{E}_n(f)_\infty \leq 2E_n(f)_\infty \tag{1.3}$$

for any $f \in \mathbf{C}[-1,1]$, and

$$E_n(f)_\infty \leq E_n^{(0)}(f)_\infty \leq 2E_n(f)_\infty \tag{1.4}$$

for $f \in \mathbf{C}[-1,1] \cap \Delta^0$.

At the same time, if $1 \leq p < \infty$, then the situation is quite different. It was shown by Stojanova [31] (see also Hristov and Ivanov [9], [10], and [11]) that for any bounded and measurable function on $[-1,1]$, and $m \in N$,

$$\tilde{E}_n(f)_p \leq C(m)\tau^m(f, n^{-1})_p, \quad 1 \leq p < \infty. \tag{1.5}$$

In fact, the estimates obtained in [31] were given in terms of $\tau^m\big(f, \Delta_n(x)\big)_p$, which is smaller than $\tau^m(f, n^{-1})_p$, and which is, in a sense, "the right" quantity for estimation of degree of onesided approximation.

We also remark that τ is the "correct" modulus in (1.5) (*i.e.*, it can not be replaced by ω or ω_φ), since the estimate $\tilde{E}_n(f)_p \leq C\|f\|_p$, certainly, can not be correct for all $f \in \mathbf{L}_p[-1,1]$, $p < \infty$. To see this it is sufficient to consider the function f such that $f(0) = 1$ and $f(x) = 0$, $x \neq 0$. Then $\|f\|_p = 0$ and $\tilde{E}_n(f)_p > 0$.

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An immediate consequence of (1.5) and (1.2) is the fact that if $f \in \mathbf{W}_p^1[-1,1]$, then

$$\tilde{E}_n(f)_p \le C(m)n^{-1}\omega^m(f', n^{-1})_p, \quad 1 \le p < \infty.$$

Moreover, it was shown in [31] that for any $f \in \mathbf{W}_p^1[-1,1]$

$$\tilde{E}_n(f)_p \le Cn^{-1}E_{n-1}(f')_p, \quad 1 \le p < \infty. \tag{1.6}$$

(Though, the estimate (1.6) was not explicitly stated in [31], it immediately follows from the proof of Corollary 1 in that paper.)

As for positive approximation, it was shown in [13] (see also Ivanov [17]) that for any $f \in \mathbf{L}_p[-1,1] \cap \Delta^0$ and $0 < p < \infty$

$$E_n^{(0)}(f)_p \le C\omega_\varphi(f, n^{-1})_p.$$

At the same time, for every $n \in N$, $0 < p < \infty$ and $A > 0$ there exists a function $f \in \mathbf{L}_p[-1,1] \cap \Delta^0$ such that

$$E_n^{(0)}(f)_p > A\omega^2(f, 1)_p.$$

In this paper we show, in particular, that pointwise estimates in terms of $\omega^m\big(f, \Delta_n(x)\big)_\infty$ are true for onesided (and, therefore, for positive) approximation in $C[-1,1]$, thus, in a sense, completing the investigation of these types of approximation. (Of course, some improvements are possible if measures of smoothness different from those considered here are used.)

THEOREM 1 (ONESIDED APPROXIMATION). *Let $f \in \mathbf{C}[-1,1]$ and $m \in N$. Then for every $n \ge m-1$ there exist polynomials $P, Q \in \mathbf{P}_n$ such that $P(x) \ge f(x) \ge Q(x)$, $-1 \le x \le 1$, and*

$$|P(x) - Q(x)| \le C(m)\omega^m\big(f, \Delta_n(x)\big)_\infty. \tag{1.7}$$

COROLLARY 2 (POSITIVE APPROXIMATION). *Let $m \in N$ and $f \in \mathbf{C}[-1,1]$ be such that $f(x) \ge 0$, $-1 \le x \le 1$. Then for every $n \ge m-1$ there exist a polynomial $P \in \mathbf{P}_n$, $P(x) \ge 0$, $-1 \le x \le 1$ satisfying*

$$|f(x) - P(x)| \le C(m)\omega^m\big(f, \Delta_n(x)\big)_\infty. \tag{1.8}$$

While preparing this paper for publication, the authors learned that Corollary 2 was also recently proved by G. Dzyubenko [8].

The above results can be summarized as follows.

Onesided approximation		
$p = \infty$		
$f \in \mathbf{C}$	$\exists\, P_n, Q_n$: $P_n(x) \ge f(x) \ge Q_n(x)$, such that $\lvert P_n(x) - Q_n(x)\rvert \le C\omega^m\big(f, \Delta_n(x)\big)_\infty$	Theorem 1
$1 \le p < \infty$		
$f \in \mathbf{L}_p$	$\tilde{E}_n(f)_p \le C\tau^m(f, n^{-1})_p$	Stojanova [31], see also Hristov and Ivanov [10]
	$\tilde{E}_n(f)_p \not\le C\lVert f\rVert_p$	obvious
$f \in \mathbf{W}_p^1$	$\tilde{E}_n(f)_p \le Cn^{-1}E_{n-1}(f')_p$	Stojanova [31]

Positive approximation		
$p = \infty$		
$f \in \mathbf{C}$	$\exists\, P_n$, $P_n(x) \geq 0$, such that $\lvert f(x) - P_n(x)\rvert \leq C\omega^m\big(f, \Delta_n(x)\big)_\infty$	Corollary 2, see also Dzyubenko [8]
$1 \leq p < \infty$		
$f \in \mathbf{L}_p$	$E_n^{(0)}(f)_p \leq C\tau^m(f, n^{-1})_p$	Stojanova [31]
	$E_n^{(0)}(f)_p \leq C\omega_\varphi(f, n^{-1})_p$	[13], see also Ivanov [17]
	$E_n^{(0)}(f)_p \not\leq C\omega^2(f, 1)_p$	[13]
$f \in \mathbf{W}_p^1$	$E_n^{(0)}(f)_p \leq Cn^{-1}E_{n-1}(f')_p$	Stojanova [31]

1.2. *Copositive and intertwining approximations.* Copositive approximation was extensively studied in recent years. A number of results were obtained (see [8], [12], [13], [14], [15], [16], [20], [24], [25], [26], [28], [32], [33], [34], for example). Recently, Kopotun [20] showed that if $f \in \mathbf{C}[-1,1] \cap \Delta^0(Y_s)$, then

$$E_n^{(0)}(f, Y_s)_\infty \leq C(Y_s)\omega_\varphi^3(f, n^{-1})_\infty, \quad n \geq 2. \tag{1.9}$$

(See Hu and Yu [16], Hu, Leviatan and Yu [14, 15] for weaker but earlier results.) This is the best possible estimate in the sense that ω_φ^3 in (1.9) can not be replaced by ω^4 (Zhou [33]). If f is continuously differentiable, Hu, Leviatan and Yu [15] gave an estimate in terms of higher order modulus of f', *i.e.*, for any function $f \in \mathbf{C}^1[-1,1] \cap \Delta^0(Y_s)$

$$E_n^{(0)}(f, Y_s)_\infty \leq C_2 n^{-1}\omega^m(f', n^{-1})_\infty, \quad n \geq C_1, \tag{1.10}$$

where the constants C_1 and C_2 depend only on m and Y_s. In fact, using a slight modification of the proof in [15], one can show that ω^m in (1.10) can be replaced by ω_φ^m. In this paper we use a different method to show that, and also obtain pointwise estimates improving (1.10) (see Corollary 6).

As for $f \in \mathbf{L}_p \cap \Delta^0(Y_s)$, the authors [12, 13] have shown that the copositive approximation is quite different from other kinds of constrained approximation such as monotone or convex approximation, with which we have seen similarities between approximations in $\mathbf{L}_p$ and in $\mathbf{C}$. For example, DeVore, Hu and Leviatan [3] recently proved that the degree of convex polynomial approximation in $\mathbf{L}_p$, $0 < p < \infty$, has order $\omega_\varphi^3(f, n^{-1})_p$, which is a natural extension of Kopotun's result in [19] for the space $\mathbf{C}$. By contrast, if $0 < p < \infty$, the degree of copositive polynomial approximation is merely given by (see [13])

$$E_n^{(0)}(f, Y_s)_p \leq C\omega_\varphi(f, n^{-1})_p, \tag{1.11}$$

which is significantly lower than (1.9). We will show in this paper that ω_φ in (1.11) is the best possible in the sense that it can not be replaced even by $\omega^2(f, 1)_p$ (see also Zhou [34], where a similar result was proved for $1 < p < \infty$), and that an analogue of this holds true for splines. (As was shown in [13] and mentioned above, this is also the case for positive approximation.)

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Also, we extend our investigation of copositive approximation to the Sobolev spaces $\mathbf{W}_p^k$, $p \geq 1$, and consider intertwining approximation in $\mathbf{W}_p^k$ as well, obtaining the estimates which are exact in the sense of the orders of moduli of smoothness. We prove that if $f \in \mathbf{W}_p^2$, then the degree of intertwining (and, hence, copositive) polynomial approximation has the order $n^{-2}\omega_\varphi^m(f'', n^{-1})_p$ for any positive integer m. If f is merely in $\mathbf{W}_p^1$, then it deteriorates to $n^{-1}\tau^m(f', n^{-1})_p$, which is the best in the sense that one can not replace $n^{-1}\tau^m(f', n^{-1})_p$ even by $\|f'\|_p$. At the same time, the degree of copositive approximation does not deteriorate that bad. In particular, the estimate $E_n^{(0)}(f, Y_s)_p \leq Cn^{-1}\omega_\varphi^2(f', n^{-1})_p$ holds true. (This estimate is exact in the sense that $n^{-1}\omega_\varphi^2(f', n^{-1})_p$ can not be replaced by $\omega^3(f', 1)_p$.) Analogues of these again are true for splines.

The investigation of constrained approximation in $\mathbf{L}_p$, $0 < p < 1$ quasi-norm is not our goal in this paper. (We prove some of our results in the case $p < 1$ as well. However, it is done only if the proof is similar to that for $p \geq 1$, and no extra effort or discussions are required.) It is known that for unconstrained approximation the usual Jackson type estimates, involving the first derivatives of functions, are no longer valid if $p < 1$ (see Kopotun [21], for example). However, it does not guarantee that the same is true in constrained case, since the functions satisfying some shape preserving constraint form a proper subset of $\mathbf{W}_p^k$. In fact, it was shown in [21] that for convex polynomial approximation one can get estimates which are not true in the general (unconstrained) case. At the same time, the restriction $f \in \Delta^0(Y_s)$, for example, is not as "strong" as $f \in \Delta^2$ (*i.e.*, f is convex), and does not eliminate those functions f which "bring anomalous properties" into $\mathbf{L}_p$ for $p < 1$. (See [21] for further discussions. We only mention that the proof of Theorem 3 of [21] can be used to show that for every $A > 0$, $B > 0$, $0 < p < 1$, $n \in N$ and a set Y_s, there exists a function $f \in AC[-1, 1] \cap \Delta^0(Y_s)$ such that $E_n^{(0)}(f, Y_s)_p > An^B\|f'\|_p$.)

