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# Riemann Extensions of Torsion-Free Connections with Degenerate Ricci Tensor

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*Abstract.* Correspondence between torsion-free connections with nilpotent skew-symmetric curvature operator and IP Riemann extensions is shown. Some consequences are derived in the study of four-dimensional IP metrics and locally homogeneous affine surfaces.

## 1 Introduction

Manifolds whose Riemann curvature has a high degree of symmetry are important in many contexts. Usually this symmetry arises from an underlying symmetry of the metric tensor. Locally homogeneous and locally symmetric spaces are typical examples. Also, symmetries arise from properties of natural operators associated with the curvature tensor. In considering the spectral geometry of the Riemann curvature tensor, one studies when a certain natural operator associated with the curvature has constant Jordan normal form on the natural domain of definition. A pseudo-Riemannian manifold $(\mathcal{M}, g)$ is said to be Osserman if the spectrum of the Jacobi operators is constant on the unit pseudo-sphere bundles. Any isotropic pseudo-Riemannian manifold is Osserman, but the converse does not hold. (See [10, 11] for more detailed information.)

Let $R$ be the curvature tensor of a pseudo-Riemannian manifold $(\mathcal{M}, g)$. Let $\pi$ be an oriented non-degenerate 2-plane in the tangent space over a point $p \in \mathcal{M}$ with an oriented orthonormal basis $\{X, Y\}$. Then the skew-symmetric curvature operator is defined as the operator $\mathcal{R}(\pi)\colon Z \mapsto R(X, Y)Z$. $(\mathcal{M}, g)$ is called IP (Ivanov–Petrova) if the eigenvalues of $\mathcal{R}(\pi)$ depend only on the basepoint $p \in \mathcal{M}$, but not on the choice of $\pi \subset T_p\mathcal{M}$ [15]. Metrics of constant curvature are IP, but there are IP metrics that do not have constant sectional curvature [11, 12, 14]. Three-dimensional IP metrics have been investigated in the Riemannian and Lorentzian setting (cf. [8, 15]) where a complete algebraic description is available. In the affine setting, one says that $(M, D)$ is affine IP if $R(X, Y)$ is nilpotent for any linearly independent $\{X, Y\}$.

Our aim in this work is twofold. First, we investigate IP metrics in signature $(2, 2)$, where only partial results are known [3, 7, 13], with special attention to Walker metrics. Our second purpose is to study locally symmetric and locally homogeneous affine surfaces with symmetric and degenerate Ricci tensor. Locally homogeneous affine surfaces were described by Opozda [20] (see also [17, 18]). It is shown in [20]

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that any such connection is either the Levi-Civita connection of a metric of constant curvature or corresponds to one of two families A and B where all the Christoffel symbols are explicitly given by (5.1) and (5.2), respectively. We consider the equivalence problem for such connections, showing that besides the flat ones, there is a explicit family of locally homogeneous torsion-free connections that is of both types A and B, thus answering to a question posed by Kowalski. Such a family is contained in the class of projectively flat and recurrent homogeneous connections with symmetric and degenerate Ricci tensor (cf. Theorem 5.12).

The paper is organized as follows. After recalling some preliminaries in Section 2, Riemann extensions of torsion-free connections are studied in Section 3, where a correspondence between the pseudo-Riemannian and the affine IP conditions is shown (cf. Theorem 3.1). Riemann extensions have a special significance in dimension four, where all such metrics are self-dual. As a partial converse, it is shown in Section 4 that any four-dimensional self-dual IP Walker metric is necessarily a Riemann extension of a torsion-free affine surface (cf. Theorem 4.2). Affine IP surfaces are characterized by having symmetric and degenerate Ricci tensor. This is a large class of affine surfaces, and our purpose in Section 5 is to develop a systematic study of the locally symmetric and locally homogeneous ones.

## 2 Preliminaries

Throughout this paper we adopt the following general notational conventions. Let $(\mathcal{M}, g)$ be a pseudo-Riemannian manifold and $(M, D)$ and affine manifold, *i.e.*, $M$ is a differentiable manifold equipped with an affine connection $D$ that is assumed to be torsion-free. $M \equiv \Sigma$ is used in the particular case of an affine surface. We choose the following convention for the curvature tensor, $R(X,Y) = \Xi_{[X,Y]} - [\Xi_X, \Xi_Y]$, where $\Xi$ denotes the Levi-Civita connection associated with $g$ or with the torsion-free affine connection $D$ and, as usual, $\rho(X,Y) = \operatorname{Tr}\{U \rightsquigarrow R(X,U)Y\}$ is the Ricci tensor.

For any oriented, non-degenerate 2-plane $\pi$ on $(\mathcal{M}, g)$, the *skew-symmetric curvature operator*, defined by

$$\mathcal{R}(\pi) = \left|g(X,X)g(Y,Y) - g(X,Y)^2\right|^{-1/2} R(X,Y),$$

is a skew-adjoint operator that is independent of the oriented basis $\{X,Y\}$ of $\pi$. $(\mathcal{M}, g)$ is said to be spacelike (respectively, timelike or mixed) *Ivanov–Petrova* (IP for short) if the eigenvalues of $\mathcal{R}(\pi)$ are constant on the Grassmannian of all oriented non-degenerate spacelike (respectively, timelike or mixed) 2-planes. In signature $(p,q)$, the following are equivalent conditions [11]:

(i) *spacelike IP* if $p \geq 2$, (so there are spacelike 2-planes),
(ii) *mixed IP* if $p \geq 1$, $q \geq 1$, (so there are mixed 2-planes),
(iii) *timelike IP* if $q \geq 2$, (so there are timelike 2-planes).