We now state our results on copositive and intertwining approximations, and begin with a theorem on splines. We give local estimates in the theorems because they are stronger than the corresponding global estimates (also, this is the form needed in the proof of theorems on polynomial approximation). The global estimates follow immediately from the inequality (which can be shown directly from the definition of the τ-modulus)

$$\sum_i \tau^m(f, t, I_i)_p^p \leq k\tau^m(f, t, I)_p^p, \tag{1.12}$$

where $\cup I_i = I$, and each x in the interval I is contained in at most k subintervals I_i. Moreover, if f is smooth, and the partition $I = \cup I_i$ is (close to) the one formed by zeros of Chebyshev's polynomial $\cos(n \arccos x)$, then the global estimates can be further improved since

$$\sum_i \tau^m(f, |I_i|, I_i)_p^p \leq C\sum_i |I_i|^p \omega^{m-1}(f', |I_i|, I_i)_p^p \leq Cn^{-p}\omega_\varphi^{m-1}(f', n^{-1})_p^p. \tag{1.13}$$

Let $\mathbf{T}_n := \{z_0, \ldots, z_n \mid -1 := z_0 < z_1 < \cdots < z_{n-1} < z_n := 1\}$, $n \geq 1$, be a given knot sequence on $[-1, 1]$, and set $z_i := -1$, $i < 0$, and $z_i := 1$, $i > n$. For $i = -1, \ldots, n$, let $J_i := [z_i, z_{i+1}]$. With this notation we have

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THEOREM 3 (INTERTWINING SPLINE APPROXIMATION, $1 \le p \le \infty$). *Let* $f \in \mathbf{W}_p^1[-1,1]$, $1 \le p \le \infty$, $Y_s := \{y_1, \ldots, y_s \mid y_0 := -1 < y_1 < y_2 < \cdots < y_s < 1 =: y_{s+1}\}$, $s \ge 0$, *and let* $r \ge 2$ *be an integer. Let* $\mathbf{T}_n$ *be a given knot sequence such that there are at least* $4(r-1)^2$ *knots in each open interval* (y_j, y_{j+1}), $j = 1, \ldots, s-1$. *Then there exists an intertwining pair of splines* $\{\overline{S}, S\}$ *of order r on the knot sequence* $\mathbf{T}_n$ *(i.e.,* $S, \overline{S} \in \mathbf{C}^{r-2}[-1,1]$ *and* $\overline{S} - f, f - S \in \Delta^0(Y_s)$*) such that for* $i = 0, \ldots, n-1$

$$\|\overline{S} - S\|_{\mathbf{L}_p(J_i)} \le C|J_i|\tau^{r-1}(f', |J_i|, J_i)_p, \tag{1.14}$$

where C is a constant depending on r and the maximum ratio $\rho := \max_{i=0}^{n-1} |J_{i\pm1}|/|J_i|$, *and* J_i *is an interval such that* $J_i \subset J_i \subseteq \left[z_{i-6(r-1)^2}, z_{i+6(r-1)^2}\right]$. *Consequently, if in addition* $f \in \mathbf{W}_p^2$, *then*

$$\|\overline{S} - S\|_{\mathbf{L}_p(J_i)} \le C|J_i|^2\omega^{r-2}(f'', |J_i|, J_i)_p. \tag{1.15}$$

We establish our results on polynomial approximation by proving that errors of constrained polynomial approximants are no worse than those of their spline counterparts. More precisely, the following theorem plays a main role in this paper.

THEOREM 4. *Let* $Y_s := \{y_1, \ldots, y_s \mid y_0 := -1 < y_1 < \cdots < y_s < 1 =: y_{s+1}\}$, $s \ge 0$, $m \in N \cup \{0\}$, $\mu \ge 2m + 30$, $0 < p \le \infty$, *and let* $S(x)$ *be a spline of an odd order* r $(r = 2m+1)$ *on the knot sequence* $\{x_j = \cos\frac{j\pi}{n}\}_{j \in J(Y_s)}$, *where* $n > C(Y_s)$ *is such that there are at least* 4 *knots* x_j *in each interval* (y_i, y_{i+1}), $i = 0, \ldots, s$, *and* $J(Y_s) = \{1, \ldots, n\} \setminus \{j, j-1 \mid x_j \le y_i < x_{j-1}$ *for some* $1 \le i \le s\}$. *Then there exists an intertwining pair of polynomials* $\{P_1, P_2\} \subset \mathbf{P}_{C(\mu)n}$ *for S with respect to* Y_s *such that*

$$\|P_1 - P_2\|_p^p \le C(r, \mu, s)^p \sum_{j=1}^{n-1} E_{r-1}(S, I_j \cup I_{j+1})_p^p, \quad \text{if } 0 < p < \infty, \tag{1.16}$$

and

$$|P_1(x) - P_2(x)| \le C(r, \mu, s) \sum_{j=1}^{n-1} E_{r-1}(S, I_j \cup I_{j+1})_\infty \left(\frac{|I_j|}{|x - x_j| + |I_j|}\right)^\mu, \quad \text{if } p = \infty, \tag{1.17}$$

where $I_j := [x_j, x_{j-1}]$ *and* $E_n(f, [a,b])_p := \inf_{P_n \in \mathbf{P}_n} \|f - P_n\|_{\mathbf{L}_p[a,b]}$.

REMARK. We emphasize that Theorem 4 is not true in general for a spline S on the knot sequence $\{x_j\}_{j=1}^{n-1}$ (*i.e.*, if the knots x_j which are "too close" to y_i's are not removed). For example, if $m = 0$, $s = 1$, n is even, $y_1 = 0$, and $S(x) = \begin{cases} 1 & \text{if } x \ge 0, \\ 0 & \text{otherwise,} \end{cases}$ then, clearly, no polynomial $P(x)$ satisfies $P(x) \le S(x) = 0$ for $x < 0$, and $P(x) \ge S(x) = 1$ for $x \ge 0$. (In other words, an intertwining pair of polynomials for S with respect to $\{y_1\}$ simply does not exist in this case.)

As shown in Section 3 the following results are almost straightforward consequences of the above theorem and direct estimates for constrained spline approximation.

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THEOREM 5 (INTERTWINING POLYNOMIAL APPROXIMATION IN $\mathbf{C}^1[-1,1]$). *Let* $f \in \mathbf{C}^1[-1,1]$, $m \in N$, *and* $Y_s := \{y_1, \dots, y_s \mid y_0 := -1 < y_1 < y_2 < \cdots < y_s < 1 =: y_{s+1}\}$, $s \geq 0$. *Then*

$$\tilde{E}_n(f, Y_s)_\infty \leq C(m,s) n^{-1} \omega_\varphi^m(f', n^{-1})_\infty, \quad n \geq C(Y_s). \tag{1.18}$$

Also, there exists an intertwining pair of polynomials $\{P_n, Q_n\} \subset \mathbf{P}_n$ *such that*

$$|P_n(x) - Q_n(x)| \leq C(m,s)\Delta_n(x)\omega^m\big(f', \Delta_n(x)\big)_\infty, \quad n \geq C(Y_s). \tag{1.19}$$

COROLLARY 6 (COPOSITIVE POLYNOMIAL APPROXIMATION IN $\mathbf{C}^1[-1,1]$). *Let* $f \in \mathbf{C}^1[-1,1] \cap \Delta^0(Y_s)$, $Y_s := \{y_1, \dots, y_s \mid y_0 := -1 < y_1 < y_2 < \cdots < y_s < 1 =: y_{s+1}\}$, $s \geq 0$, *and* $m \in N$. *Then*

$$E_n^{(0)}(f, Y_s)_\infty \leq C(m,s) n^{-1} \omega_\varphi^m(f', n^{-1})_\infty, \quad n \geq C(Y_s). \tag{1.20}$$

Also, there exists a polynomial $P_n \in \mathbf{P}_n \cap \Delta^0(Y_s)$ *such that*

$$|f(x) - P_n(x)| \leq C(m,s)\Delta_n(x)\omega^m\big(f', \Delta_n(x)\big)_\infty, \quad n \geq C(Y_s). \tag{1.21}$$

The following result does not hold for intertwining approximation (see Theorem 13), and, therefore, copositive case is considered separately.

THEOREM 7 (COPOSITIVE POLYNOMIAL APPROXIMATION IN $\mathbf{C}[-1,1]$). *Let* $f \in \mathbf{C}[-1,1] \cap \Delta^0(Y_s)$, $Y_s := \{y_1, \dots, y_s \mid y_0 := -1 < y_1 < y_2 < \cdots < y_s < 1 =: y_{s+1}\}$, $s \geq 0$. *Then there is a polynomial* $P_n \in \mathbf{P}_n$, *copositive with* f, *such that*

$$|f(x) - P_n(x)| \leq C(s)\omega^3\big(f, \Delta_n(x)\big)_\infty, \quad n \geq C(Y_s). \tag{1.22}$$

Now, we state the results on approximation of functions in $\mathbf{L}_p[-1,1]$, $1 \leq p < \infty$ norm.

THEOREM 8 (INTERTWINING POLYNOMIAL APPROXIMATION, $1 \leq p < \infty$). *Let* $f \in \mathbf{W}_p^1[-1,1]$, $1 \leq p < \infty$, $m \in N$, *and* $Y_s := \{y_1, \dots, y_s \mid y_0 := -1 < y_1 < y_2 < \cdots < y_s < 1 =: y_{s+1}\}$, $s \geq 0$. *Then*

$$\tilde{E}_n(f, Y_s)_p \leq C(m,s) n^{-1} \tau^m(f', n^{-1})_p, \quad n \geq C(Y_s). \tag{1.23}$$

Moreover, if f *is also in* $\mathbf{W}_p^2[-1,1]$, *then*

$$\tilde{E}_n(f, Y_s)_p \leq C(m,s) n^{-2} \omega_\varphi^{m-1}(f'', n^{-1})_p, \quad n \geq C(Y_s). \tag{1.24}$$

COROLLARY 9 (COPOSITIVE POLYNOMIAL APPROXIMATION, $1 \leq p < \infty$). *Let* $f \in \mathbf{W}_p^1[-1,1] \cap \Delta^0(Y_s)$, $1 \leq p < \infty$. *Then*

$$E_n^{(0)}(f, Y_s)_p \leq C(m,s) n^{-1} \tau^m(f', n^{-1})_p \quad n \geq C(Y_s). \tag{1.25}$$

Moreover, if f *is also in* $\mathbf{W}_p^2[-1,1]$, *then*

$$E_n^{(0)}(f, Y_s)_p \leq C(m,s) n^{-2} \omega_\varphi^{m-1}(f'', n^{-1})_p, \quad n \geq C(Y_s). \tag{1.26}$$

Again, the following estimates are true for copositive approximation and false in the intertwining case.

THEOREM 10 (COPOSITIVE POLYNOMIAL APPROXIMATION, $1 \le p < \infty$). *Let* $f \in \mathbf{L}_p[-1,1] \cap \Delta^0(Y_s)$, $1 \le p < \infty$. *Then*

$$E_n^{(0)}(f, Y_s)_p \le C(s)\tau^3(f, n^{-1})_p, \quad n \ge C(Y_s). \tag{1.27}$$

Moreover, if $f \in \mathbf{W}_p^1[-1,1]$, *then*

$$E_n^{(0)}(f, Y_s)_p \le C(s)n^{-1}\omega_\varphi^2(f', n^{-1})_p, \quad n \ge C(Y_s). \tag{1.28}$$

As mentioned earlier, (1.28) is the best one can get for $f \in \mathbf{W}_p^1$. We back this assertion by proving the following counterexample.

THEOREM 11. *For every* $n \in N$, $0 < p < \infty$, $0 < \varepsilon \le 1$ *and* $A > 0$, *there exists a monotone increasing function* $f \in \mathbf{C}^\infty[-1,1]$ *with* $f(0) = 0$ *such that for every polynomial* $P_n \in \mathbf{P}_n$ *with* $P_n(0) = 0$ *and* $P_n(x) \ge 0$ *for* $x \in [0, \varepsilon]$, *the following inequality holds:*

$$\|f - P_n\|_{\mathbf{L}_p[0,\varepsilon]} > A\omega^3(f', 1)_p. \tag{1.29}$$

It also follows from Theorem 11 that the estimate

$$\|f - S_n\|_p \le C\delta\omega^2(f', \delta)_p,$$

where δ is the mesh size of the knot sequence $\mathbf{T}_n$, is the best possible for copositive spline approximation in the $\mathbf{L}_p$ metric. Also, (1.29) and (1.2) imply that τ^3 in (1.27) can not be replaced by τ^4.

In the second counterexample, we show that (1.11) and its spline analogue proved in [12, 13] are best possible.

THEOREM 12. *For every* $n \in N$, $0 < p < \infty$, $0 < \varepsilon \le 1$ *and* $A > 0$, *there exists a function* $f \in \mathbf{C}^\infty[-1,1]$ *satisfying* $xf(x) \ge 0$, $x \in [-1,1]$, *and such that for every polynomial* $P_n \in \mathbf{P}_n$ *with* $P_n(0) \ge 0$, *the following inequality holds:*

$$\|f - P_n\|_{\mathbf{L}_p[0,\varepsilon]} > A\omega^2(f, 1)_p. \tag{1.30}$$

The following theorem shows that the estimates in terms of $\omega(f,1)_p$ or $\tau(f,1)_p$ are impossible for intertwining approximation. (Note, that $\omega(f,1)_p \le C\|f\|_p$ and $\tau(f,1)_p \le C\|f'\|_p$.)

THEOREM 13. *For every* $n \in N$, $0 < p \le \infty$, $0 < \varepsilon \le 1$ *and* $A > 0$, *there exists a monotone increasing function* $f \in \mathbf{C}^\infty[-1,1]$ *with* $f(0) = 0$ *such that for every polynomial* $P_n \in \mathbf{P}_n$ *with* $P_n(0) = 0$ *and* $P_n(x) \ge f(x)$, $0 \le x \le \varepsilon$, *the following inequality holds:*

$$\|f - P_n\|_{\mathbf{L}_p[0,\varepsilon]} > A\|f\|_{\mathbf{L}_p[-1,1]}. \tag{1.31}$$

Under the same conditions except that $0 < p < \infty$, *there also exists such a function* f *that*

$$\|f - P_n\|_{\mathbf{L}_p[0,\varepsilon]} > A\|f'\|_{\mathbf{L}_p[-1,1]}. \tag{1.32}$$

Finally, we summarize all the results discussed in this section in the form of the following two tables.

Intertwining approximation		
$p = \infty$		
$f \in \mathbf{C}$	$\tilde{E}_n(f, Y_s)_\infty \not\leq C\|f\|_\infty$	Theorem 13
$f \in \mathbf{C}^1$	$\tilde{E}_n(f, Y_s)_\infty \leq Cn^{-1}\omega_\varphi^m(f', n^{-1})_\infty$	Theorem 5
	$\exists$ an intertwining pair $\{P_n, Q_n\}$ for f satisfying $\lvert P_n(x) - Q_n(x)\rvert \leq C\Delta_n(x)\omega^m\big(f', \Delta_n(x)\big)_\infty$	Theorem 5
$1 \leq p < \infty$		
$f \in \mathbf{L}_p$	$\tilde{E}_n(f, Y_s)_p \not\leq C\|f\|_p$	Theorem 13
	$\tilde{E}_n(f, Y_s)_p \not\leq C\tau(f, 1)_p$	Theorem 13 and ineq. (1.2)
$f \in \mathbf{W}_p^1$	$\tilde{E}_n(f, Y_s)_p \not\leq C\|f'\|_p$	Theorem 13
	$\tilde{E}_n(f, Y_s)_p \leq Cn^{-1}\tau^m(f', n^{-1})_p$	Theorem 8
$f \in \mathbf{W}_p^2$	$\tilde{E}_n(f, Y_s)_p \leq Cn^{-2}\omega_\varphi^m(f'', n^{-1})_p$	Theorem 8

Copositive approximation		
$p = \infty$		
$f \in \mathbf{C}$	$E_n^{(0)}(f, Y_s)_\infty \leq C\omega_\varphi^3(f, n^{-1})_\infty$	Kopotun [20]
	$\exists\, P_n$, copositive with f, such that $\lvert f(x) - P_n(x)\rvert \leq C\omega^3\big(f, \Delta_n(x)\big)_\infty$	Theorem 7
	$E_n^{(0)}(f, Y_s)_\infty \not\leq C\omega^4(f, n^{-1})_\infty$	Zhou [34]
$f \in \mathbf{C}^1$	$E_n^{(0)}(f, Y_s)_\infty \leq Cn^{-1}\omega_\varphi^m(f', n^{-1})_\infty$	Corollary 6, see also Hu, Leviatan and Yu [15]
	$\exists\, P_n$, copositive with f, such that $\lvert f(x) - P_n(x)\rvert \leq C\Delta_n(x)\omega^m\big(f', \Delta_n(x)\big)_\infty$	Corollary 6 see also Dzyubenko [8]
$1 \leq p < \infty$		
$f \in \mathbf{L}_p$	$E_n^{(0)}(f, Y_s)_p \leq C\tau^3(f, n^{-1})_p$	Theorem 10
	$E_n^{(0)}(f, Y_s)_p \leq C\omega_\varphi(f, n^{-1})_p$	[13]
	$E_n^{(0)}(f, Y_s)_p \not\leq C\omega^2(f, 1)_p$	Theorem 12, see also Zhou [34]
	$E_n^{(0)}(f, Y_s)_p \not\leq C\tau^4(f, 1)_p$	Theorem 11 and ineq. (1.2)
$f \in \mathbf{W}_p^1$	$E_n^{(0)}(f, Y_s)_p \leq Cn^{-1}\omega_\varphi^2(f', n^{-1})_p$	Theorem 10
	$E_n^{(0)}(f, Y_s)_p \leq Cn^{-1}\tau^m(f', n^{-1})_p$	Corollary 9
	$E_n^{(0)}(f, Y_s)_p \not\leq C\omega^3(f', 1)_p$	Theorem 11
$f \in \mathbf{W}_p^2$	$E_n^{(0)}(f, Y_s)_p \leq Cn^{-2}\omega_\varphi^m(f'', n^{-1})_p$	Corollary 9

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The rest of this paper is organized as follows. The affirmative results for splines, including the proof of Theorem 3, are given in Section 2, and those for polynomials in Section 3. The last section is devoted to proving Theorems 11–13.

2. Constrained spline approximation.

2.1. *Intertwining spline approximation.*

PROOF OF THEOREM 3. We only consider the case for $1 \le p < \infty$. The proof for $p = \infty$ is almost identical. We use some idea in the proof of [16, Theorem 3]. Let $d := 2(r-1)^2$, $m := [(n+d-1)/d]$ and $\bar{z}_i := z_{di}$. Note that $\bar{z}_i = -1$ for $i \le 0$ and $\bar{z}_i = 1$ for $i \ge m$. We first construct overlapping polynomial pieces of degree $< r$ on the coarser partition $\overline{\mathbf{T}_n} := \{\bar{z}_i\}_{i=0}^m$. We call the interval $\bar{I}_i := [\bar{z}_i, \bar{z}_{i+1}]$ contaminated if $\bar{z}_i < y_j \le \bar{z}_{i+1}$ for some point $y_j \in Y_s$. By assumption, there exists exactly one y_j in each of the contaminated intervals $\bar{I}_{m_j}$, $j = 1, \ldots, s$, and there is at least one non-contaminated interval between $\bar{I}_{m_j}$ and $\bar{I}_{m_{j+1}}$, that is,

$$m_j < m_j + 2 \le m_{j+1}, \quad j = 1, \ldots, s-1. \tag{2.1}$$

For convenience we also denote $m_0 := -1$, and $m_{s+1} := m$. Note that there are no y_i's between $\bar{I}_{m_j}$ and $\bar{I}_{m_{j+1}}$, $j = 0, \ldots, s$.

If $m_{j+1} = m_j + 2$ (*i.e.*, if there is only one non-contaminated interval between $\bar{I}_{m_j}$ and $\bar{I}_{m_{j+1}}$), then the following construction is not needed, and the next two paragraphs can be skipped.

In the case $m_{j+1} > m_j + 2$, by Whitney's Theorem for onesided polynomial approximation (see Theorem 2.6 of [12], but most of the credit goes to V. H. Hristov and K. G. Ivanov), on each of the intervals $[\bar{z}_i, \bar{z}_{i+2}]$, $i = m_j + 1, \ldots, m_{j+1} - 2$, there exist two polynomials P_i and Q_i of degree $< r$ such that

$$\begin{aligned} &P_i(x) \ge f(x) \ge Q_i(x), \quad \forall x \in [\bar{z}_i, \bar{z}_{i+2}] \\ &\|P_i - Q_i\|_{\mathbf{L}_p[\bar{z}_i, \bar{z}_{i+2}]} \le C\tau^r(f, |\bar{I}_i|, [\bar{z}_i, \bar{z}_{i+2}])_p. \end{aligned} \tag{2.2}$$

We define p_i and q_i on $[\bar{z}_i, \bar{z}_{i+2}]$ by $p_i := P_i$ and $q_i := Q_i$ if $(-1)^{s-j} > 0$, and $p_i := Q_i$ and $q_i := P_i$ if $(-1)^{s-j} < 0$. Hence, $(-1)^{s-j}\big(p_i(x) - f(x)\big) \ge 0$, $(-1)^{s-j}\big(q_i(x) - f(x)\big) \le 0$, and

$$\begin{aligned} \|p_i - q_i\|_{\mathbf{L}_p[\bar{z}_i, \bar{z}_{i+2}]} &= \|P_i - Q_i\|_{\mathbf{L}_p[\bar{z}_i, \bar{z}_{i+2}]} \\ &\le C\tau^r(f, |\bar{I}_i|, [\bar{z}_i, \bar{z}_{i+2}])_p \le C|\bar{I}_i|\tau^{r-1}(f', |\bar{I}_i|, [\bar{z}_i, \bar{z}_{i+2}])_p, \end{aligned} \tag{2.3}$$

where, in the last step, we have used the inequality (see [29])

$$\tau^m(f, t)_p \le C_m t \tau^{m-1}(f', t)_p, \quad t \ge 0.$$

We should emphasize that when we speak of a polynomial on an interval, we mean the restriction of the polynomial to the interval, hence it is considered undefined outside.

Near each point y_j, we construct local polynomials differently. More precisely, we approximate f' on $[\bar{z}_{m_j-1}, \bar{z}_{m_j+2}]$, $j = 1, \ldots, s$, from above and below by two polynomials $\tilde{P}_{m_j}$ and $\tilde{Q}_{m_j}$ of degree $< r-1$. Then

$$\tag{2.4} \begin{gathered} \tilde{P}_{m_j}(x) \geq f'(x) \geq \tilde{Q}_{m_j}(x), \quad \forall x \in [\bar{z}_{m_j-1}, \bar{z}_{m_j+2}] \\ \left\| \tilde{P}_{m_j} - \tilde{Q}_{m_j} \right\|_{\mathbf{L}_p[\bar{z}_{m_j-1}, \bar{z}_{m_j+2}]} \leq C\tau^{r-1}(f', |\bar{I}_{m_j}|, [\bar{z}_{m_j-1}, \bar{z}_{m_j+2}])_p. \end{gathered}$$

Define $\tilde{p}_{m_j} := \tilde{P}_{m_j}$ and $\tilde{q}_{m_j} := \tilde{Q}_{m_j}$ if $(-1)^{s-j} > 0$, and $\tilde{p}_{m_j} := \tilde{Q}_{m_j}$ and $\tilde{q}_{m_j} := \tilde{P}_{m_j}$ otherwise. It is easy to check that

$$p_{m_j}(x) := \int_{y_j}^{x} \tilde{p}_{m_j}(t)\, dt + f(y_j)$$

and

$$q_{m_j}(x) := \int_{y_j}^{x} \tilde{q}_{m_j}(t)\, dt + f(y_j)$$

satisfy the inequalities

$$(-1)^{s-j}\big(p_{m_j}(x) - f(x)\big) \operatorname{sgn}(x - y_j) \geq 0,$$

$$(-1)^{s-j}\big(q_{m_j}(x) - f(x)\big) \operatorname{sgn}(x - y_j) \leq 0,$$

and

$$\tag{2.5} \begin{aligned} \|p_{m_j} - q_{m_j}\|_p &= \left\| \int_{y_j}^{x} \big(\tilde{P}_{m_j}(t) - \tilde{Q}_{m_j}(t)\big)\, dt \right\|_p \leq \left\| \int_{\bar{z}_{m_j-1}}^{\bar{z}_{m_j+2}} \big(\tilde{P}_{m_j}(t) - \tilde{Q}_{m_j}(t)\big)\, dt \right\|_p \\ &\leq C|\bar{I}_{m_j}|\, \|\tilde{P}_{m_j} - \tilde{Q}_{m_j}\|_p \leq C|\bar{I}_{m_j}| \tau^{r-1}\big(f', |\bar{I}_{m_j}|, [\bar{z}_{m_j-1}, \bar{z}_{m_j+2}]\big)_p, \end{aligned}$$

where all norms are taken over $[\bar{z}_{m_j-1}, \bar{z}_{m_j+2}]$.

Having constructed the overlapping local polynomials which are "intertwining" with f and have the right approximation order, we now blend them for smooth spline approximants $\bar{S}$ and S on the original knot sequence $\mathbf{T}_n$ with the same properties. If both $\bar{I}_{i-1}$ and $\bar{I}_i$ are non-contaminated and $i < m$, then p_{i-1} and p_i overlap on $\bar{I}_i$, which contains $d-1$ interior knots from $\mathbf{T}_n$. By Beatson's Lemma (see Lemma 3.2 of [2]), there exists a spline $\bar{S}_i$ of order r on $\bar{I}_i$ on these knots that connects with p_{i-1} and p_i in a $\mathbf{C}^{r-2}$ manner at $\bar{z}_i = z_{di}$ and $\bar{z}_{i+1} = z_{d(i+1)}$, respectively. Moreover, the graph of $\bar{S}_i$ lies between those of p_{i-1} and p_i, and, hence, $\operatorname{sgn}\big(p_{i-1}(x) - f(x)\big) = \operatorname{sgn}\big(p_i(x) - f(x)\big) = \operatorname{sgn}\big(\bar{S}_i(x) - f(x)\big)$, $x \in \bar{I}_i$.

Similarly, considering the overlapping polynomials q_{i-1} and q_i, we construct a spline S_i satisfying $\operatorname{sgn}\big(q_{i-1}(x) - f(x)\big) = \operatorname{sgn}\big(q_i(x) - f(x)\big) = \operatorname{sgn}\big(S_i(x) - f(x)\big)$, $x \in \bar{I}_i$. Also,

$$\int_{\bar{I}_i} |\bar{S}_i - S_i|^p \leq 2^p \Big(\int_{\bar{I}_i} |p_{i-1} - q_{i-1}|^p + \int_{\bar{I}_i} |p_i - q_i|^p \Big).$$

By (2.3) and a property of the τ-modulus, this gives

$$\tag{2.6} \|\bar{S}_i - S_i\|_{\mathbf{L}_p(\bar{I}_i)} \leq C\tau^r(f, |\bar{I}_i|, [\bar{z}_{i-1}, \bar{z}_{i+2}])_p \leq C|\bar{I}_i| \tau^{r-1}(f', |\bar{I}_i|, [\bar{z}_{i-1}, \bar{z}_{i+2}])_p.$$

The blending of overlapping polynomial pieces involving contaminated intervals can be done in the same way. The spline pieces $\overline{S}_i$ and S_i thus produced also satisfy the estimate above with a slightly larger interval in place of $[\overline{z}_{i-1}, \overline{z}_{i+2}]$ on the right-hand side, ($[\overline{z}_{i-2}, \overline{z}_{i+3}]$ at worst), which will make no difference in the rest of the proof.

We define the final spline $\overline{S}$ on each $\overline{I}_i$ as follows: if there is only one local polynomial p_i over $\overline{I}_i$, set $\overline{S}$ to this polynomial; if there are two polynomials overlapping on $\overline{I}_i$, then there must be a blending local spline $\overline{S}_i$, set $\overline{S}$ to $\overline{S}_i$. It is clear from its construction that $\overline{S} - f \in \Delta^0(Y_s)$ on the whole interval $[-1, 1]$, and $\overline{S} \in \mathbf{C}^{r-2}$. Similarly, we construct $S \in \mathbf{C}^{r-2}$ such that $f - S \in \Delta^0(Y_s)$. Now, recall that all neighboring intervals $I_i := [z_i, z_{i+1}]$ in the original partition $\mathbf{T}_n$ are comparable in size and each interval $\overline{I}_i = [z_{di}, z_{d(i+1)}]$ contains no more than d such intervals. Therefore, (1.14) follows directly from (2.3) and (2.6). Now (1.15) is a direct consequence of (1.14) and (1.2). ∎

The above proof also yields the following result on onesided spline approximation.

LEMMA 14. *Let $f \in \mathbf{L}_p[-1, 1]$, $1 \le p \le \infty$, and let $r \ge 2$ be an integer. Then there exist splines $\overline{S}_n$ and S_n of order r on the knot sequence $\mathbf{T}_n$ such that $\overline{S}_n(x) \ge f(x) \ge S_n(x)$, $x \in [-1, 1]$, and for $i = 1, \ldots, n-1$*

$$\|\overline{S}_n - S_n\|_{\mathbf{L}_p(J_i)} \le C\tau^r(f, |J_i|, J_i)_p, \tag{2.7}$$

where C is a constant depending on r and the maximum ratio $\rho := \max_{i=0}^{n-1} |J_{i\pm 1}| / |J_i|$, and J_i is an interval such that $J_i \subset J_i \subseteq \left[z_{i-6(r-1)^2}, z_{i+6(r-1)^2}\right]$.

(Note that this lemma is probably known. For example, a similar result follows from Andreev, Popov and Sendov [1] and Popov [27].)

2.2. *Copositive spline approximation.* The next theorem is an improvement to Theorem 4 in Hu [12], where it was proved for $\mathbf{C}^1$ quadratic splines on equidistant knots. The improvement is needed in the proof of Theorem 10.

THEOREM 15. *Let $f \in \mathbf{L}_p[-1, 1] \cap \Delta^0(Y_s)$, $1 \le p < \infty$, $s \ge 0$, and let $r \ge 3$ be an integer. Let $\mathbf{T}_n$ be a given knot sequence such that there are at least $4(r-1)^2$ knots in each open interval (y_j, y_{j+1}), $j = 0, \ldots, s$. Then there exists a spline $S_n \in \mathbf{C}^{r-2}[-1, 1] \cap \Delta^0(Y_s)$ of order r on the knot sequence $\mathbf{T}_n$ such that for $i = 1, \ldots, n-1$*

$$\|f - S_n\|_{\mathbf{L}_p(J_i)} \le C\tau^3(f, |J_i|, J_i)_p, \tag{2.8}$$

where C and J_i are as in Theorem 3.

PROOF. The theorem is proved in Hu [12] for the case of quadratic splines with equidistant knots. We now generalize it to splines of any order $r \ge 3$ on unequally spaced knots. The proof is similar to that of Theorem 3. We shall use most of the notation and only indicate the differences of the two proofs.

The first two paragraphs in the poof of Theorem 3 also work here, except that we need to change (2.3) to:

$$\|f - p_i\|_{\mathbf{L}_p[\overline{z}_i, \overline{z}_{i+2}]} \le C\tau^r(f, |\overline{I}_i|, [\overline{z}_i, \overline{z}_{i+2}])_p \le C\tau^3(f, |\overline{I}_i|, [\overline{z}_i, \overline{z}_{i+2}])_p, \tag{2.9}$$

https://doi.org/10.4153/CJM-1997-004-2 Published online by Cambridge University Press

The main change of the proof is near each point y_j of sign change of f, $j = 1, \ldots, s$, where we now construct a local quadratic polynomial p_{m_j} on $[\bar{z}_{m_j-1}, \bar{z}_{m_j+2}]$ interpolating f at $\bar{z}_{m_j-1}$, y_j and $\bar{z}_{m_j+2}$. Since (2.1) is now true for $j = 0, \ldots, s$, these points are separated by at least $d = 2(r-1)^2$ knots in $\mathbf{T}_n$. It is proved in [12] that p_{m_j} is copositive with f and satisfies

(2.10) $$\|f - p_{m_j}\|_{\mathbf{L}_p[\bar{z}_i, \bar{z}_{i+2}]} \le C\tau^3\big(f, |\bar{I}_{m_j}|, [\bar{z}_{m_j-1}, \bar{z}_{m_j+2}]\big)_p.$$

The rest of the proof is analogous to that of Theorem 3. ∎

3. Constrained polynomial approximation.

3.1. *Intertwining approximation of truncated power functions.* In this section, we consider intertwining polynomial approximation of the truncated power functions $(x-\Lambda)_+^{2k}$, *i.e.*, we construct polynomials P_1 and P_2 which sufficiently approximate $(x-\Lambda)_+^{2k}$ and such that $P_1(x) - (x-\Lambda)_+^{2k} \in \Delta^0(Y_s)$ and $(x-\Lambda)_+^{2k} - P_2(x) \in \Delta^0(Y_s)$. After that we use the well known procedure involving analytic representations of splines to construct an intertwining pair of polynomials for an arbitrary function f.

Let

$$x_j := \cos\frac{j\pi}{n}, \quad 0 \le j \le n; \ \bar{x}_j := \cos\Big(\frac{j\pi}{n} - \frac{\pi}{2n}\Big), \quad 1 \le j \le n;$$
$$x_j^0 := \cos\Big(\frac{j\pi}{n} - \frac{\pi}{4n}\Big) \quad \text{if } j < n/2, \ x_j^0 := \cos\Big(\frac{j\pi}{n} - \frac{3\pi}{4n}\Big) \quad \text{if } j \ge n/2;$$
$$I_j := [x_j, x_{j-1}], \ h_j := x_{j-1} - x_j, \quad 1 \le j \le n$$

(note that $h_{j\pm1} < 3h_j$ and $\Delta_n < h_j < 5\Delta_n$ for $x \in I_j$).

Also,

$$t_j(x) := (x - x_j^0)^{-2}\cos^2 2n\arccos x + (x - \bar{x}_j)^{-2}\sin^2 2n\arccos x$$

is the algebraic polynomial of degree $4n-2$ (see [30], for example).

We also denote

$$\psi_j(x) := \frac{h_j}{|x - x_j| + h_j}, \quad \chi[a,b](x) := \begin{cases} 1 & \text{if } x \in [a,b], \\ 0 & \text{otherwise,} \end{cases}$$
$$\chi_j(x) := \chi[x_j, 1](x), \quad \operatorname{sgn}\big(f(x)\big) := \begin{cases} -1 & \text{if } f(x) < 0, \\ 0 & \text{if } f(x) = 0, \\ 1 & \text{if } f(x) > 0. \end{cases}$$

Now, let

$$T_{\alpha,\beta}(x) := T_{\alpha,\beta}(j,n,\mu,\xi,\zeta)(x) := \frac{\int_{-1}^x (y-\alpha)(\beta-y)(y-x_j)^\xi (x_{j-1}-y)^\zeta t_j(y)^\mu\,dy}{\int_{-1}^1 (y-\alpha)(\beta-y)(y-x_j)^\xi (x_{j-1}-y)^\zeta t_j(y)^\mu\,dy},$$

where $\alpha \in [-1, x_j]$ and $\beta \in [x_{j-1}, 1]$. If μ is sufficiently large in comparison with ξ and ζ (for example, $\mu \ge 5\xi + 5\zeta + 15$ will do), then $T_{\alpha,\beta}$ is a polynomial of degree $\le c(\mu)n$, and the denominator $\int_{-1}^1 (y-\alpha)(\beta-y)(y-x_j)^\xi(x_{j-1}-y)^\zeta t_j(y)^\mu\,dy$ is a positive number (see [18], for example).

For the sake of brevity, we introduce the following convention:

$$\tau(y) := (y - x_j)^{\xi}(x_{j-1} - y)^{\zeta} t_j(y)^{\mu}.$$

First, we consider intertwining approximation of the step function $\chi_j(x)$. The following lemma and its corollary contain the basis for all our further constructions.

LEMMA 16. *For any $A \in [-1, x_j]$ and $B \in [x_{j-1}, 1]$ there exist $\alpha \in [-1, A]$ and $\beta \in [B, 1]$ such that*

$$T_{\alpha,\beta}(A) = 0 \tag{3.1}$$

and

$$T_{\alpha,\beta}(B) = 1. \tag{3.2}$$

PROOF. If $A = -1$ or/and $B = 1$, then the choice of α and β is obvious. Now, let $A \in (-1, x_j]$ and $B \in [x_{j-1}, 1)$ be fixed. Then for any $\beta \in [B, 1]$ there exists a unique $\alpha \in [-1, A]$ such that (3.1) is satisfied. Indeed,

$$T_{\alpha,\beta}(A) = 0 \iff \alpha = \gamma(\beta) := \frac{\int_{-1}^{A} y(\beta - y)\tau(y)\,dy}{\int_{-1}^{A} (\beta - y)\tau(y)\,dy}.$$

Since $\int_{-1}^{A}(\beta - y)\tau(y)\,dy \neq 0$ for $\beta \in [B, 1]$, then the function $\gamma \in \mathbf{C}[B, 1]$. Also, for any $\beta \in [B, 1]$ we have $-1 \le \gamma(\beta) \le A$. The first inequality is obvious. The second one holds since

$$\gamma(\beta) \le A \iff \frac{\int_{-1}^{A}(A - y)(\beta - y)\tau(y)\,dy}{\int_{-1}^{A}(\beta - y)\tau(y)\,dy} \ge 0,$$

which is true since

$$\operatorname{sgn}\Big(\int_{-1}^{A}(A - y)(\beta - y)\tau(y)\,dy\Big) = \operatorname{sgn}\Big(\int_{-1}^{A}(\beta - y)\tau(y)\,dy\Big) = (-1)^{\xi}.$$

Similarly, it can be shown that for any $\alpha \in [-1, A]$ there exists a unique $\beta \in [B, 1]$ such that (3.2) is satisfied, *i.e.*, there exists a function $\delta \in \mathbf{C}[-1, A]$ with the range $[B, 1]$ such that (3.2) is satisfied for $\beta = \delta(\alpha)$.

Thus, there exists $(\alpha, \beta) \in [-1, A] \times [B, 1]$ such that $\alpha = \gamma(\beta)$ and $\beta = \delta(\alpha)$, *i.e.*, (3.1) and (3.2) are satisfied simultaneously. The proof of the lemma is complete. ∎

COROLLARY 17. *Let an index $1 \le j \le n - 1$ be fixed. For any $A \in [-1, x_{j+1}]$ and $B \in [x_{j-1}, 1]$ there exist polynomials $T_i(A, B)(x)$, $i = 1, 2, 3, 4$ of degree $\le C(\mu)n$ such that $T_i(A, B)(x) = \chi_j(x)$ for $x = -1, A, B, 1$, and also satisfying*

$$|\chi_j(x) - T_i(A, B)(x)| \le 1/3, \quad x \in [-1, 1] \setminus [x_{j+1}, x_{j-1}], \quad i = 1, 2, 3, 4, \tag{3.3}$$

$$|\chi_j(x) - T_i(A, B)(x)| \le C\psi_j(x)^{\mu}, \quad x \in [-1, 1],\ i = 1, 2, 3, 4, \tag{3.4}$$

(3.5)

$$\operatorname{sgn}\{T_1(A, B)(x) - \chi_j(x)\} = \operatorname{sgn}\{-(x - A)(x - x_j)(B - x)\}, \quad x \in (-1, 1) \setminus \{x_j\},$$

(3.6)

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$T_1(A,B)(x)$ is increasing on $[A,B]$ and, thus, $0 < T_1(A,B)(x) < 1, \quad x \in (A,B)$,

$$\operatorname{sgn}\{T_2(A,B)(x) - \chi_j(x)\} = \operatorname{sgn}\{(x-A)(B-x)\}, \quad x \in (-1,1), \tag{3.7}$$

$$\operatorname{sgn}\{T_3(A,B)(x) - \chi_j(x)\} = \operatorname{sgn}\{-(x-A)(B-x)\}, \quad x \in (-1,1), \tag{3.8}$$

(3.9)

$$\operatorname{sgn}\{T_4(A,B)(x) - \chi_j(x)\} = \operatorname{sgn}\{(x-A)(x-x_j)(B-x)\}, \quad x \in (-1,x_j) \cup [x_{j-1},1).$$

The polynomial $T_4(x)$ will not be used in the construction later on, but is included in the above statement for completeness, since the authors believe that Corollary 17 is an interesting and important result by itself.

PROOF. It is sufficient to choose

$$\begin{aligned} T_1(A,B)(x) &:= T_{\alpha,\beta}(Mj, Mn, \mu, 0, 0)(x), \\ T_2(A,B)(x) &:= T_{\alpha,\beta}(Mj+1, Mn, \mu, 0, 1)(x), \\ T_3(A,B)(x) &:= T_{\alpha,\beta}(Mj, Mn, \mu, 1, 0)(x) \end{aligned}$$

and

$$T_4(A,B)(x) := T_{\alpha,\beta}(Mj, Mn, \mu, 1, 1)(x),$$

with sufficiently large constant $M = M(\mu) \in N$.

The equalities (3.5)–(3.9) are obvious, and (3.4) and (3.3) follow, respectively, from [18, Lemma 5] and the proof of [18, Lemma 6]. ∎

LEMMA 18. *Let an index $1 \le j \le n-1$ be fixed, and let the numbers $\{a_i\}_{i=1}^k$ and b be such that $-1 \le a_1 < a_2 < \cdots < a_k \le x_{j+1}$ and $x_{j-1} \le b \le 1$. Then there exist polynomials $Q_i(x) := Q_i(x_j, \mu; a_1, \ldots, a_k; b)(x)$, $i = 1,2,3$, of degree $\le C(\mu)n$ such that $Q_i(x) = \chi_j(x)$, $i = 1,2,3$, for $x = -1, a_1, \ldots, a_k, b, 1$, $Q_i(x) \ge 0$, $x \ge b$, $Q_i(x) \le 1$, $x \le a_k$, and also satisfying*

$$|\chi_j(x) - Q_i(x)| \le C(\mu,k)\psi_j(x)^{\mu}, \quad i = 1,2,3, \tag{3.10}$$

$$\begin{aligned} &\operatorname{sgn}\{Q_1(x_j,\mu;a_1,\ldots,a_k;b)(x) - \chi_j(x)\} \\ &\quad = \operatorname{sgn}\Big\{-\prod_{\nu=1}^k (x-a_\nu)(x-x_j)(b-x)\Big\}, \quad x \in [-1,1] \setminus \{x_j\}, \end{aligned} \tag{3.11}$$

$$\begin{aligned} &\operatorname{sgn}\{Q_2(x_j,\mu;a_1,\ldots,a_k;b)(x) - \chi_j(x)\} \\ &\quad = \operatorname{sgn}\Big\{-\prod_{\nu=1}^k (x-a_\nu)(b-x)\Big\}, \quad x \in [-1,1], \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} &\operatorname{sgn}\{Q_3(x_j,\mu;a_1,\ldots,a_k;b)(x) - \chi_j(x)\} \\ &\quad = \operatorname{sgn}\Big\{\prod_{\nu=1}^k (x-a_\nu)(b-x)\Big\}, \quad x \in [-1,1]. \end{aligned} \tag{3.13}$$

PROOF. Let

$$Q_1(x_j,\mu;a_1,\dots,a_k;b)(x) := \prod_{\nu=1}^{k} T_1(a_\nu,b)(x),$$

$$Q_2(x_j,\mu;a_1,\dots,a_k;b)(x) := \prod_{\nu=1}^{k-1} T_1(a_\nu,b)(x)T_3(a_k,b)(x)$$

and

$$Q_3(x_j,\mu;a_1,\dots,a_k;b)(x) := \prod_{\nu=1}^{k} T_2(a_\nu,b)(x).$$

It is straightforward to check that the inequalities (3.11)–(3.13) are satisfied. Inequality (3.10) follows from the observation that if $\tilde{q}(x) = \prod_{\nu=1}^{k} q_\nu(x)$ and $|\chi_j(x) - q_\nu(x)| \le C(\mu)\psi_j(x)^\mu$, $x \in [-1,1]$, then $|\chi_j(x) - \tilde{q}(x)| \le C(\mu,k)\psi_j(x)^\mu$ for all $x \in [-1,1]$. Indeed, for $x \le x_j$ we have

$$|\tilde{q}(x) - \chi_j(x)| = \prod_{\nu=1}^{k} |q_\nu(x)| \le C(\mu,k)\psi_j(x)^{k\mu}.$$

If $x \ge x_j$, then

$$|\tilde{q}(x) - \chi_j(x)| = \Big|\prod_{\nu=1}^{k} q_\nu(x) - 1\Big| = \Big|\sum_{\nu=1}^{k-1}\Big(\prod_{i=1}^{\nu} q_i(x)\Big)(q_{\nu+1}(x)-1) + q_1(x) - 1\Big| \le C(\mu,k)\psi_j(x)^\mu.$$

∎

Now, consider the "flipped" functions $\bar{Q}_i(x_j,\mu;a;b_1,\dots,b_k)(x)$, $i = 1,2,3$, defined as follows:

$$\bar{Q}_i(x_j,\mu;a;b_1,\dots,b_k)(x) := 1 - Q_i(x_{n-j},\mu;-b_k,\dots,-b_1;-a)(-x)$$

for $-1 \le a \le x_{j+1}$ and $x_{j-1} \le b_1 < b_2 < \cdots < b_k \le 1$.

Then $\bar{Q}_i(x) = \chi_j(x)$, $i = 1,2,3$, for $x = -1, a, b_1, \dots, b_k, 1$, $\bar{Q}_i(x) \ge 0$, $x \ge b_1$, $\bar{Q}_i(x) \le 1$, $x \le a$, and also

$$|\chi_j(x) - \bar{Q}_i(x)| \le C(\mu,k)\psi_j(x)^\mu, \quad i = 1,2,3, \tag{3.14}$$

$$\begin{aligned}&\operatorname{sgn}\{\bar{Q}_1(x_j,\mu;a;b_1,\dots,b_k)(x) - \chi_j(x)\}\\ &\qquad = \operatorname{sgn}\Big\{-(x-a)(x-x_j)\prod_{\nu=1}^{k}(b_\nu - x)\Big\}, \quad x \in [-1,1]\setminus\{x_j\},\end{aligned} \tag{3.15}$$

$$\begin{aligned}&\operatorname{sgn}\{\bar{Q}_2(x_j,\mu;a;b_1,\dots,b_k)(x) - \chi_j(x)\}\\ &\qquad = \operatorname{sgn}\Big\{(x-a)\prod_{\nu=1}^{k}(b_\nu - x)\Big\}, \quad x \in [-1,1]\end{aligned} \tag{3.16}$$

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and

$$\operatorname{sgn}\{\bar{Q}_3(x_j,\mu;a;b_1,\ldots,b_k)(x)-\chi_j(x)\}$$
$$= \operatorname{sgn}\Big\{-(x-a)\prod_{\nu=1}^{k}(b_\nu - x)\Big\}, \quad x\in[-1,1]. \tag{3.17}$$

Let

$$\begin{aligned} R_1(x) &:= R_1(x_j,\mu;a_1,\ldots,a_k;b_1,\ldots,b_l)(x) \\ &:= \prod_{\nu=1}^{k-1}\bar{Q}_1(x_j,\mu;a_\nu;b_1,\ldots,b_l)(x)\bar{Q}_3(x_j,\mu;a_k;b_1,\ldots,b_l)(x) \end{aligned}$$

and

$$R_2(x) := R_2(x_j,\mu;a_1,\ldots,a_k;b_1,\ldots,b_l)(x) := \prod_{\nu=1}^{k}\bar{Q}_2(x_j,\mu;a_\nu;b_1,\ldots,b_l)(x).$$

Then

$$|\chi_j(x)-R_i(x)| \le C(\mu,k,l)\psi_j(x)^\mu, \quad i=1,2, \tag{3.18}$$

$$\operatorname{sgn}\{R_1(x_j,\mu;a_1,\ldots,a_k;b_1,\ldots,b_l)(x)-\chi_j(x)\}$$
$$= \operatorname{sgn}\Big\{-\prod_{\nu=1}^{k}(x-a_\nu)\prod_{\tilde\nu=1}^{l}(b_{\tilde\nu}-x)\Big\}, \quad x\in[-1,1] \tag{3.19}$$

and

$$\operatorname{sgn}\{R_2(x_j,\mu;a_1,\ldots,a_k;b_1,\ldots,b_l)(x)-\chi_j(x)\}$$
$$= \operatorname{sgn}\Big\{\prod_{\nu=1}^{k}(x-a_\nu)\prod_{\tilde\nu=1}^{l}(b_{\tilde\nu}-x)\Big\}, \quad x\in[-1,1]. \tag{3.20}$$

Finally, multiplying R_i, $i=1,2$, by $(x-\Lambda)^{2m}$ we obtain an intertwining pair of polynomials for $(x-\Lambda)_+^{2m}$ with respect to $\{a_i\}_{i=1}^k \cup \{b_i\}_{i=1}^l$ with good approximation properties.

LEMMA 19. *Let an index $1\le j\le n-1$ be fixed, $\{a_i\}_{i=1}^k$ and $\{b_i\}_{i=1}^l$ be such that $-1\le a_1<a_2<\cdots<a_k\le x_{j+1}<x_{j-2}\le b_1<\cdots<b_l\le 1$, $\Lambda\in[x_j,x_{j-1}]$ and $m\in\mathbb{N}\cup\{0\}$. Then the polynomials*

$$\begin{aligned} \bar{R}_i(x) &:= \bar{R}_i(x_j,\Lambda,\mu;a_1,\ldots,a_k;b_1,\ldots,b_l)(x) \\ &:= (x-\Lambda)^{2m}R_i(x_{j+i-2},\mu;a_1,\ldots,a_k;b_1,\ldots,b_l)(x), \quad i=1,2 \end{aligned}$$

of degree $\le C(\mu)n$ are such that

$$|\bar{R}_i(x)-(x-\Lambda)_+^{2m}| \le C(\mu,k,l)\psi_j(x)^{\mu-2m}h_j^{2m}, \quad x\in[-1,1],\ i=1,2, \tag{3.21}$$

$$(-1)^{l+1}\Big(\bar{R}_1(x)-(x-\Lambda)_+^{2m}\Big)\in\Delta^0(\{a_i\}_{i=1}^k\cup\{b_i\}_{i=1}^l) \tag{3.22}$$

and

$$(-1)^{l}\Big(\bar{R}_2(x)-(x-\Lambda)_+^{2m}\Big)\in\Delta^0(\{a_i\}_{i=1}^k\cup\{b_i\}_{i=1}^l). \tag{3.23}$$

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PROOF. The assertion of the lemma follows from (3.18)–(3.20), the observation that

$$
\begin{aligned}
&\overline{R}_i(x) - (x-\Lambda)_+^{2m} \\
&\quad = (x-\Lambda)^{2m}\big(R_i(x_{j+i-2}, \mu, a_1, \ldots, a_k; b_1, \ldots, b_l)(x) - \chi[\Lambda, 1](x)\big) \\
&\quad = (x-\Lambda)^{2m}\big(R_i(x_{j+i-2}, \mu, a_1, \ldots, a_k; b_1, \ldots, b_l)(x) - \chi_{j+i-2}(x)\big), \quad i = 1, 2,
\end{aligned}
$$

for $x \in [-1,1] \setminus [x_j, x_{j-1}]$, the inequalities

$$
\begin{aligned}
R_2(x_j, \mu, a_1, \ldots, a_k; b_1, \ldots, b_l)(x) &\geq \chi_j(x) \geq \chi[\Lambda, 1](x) \geq \chi_{j-1}(x) \\
&\geq R_1(x_{j-1}, \mu, a_1, \ldots, a_k; b_1, \ldots, b_l)(x)
\end{aligned}
$$

for $x \in [x_j, x_{j-1}]$, and the fact that $\psi_j(x) \sim \psi_{j\pm1}(x)$ for $x \in [-1,1]$. ∎

3.2. *Proof of Theorem 4.* For the proof of Theorem 4 we need the following classical result on analytic representation of splines in terms of the truncated power functions. Its proof can be found in Kornejchuk [23], for example.

LEMMA 20. *Let $S(t)$ be a spline of order r on the knot sequence $-1 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$. Then, for $t \in [-1,1]$,*

$$
S(t) = \sum_{\nu=0}^{r-1} \frac{S^{(\nu)}(-1)}{\nu!}(t+1)^\nu + \sum_{j=1}^{n-1} \frac{S^{(r-1)}(t_j+) - S^{(r-1)}(t_j-)}{(r-1)!}(t-t_j)_+^{r-1}. \tag{3.24}
$$

Let S satisfy the assertion of Theorem 4. Then

$$
\begin{aligned}
S(x) &= \sum_{\nu=0}^{2m} \frac{S^{(\nu)}(-1)}{\nu!}(x+1)^\nu + \sum_{j\in J(Y_s)} \alpha_j (x-x_j)_+^{2m} \\
&= \sum_{\nu=0}^{2m} \frac{S^{(\nu)}(-1)}{\nu!}(x+1)^\nu + \sum_{j\in J^+} |\alpha_j|(x-x_j)_+^{2m} - \sum_{j\in J^-} |\alpha_j|(x-x_j)_+^{2m},
\end{aligned}
$$

where $\alpha_j := \frac{S^{(2m)}(x_j+) - S^{(2m)}(x_j-)}{(2m)!}$, $J^+ := \{j \mid \alpha_j \geq 0\} \cap J(Y_s)$, and $J^- := \{j \mid \alpha_j < 0\} \cap J(Y_s)$. For each $j \in J(Y_s)$ we define

$$
u(j) := \min\{i \mid y_i \geq x_j\}
$$

(*i.e.*, $y_{u(j)}$ is the first point y_i on the right of x_j), and

$$
\nu(j) := \begin{cases} j, & \text{if } y_{u(j)} \geq x_{j-2}, \\ j+1, & \text{if } x_{j-1} \leq y_{u(j)} < x_{j-2}. \end{cases}
$$

(In other words, we define $\nu(j)$ to be an index satisfying $|\nu(j) - j| \leq 1$ and such that the interval $[x_{\nu(j)}, x_{\nu(j)-1}]$ is not "too close" to y_i's (recall that $j \in J(Y_s)$). This is a technicality which is needed because a_i's and b_i's in the assertion of Lemma 19 "should be far" from x_j. Also, we would like to emphasize that if $\nu(j) = j+1$, then $y_{u(j)-1} < x_{j+2}$, since there are at least 4 points x_j between $y_{u(j)-1}$ and $y_{u(j)}$.)

We show that the polynomials

$$P_i(x) = \sum_{\nu=0}^{2m} \frac{S^{(\nu)}(-1)}{\nu!}(x+1)^\nu + \sum_{j\in J(Y_s)} \alpha_j \tilde{R}_{i,j}(Y_s)(x), \quad i = 1, 2,$$

where $\tilde{R}_{i,j}(Y_s)(x) \equiv \bar{R}_i(x_{\nu(j)}, x_j, 2m+30; Y_s)(x)$ if $j \in J^+$ and $(-1)^{s-u(j)} > 0$, or $j \in J^-$ and $(-1)^{s-u(j)} < 0$, and $\tilde{R}_{i,j}(Y_s)(x) \equiv \bar{R}_{3-i}(x_{\nu(j)}, x_j, 2m+30; Y_s)(x)$ otherwise, $i = 1, 2$, form an intertwining pair for S satisfying the estimate (1.16).

It is relatively straightforward to verify (1.16). Using Markov's inequality first and then Jensen's inequality (note that $\sum_{j=1}^{n-1} \psi_j(x)^{\mu-2m} < C$), as well as the fact that $h_j \sim h_{j\pm 1}$ and $\psi_j(x) \sim \psi_{j\pm 1}(x)$, we write for $0 < p < \infty$:

$$\begin{aligned}
\|P_1 - P_2\|_p^p &\le C^p \int_{-1}^1 \Big(\sum_{j\in J(Y_s)} \|S_j^{(2m)}(x) - S_{j+1}^{(2m)}(x)\|_{\mathbf{C}(I_j)} \Big|\tilde{R}_{1,j}(Y_s)(x) - \tilde{R}_{2,j}(Y_s)(x)\Big|\Big)^p dx \\
&\le C^p \int_{-1}^1 \Big(\sum_{j=1}^{n-1} \|S_j(x) - S_{j+1}(x)\|_{\mathbf{C}(I_j)} \psi_j(x)^{\mu-2m}\Big)^p dx \\
&\le C^p \sum_{j=1}^{n-1} h_j^{-1} \|S_j - S_{j+1}\|_{\mathbf{L}_p(I_j)}^p \int_{-1}^1 \psi_j(x)^{(\mu-2m)\min\{1,p\}} dx \\
&\le C^p \sum_{j=1}^{n-1} \|S_j - S_{j+1}\|_{\mathbf{L}_p(I_j)}^p,
\end{aligned}$$

where S_j denotes a polynomial from $\mathbf{P}_{r-1}$ such that $S_j(x) \equiv S|_{I_j}(x)$, $x \in I_j$ (*i.e.*, S_j is a polynomial of degree $\le r-1$ which coincides with S on I_j). A similar estimate is true in the case $p = \infty$ as well (see [19], for example). It remains to show that

$$\|S_j - S_{j+1}\|_{\mathbf{L}_p(I_j)} \le C E_{r-1}(S, I_j \cup I_{j+1})_p, \quad 0 < p \le \infty. \tag{3.25}$$

Indeed, using the observation that S_{j+1} is the best approximant to S on I_{j+1} from $\mathbf{P}_{r-1}$ (this is, of course, true since $S_{j+1} \equiv S|_{I_{j+1}}$) we write

$$\begin{aligned}
\|S_{j+1} - S_j\|_{\mathbf{L}_p(I_j)} &= \|S_{j+1} - S\|_{\mathbf{L}_p(I_j)} \\
&\le \|S_{j+1} - S\|_{\mathbf{L}_p(I_j \cup I_{j+1})} \\
&\le C E_{r-1}(S, I_j \cup I_{j+1})_p.
\end{aligned}$$

In the last inequality, we used the fact that S_{j+1} is also a near-best $\mathbf{L}_p$ approximant to S on $I_j \cup I_{j+1}$ from $\mathbf{P}_{r-1}$ (see DeVore and Popov [6, Lemma 3.3]).

Finally, we verify that $P_1(x) - S(x) \in \Delta^0(Y_s)$ (the proof of the inclusion $S(x) - P_2(x) \in \Delta^0(Y_s)$ is similar). Let $x \in [y_i, y_{i+1})$ be fixed. Then denoting $\bar{R}_{i,j}(x) := \bar{R}_i(x_{\nu(j)}, x_j, 2m + 30; Y_s)(x)$, $i = 1, 2$, and using Lemma 19 with $l = s - u(j) + 1$ for each $j \in J(Y_s)$, we have

$$\begin{aligned}
&P_1(x) - S(x) \\
&\quad = \sum_{j\in J(Y_s)} \alpha_j \big(\tilde{R}_{1,j}(Y_s)(x) - (x - x_j)_+^{2m}\big)
\end{aligned}$$

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$$= \sum_{j\in J^+,(-1)^{s-u(j)}>0 \text{ or } j\in J^-,(-1)^{s-u(j)}<0} |\alpha_j|(-1)^{s-u(j)}\Big(\bar{R}_{1,j}(Y_s)(x) - (x-x_j)_+^{2m}\Big)$$
$$+ \sum_{j\in J^+,(-1)^{s-u(j)}<0 \text{ or } j\in J^-,(-1)^{s-u(j)}>0} |\alpha_j|(-1)^{s-u(j)+1}\Big(\bar{R}_{2,j}(Y_s)(x) - (x-x_j)_+^{2m}\Big)$$
$$\in \Delta^0(Y_s),$$

since if $f_j \in \Delta^0(Y_s)$ and $\beta_j \geq 0$ for all j, then $\sum_j \beta_j f_j \in \Delta^0(Y_s)$. The proof is now complete. ∎

The proofs of Theorems 1, 5, 7, 8, and 10 will follow the same scheme. Namely, using Theorems 3, 15, and Lemma 14 we construct splines satisfying the appropriate constrains and having the right approximation order. Then, we use Theorem 4 to find polynomial(s) with similar characteristics.

PROOF OF THEOREM 1. Let $f \in \mathbf{C}[-1,1]$ and $m \in N$. It follows from Lemma 14 (with $\mathbf{T}_n = \{x_j\}$) that there exist splines $\bar{S}$ and S of an odd order r, $r \geq m$ (we choose $r = 2[\frac{m}{2}] + 1$), such that $\bar{S}(x) \geq f(x) \geq S(x)$, $x \in I$, and

$$\|\bar{S} - S\|_{\mathbf{C}(I_j)} \leq C\omega^r(f, |J_j|, J_j)_\infty.$$

Since $|J_j| \sim |I_j| = h_j \sim \Delta_n(x)$ for $x \in J_j$, then

$$|\bar{S}(x) - S(x)| \leq C\omega^r(f, h_j)_\infty \leq C\omega^r\Big(f, \Delta_n(x)\Big)_\infty \leq C\omega^m\Big(f, \Delta_n(x)\Big)_\infty, \quad x \in I_j,$$

and, therefore,

$$|\bar{S}(x) - S(x)| \leq C\omega^m\Big(f, \Delta_n(x)\Big)_\infty \quad \text{for all } x \in I.$$

Also,

$$\begin{aligned} E_{r-1}(\bar{S}, I_j \cup I_{j+1})_\infty &\leq E_{r-1}(\bar{S} - f, I_j \cup I_{j+1})_\infty + E_{r-1}(f, I_j \cup I_{j+1})_\infty \\ &\leq \|\bar{S} - S\|_{\mathbf{C}(I_j\cup I_{j+1})} + C\omega^r(f, h_j)_\infty \leq C\omega^r(f, h_j)_\infty, \end{aligned}$$

and, similarly,

$$E_{r-1}(S, I_j \cup I_{j+1})_\infty \leq C\omega^r(f, h_j)_\infty.$$

Theorem 4 (where $\mu = 31m$ is chosen) implies that there exist polynomials $\bar{P}_1$, $\bar{P}_2$, P_1, and P_2 of degree $\leq C(m)n$ such that $\bar{P}_1(x) \geq \bar{S}(x) \geq \bar{P}_2(x)$, $P_1(x) \geq S(x) \geq P_2(x)$,

$$\begin{aligned} |\bar{P}_1(x) - \bar{P}_2(x)| &\leq C\sum_{j=1}^{n-1} \omega^r(f, h_j)_\infty \psi_j(x)^{r+2} \\ &\leq C\omega^r\Big(f, \Delta_n(x)\Big)_\infty \sum_{j=1}^{n-1} \psi_j(x)^2 \\ &\leq C\omega^r\Big(f, \Delta_n(x)\Big)_\infty \\ &\leq C\omega^m\Big(f, \Delta_n(x)\Big)_\infty, \end{aligned} \tag{3.26}$$

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and,

$$|P_1(x) - P_2(x)| \le C\omega^m\big(f, \Delta_n(x)\big)_\infty.$$

Here, we have used the inequality

$$h_j \le C\Delta_n(x)\psi_j(x)^{-1}, \quad x \in I, \tag{3.27}$$

which follows from the fact that $\Delta_n(y)^2 \le 4\Delta_n(x)\big(|x-y| + \Delta_n(x)\big)$ and $|x-y| + \Delta_n(x) \sim |x-y| + \Delta_n(y)$, $x, y \in I$ (see [30] or [19], for example). Now, the polynomials $\overline{P}_1$ and P_2 are what we are looking for since $\overline{P}_1 \ge \overline{S} \ge f \ge S \ge P_2$, and

$$|\overline{P}_1(x) - P_2(x)| \le |\overline{P}_1(x) - \overline{P}_2(x)| + |\overline{S}(x) - S(x)| + |P_1(x) - P_2(x)| \le C\omega^m\big(f, \Delta_n(x)\big)_\infty.$$

The proof is complete for large n ($n \ge C(m)$). For $m - 1 \le n < C(m)$ the estimate (1.7) is a trivial corollary of (1.3). ∎

PROOF OF THEOREM 5. Theorem 3 implies the existence of the intertwining pair of splines $\{\overline{S}, S\}$ of order r for f on the knot sequence $\{x_j\}_{j\in J(Y_s)}$ (recall that $J(Y_s) = \{1, \ldots, n\} \setminus \{j, j-1 \mid x_j \le y_i < x_{j-1} \text{ for some } 1 \le i \le s\}$) satisfying

$$\|\overline{S} - S\|_{\mathbf{C}(I_j)} \le Ch_j\omega^{r-1}(f', h_j, J_j)_\infty,$$

where r is an odd integer such that $m + 1 \le r \le m + 2$.

We need the following consequence of Lemma 5 of [22]:

Let $[a, b] \subset [-1, 1]$ be such that $|b - a| \sim \Delta_n(x)$ for $x \in [a, b]$. Then for any $r \in N$ there exists a constant $C(r)$ such that

$$\begin{aligned} C(r)^{-1}\omega^r\big(f, \Delta_n(x), [a, b]\big)_\infty &\le \omega_\varphi^r(f, n^{-1}, [a, b])_\infty \\ &\le C(r)\omega^r\big(f, \Delta_n(x), [a, b]\big)_\infty, \quad x \in [a, b]. \end{aligned}$$

In particular,

$$\omega^{r-1}(f, h_j, J_j)_\infty \le C(r)\omega_\varphi^{r-1}(f, n^{-1})_\infty. \tag{3.28}$$

Therefore,

$$|\overline{S}(x) - S(x)| \le C\Delta_n(x)\omega^{r-1}\big(f', \Delta_n(x)\big)_\infty, \quad x \in I_j$$

and

$$\|\overline{S} - S\|_{\mathbf{C}(I_j)} \le Cn^{-1}\omega_\varphi^{r-1}(f', n^{-1})_\infty.$$

Theorem 4 (with sufficiently large μ, say, $\mu = 31m$) implies that there exist intertwining pairs of polynomials $\{\overline{P}_1, \overline{P}_2\}$ and $\{P_1, P_2\}$ for $\overline{S}$ and S, respectively, satisfying the inequalities

$$\begin{aligned} |\overline{P}_1(x) - \overline{P}_2(x)| &\le C\Delta_n(x)\omega^{r-1}\big(f', \Delta_n(x)\big)_\infty, \\ |P_1(x) - P_2(x)| &\le C\Delta_n(x)\omega^{r-1}\big(f', \Delta_n(x)\big)_\infty, \\ \|\overline{P}_1 - \overline{P}_2\|_{\mathbf{C}(I)} &\le Cn^{-1}\omega_\varphi^{r-1}(f', n^{-1})_\infty, \end{aligned}$$

and

$$\|P_1 - P_2\|_{\mathbf{C}(I)} \leq Cn^{-1}\omega_\varphi^{r-1}(f', n^{-1})_\infty,$$

which can be verified similarly to (3.26) using (3.28), (3.27), and the fact that $\sum_{j=1}^{n-1}\psi_j(x)^2 \leq C$. Finally, $\{\overline{P}_1, P_2\}$ is an intertwining pair of polynomials for f satisfying (1.18) and (1.19). ∎

PROOF OF THEOREMS 7, 8 AND 10. The proofs of these theorems are similar to the above proofs. The needed modifications are obvious. We omit details and just emphasize that the inequalities (1.12) and (1.13) should be used. ∎

Finally, we mention that the same proofs can be used to show the validity of the results of Stojanova [31] (ineq. (1.5)), Kopotun [20] (ineq. (1.9)) and the authors [13] (ineq. (1.11)).

4. **Counterexamples.**

PROOF OF THEOREM 11. Let $n \in N$, $0 < p < \infty$, $0 < \varepsilon \leq 1$ and $A > 0$ be fixed, and define

$$g(x) := \ln(x^2 + e^{-b}), \quad f'(x) := bx^2 - \ln b - g(x)$$

and

$$f(x) := \int_0^x f'(t)\,dt = \frac{b}{3}x^3 + (2 - \ln b)x - 2e^{-b/2}\arctan(xe^{b/2}) - x\ln(x^2 + e^{-b}),$$

where $b \geq 1$ is a parameter to be chosen later. Obviously, $f \in \mathbf{C}^\infty$ and $f(0) = 0$. Basic calculus shows that f' assumes its minimum $1 - be^{-b} > 0$ at $x = \pm\sqrt{\frac{e^b - b}{be^b}} \in (-1, 1)$, thus $f'(x) > 0$ for all x, and f is strictly increasing.

Since $\ln 2 > g(x) > \ln x^2 = 2\ln|x|$ on $[-1, 1]$, we have

$$\begin{aligned}(4.1)\qquad \|g\|_p^p &= \int_{-1}^1 |g(x)|^p\,dx = 2\int_0^1 |g(x)|^p\,dx \leq 2\int_0^1 (\ln 2)^p\,dx + 2\int_0^1 |2\ln x|^p\,dx \\ &= 2(\ln 2)^p + 2^{p+1}\Gamma(p+1) =: M_1^p.\end{aligned}$$

Hence

$$(4.2)\qquad \omega^3(f', 1)_p = \omega^3(g, 1)_p \leq 8^{\max\{1,1/p\}}\|g\|_p \leq 8^{\max\{1,1/p\}}M_1 =: M_2.$$

We now prove there exists $b \geq 1$ for the given n, ε, p and A such that if any $P_n(x) = a_0 + a_1x + \cdots + a_nx^n$ satisfies

$$(4.3)\qquad \|f - P_n\|_{\mathbf{L}_p[0,\varepsilon]} \leq A\omega^3(f', 1)_p,$$

then $P_n'(0) < 0$. The theorem follows immediately because if P_n were copositive with f, $P_n'(0)$ would be nonnegative. Let $P_n(x) - (2 - \ln b)x - bx^3/3$ be denoted by $\overline{P}_n$. From (4.2)–(4.3) we have

$$\begin{aligned}(4.4)\qquad \|\overline{P}_n\|_{\mathbf{L}_p[0,\varepsilon]}^p &\leq \|f - P_n\|_{\mathbf{L}_p[0,\varepsilon]}^p + \int_{-1}^1 \left|2e^{-b/2}\arctan\left(xe^{b/2}\right)\right|^p dx \\ &\quad + \int_{-1}^1 \left|x\ln(x^2 + e^{-b})\right|^p dx \leq A^p\omega^3(f', 1)_p^p + 2\pi^p + \|g\|_p^p \\ &\leq A^pM_2^p + 2\pi^p + M_1^p =: M_3^p.\end{aligned}$$

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It follows from Markov's inequality and the equivalence of norms in $\mathbf{P}_n$ (see Theorem 2.2.7 of [5], for example) that

$$a_1 - 2 + \ln b = \bar{P}_n'(0) \le n^2 \varepsilon^{-1} \|\bar{P}_n\|_{\mathbf{C}[0,\varepsilon]} \le C n^2 \varepsilon^{-1-1/p} M_3 =: M_4,$$

or

$$a_1 \le M_4 + 2 - \ln b.$$

Since M_4 is independent of b, (though it depends on n, ε, p and A), we can choose $b > \exp(M_4 + 2)$, which gives $P_n'(0) = a_1 < 0$, as desired. ∎

PROOF OF THEOREM 12. Let $n \in \mathbb{N}$, $0 < p < \infty$, $0 < \varepsilon \le 1$ and $A > 0$ be fixed, and define

$$g(x) := \ln((x - b^{-1})^2 + e^{-b}), \quad f(x) := bx + \ln(b^{-2} + e^{-b}) - g(x),$$

where $b \ge 3$ is a parameter to be chosen later. Obviously, $f \in \mathbf{C}^\infty$ and $f(0) = 0$. Basic calculus shows that f increases on $[-1, x_1]$, where $x_1 = \frac{2-\sqrt{1-b^2e^{-b}}}{b} > 0$, and assumes a maximum at x_1; then it decreases and assumes a minimum

$$2 + \sqrt{1 - b^2 e^{-b}} + \ln(1 + b^2 e^{-b}) - \ln(2 + 2\sqrt{1 - b^2 e^{-b}}) > 2 - 2\ln 2 > 0$$

at $x_2 = \frac{2+\sqrt{1-b^2e^{-b}}}{b} \in (x_1, 1)$. On $[x_2, 1]$ it becomes monotone increasing again. Therefore, $f(x) < 0$ for $x \in [-1, 0)$ and $f(x) > 0$ for $x \in (0, 1]$.

$$\begin{aligned} \|g\|_p^p &= \int_{-1}^{1} |\ln((x - b^{-1})^2 + e^{-b})|^p\,dx = \int_{-1-b^{-1}}^{1-b^{-1}} |\ln(x^2 + e^{-b})|^p\,dx \\ &\le \int_{-2}^{1} |\ln(x^2 + e^{-b})|^p\,dx \le \int_{-2}^{1} (\ln 5)^p\,dx + \int_{-2}^{1} \big|2\ln|x|\,\big|^p\,dx =: M_1^p. \end{aligned} \tag{4.5}$$

Hence

$$\omega^2(f,1)_p = \omega^2(g,1)_p \le 4^{\max\{1,1/p\}} \|g\|_p \le 4^{\max\{1,1/p\}} M_1 =: M_2. \tag{4.6}$$

We now prove the theorem by showing that there exists $b \ge 3$ for the given n, ε, p and A such that if any $P_n(x) = a_0 + a_1 x + \cdots + a_n x^n$ satisfies

$$\|f - P_n\|_{\mathbf{L}_p[0,\varepsilon]} \le A\omega^2(f,1)_p, \tag{4.7}$$

then $P_n(0) < 0$. Let $P_n(x) - \ln(b^{-2} + e^{-b}) - bx$ be denoted by $\bar{P}_n$. From (4.5)–(4.7) we have

$$\|\bar{P}_n\|_{\mathbf{L}_p[0,\varepsilon]}^p \le \|f - P_n\|_{\mathbf{L}_p[0,\varepsilon]}^p + \|g\|_{\mathbf{L}_p[0,\varepsilon]}^p \le A^p M_2^p + M_1^p =: M_3^p. \tag{4.8}$$

It follows from the equivalence of norms in $\mathbf{P}_n$ (see Theorem 2.2.7 of [5], for example) that

$$a_0 - \ln(b^{-2} + e^{-b}) = \bar{P}_n(0) \le \|\bar{P}_n\|_{\mathbf{C}[0,\varepsilon]} \le C\varepsilon^{-1/p} M_3 =: M_4,$$

or

$$a_0 \le M_4 + \ln(b^{-2} + e^{-b}) \le M_4 + \ln(2b^{-2}) = M_4 + \ln 2 - 2\ln b.$$

Since M_4 is independent of b (though it depends on n, ε, p and A), we can choose $b > \exp((M_4 + \ln 2)/2)$, which gives $P_n(0) = a_0 < 0$, as desired. ∎

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PROOF OF THEOREM 13. Let $n \in N$, $0 < p \le \infty$, $0 < \varepsilon \le 1$ and $A > 0$ be fixed, and define

$$f(x) := \arctan(bx),$$

where $b \ge 1$ is a parameter to be chosen later. Note that $\|f\|_p \le 2^{1/p-1}\pi$.

The inequality (1.31) can be proved in much the same way as the previous proofs in the section: suppose there exist $P_n \in \mathbf{P}_n$ with $P_n(0) = 0$ and $P_n(x) \ge f(x)$, $x \in [0, \varepsilon]$, and $A \in R$ such that $\|f - P_n\|_{\mathbf{L}_p[0,\varepsilon]} \le A\|f\|_{\mathbf{L}_p[-1,1]}$, then

$$\begin{aligned} P_n'(0) &\le n^2\varepsilon^{-1}\|P_n\|_{\mathbf{C}[0,\varepsilon]} \le Cn^2\varepsilon^{-1-1/p}\|P_n\|_{\mathbf{L}_p[0,\varepsilon]} \\ &\le Cn^2\varepsilon^{-1-1/p}(A+1)\|f\|_{\mathbf{L}_p[-1,1]} \le C'n^2\varepsilon^{-1-1/p}(A+1). \end{aligned}$$

Choosing b greater than the right hand side, which is independent of b, gives the contradiction $b = f'(0) \le P_n'(0) < b$ (since $P(0) = f(0)$ and $P_n(x) \ge f(x)$, $x \in [0, \varepsilon]$ imply $P_n'(0) \ge f'(0)$).

We now suppose $0 < p < \infty$ and prove (1.32). Since

$$0 < f'(x) = \frac{b}{1 + b^2x^2} \le \begin{cases} 1, & |x| \ge \sqrt{b-1}/b, \\ b, & \text{otherwise} \end{cases}$$

we have

$$\int_{-1}^{1} |f'(x)|^p\,dx = \left\{\int_{|x|\le\sqrt{b-1}/b} + \int_{\sqrt{b-1}/b\le|x|\le1}\right\}|f'(x)|^p\,dx \le C(b^{p-1/2} + 1).$$

If some polynomial $P_n \in \mathbf{P}_n$ satisfies $\|f - P_n\|_{\mathbf{L}_p[0,\varepsilon]} \le A\|f'\|_{\mathbf{L}_p[-1,1]}$, then

$$\begin{aligned} P_n'(0) &\le n^2\varepsilon^{-1}\|P_n\|_{\mathbf{C}[0,\varepsilon]} \le Cn^2\varepsilon^{-1-1/p}\|P_n\|_{\mathbf{L}_p[0,\varepsilon]} \\ &\le Cn^2\varepsilon^{-1-1/p}(\|f - P_n\|_{\mathbf{L}_p[-1,1]} + \|f\|_{\mathbf{L}_p[-1,1]}) \\ &\le Cn^2\varepsilon^{-1-1/p}A(b^{1-1/(2p)} + 1) := M(b^{1-1/(2p)} + 1), \end{aligned}$$

therefore choosing $b > (2M)^{\max\{1,2p\}}$ gives $P_n'(0) < b$, which contradicts the facts that $P_n(0) = 0$ and $P_n(x) \ge f(x)$ for $x \in [0, \varepsilon]$. ∎

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https://doi.org/10.4153/CJM-1997-004-2 Published online by Cambridge University Press