As a consequence, one simply says that $(\mathcal{M}, g)$ is IP. Note that although the spacelike, timelike, and mixed IP conditions are equivalent, the eigenvalue structure may change among the three groups of spacelike, timelike, and mixed 2-planes. (We also refer to [3, 11, 15, 26] for more details and further references.)

In this paper we generalize this notion to the affine setting. Let $(M, D)$ be an affine manifold where $D$ is a torsion-free connection on $TM$. Let $\mathcal{R}(\pi) = R(X, Y)$ be the *skew-symmetric D-curvature operator* associated with $D$. In the pseudo-Riemannian setting, a rescaling of $R(X, Y)$ is carried out by the metric tensor as in the above. However, such normalization is not available in the affine setting, and thus one says that $(M, D)$ is *affine Ivanov-Petrova* (affine IP) if $\mathcal{R}(\pi)$ is nilpotent.

## 3 The Riemann Extension

Let $T^*M$ denote the cotangent bundle of $M$ ($\dim M = n$) and let $\pi\colon T^*M \to M$ be the projection. A point $\xi$ of the cotangent bundle is represented by an ordered pair $(p, \omega)$, where $p = \pi(\xi)$ is a point on $M$ and $\omega$ is a 1-form on $T_pM$. For each coordinate neighborhood $(U, (x_i))$ on $M$ with $p \in U$, denote by $x_{i'}$ the components of $\omega$ in the natural coframe $dx_i$. Then, for any local coordinates $(U, (x_i))$ on $M$, $(x_i, x_{i'})$ are natural induced coordinates in $\pi^{-1}(U) \subset T^*M$. (See [25, Ch. 7] for more details and further references on the geometry of cotangent bundles.)

Next, for a given symmetric connection $D$ on $M$, the cotangent bundle $T^*M$ may be equipped with a pseudo-Riemannian metric $g_D$ of signature $(n, n)$: the *Riemann extension* of $D$ [21], given by

$$g_D(X^C, Y^C) = -\gamma(D_X Y + D_Y X),$$

where $X^C, Y^C$ denote the complete lifts to $T^*M$ of vector fields $X, Y$ on $M$. Moreover for any vector field $Z$ on $M$, $Z = Z^i \partial_i$, $\gamma Z$ is the function on $T^*M$ defined by $\gamma Z = x_{i'} Z^i$. In a system of induced coordinates $(x_i, x_{i'})$ on $T^*M$, the *Riemann extension* is expressed by

$$g_D = \begin{pmatrix} -2x_{k'}\Gamma_{ij}^k & \delta_i^j \\ \delta_i^j & 0 \end{pmatrix}$$

with respect to $\{\partial_1, \dots, \partial_n, \partial_{1'}, \dots, \partial_{n'}\}$ ($i, j, k = 1, \dots, n$, $k' = k + n$), where $\Gamma_{ij}^k$ are the Christoffel symbols of the connection $D$ with respect to the coordinates $(x_i)$ on $M$.

Riemann extensions provide a link between affine and pseudo-Riemannian geometries. Some properties of the affine connection $D$ can be investigated by means of the corresponding properties of the Riemann extension $g_D$. For instance, $D$ is projectively flat if and only if $g_D$ is locally conformally flat [1] (see [6, 9, 19, 23] for more examples and further references).

From now on we will use a deformation of the Riemann extension above by means of a symmetric $(0, 2)$-tensor field $\phi$ on $M$; more precisely, we will consider $T^*M$ equipped with the metric $g_D + \pi^*\phi$, which we go on calling Riemann extension since it does not cause any confusion.

***Theorem 3.1*** *Let $(T^*M, g_D + \pi^*\phi)$ be the cotangent bundle of an affine manifold $(M, D)$ equipped with the Riemann extension. Then $(T^*M, g_D + \pi^*\phi)$ is a pseudo-Riemannian* IP *space if and only if $(M, D)$ is affine* IP *for any symmetric $(0, 2)$-tensor field $\phi$.*

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**Proof** Let $\tilde{g} = g_D + \pi^*\phi$ be a Riemann extension on $T^*M$. A straightforward calculation shows that the non-zero Christoffel symbols $\tilde{\Gamma}_{\alpha\beta}^{\gamma}$ of the Levi-Civita connection are given as follows

$$\tilde{\Gamma}_{ij}^{k} = \Gamma_{ij}^{k}, \qquad \tilde{\Gamma}_{i'j}^{k'} = -\Gamma_{jk}^{i}, \qquad \tilde{\Gamma}_{ij'}^{k'} = -\Gamma_{ik}^{j},$$

$$\tilde{\Gamma}_{ij}^{k'} = \sum_{r=1}^{n} x_{r'}\Big(\partial_k\Gamma_{ij}^{r} - \partial_i\Gamma_{jk}^{r} - \partial_j\Gamma_{ik}^{r} + 2\sum_{l=1}^{n}\Gamma_{kl}^{r}\Gamma_{ij}^{l}\Big)$$
$$+ \tfrac{1}{2}(\partial_i\phi_{jk} + \partial_j\phi_{ik} - \partial_k\phi_{ij}) - \sum_{l=1}^{n}\phi_{kl}\Gamma_{ij}^{l},$$

where $\Gamma_{ij}^{k}$ are the Christoffel symbols of $D$ and $\phi_{ij}$ denote the local components of $\phi$. Now, a long but straightforward calculation shows that the non-zero components of the curvature tensor of $(T^*M, g_D+\pi^*\phi)$ are determined (up to the usual symmetries) by the following

$$\tilde{R}_{kji}^{h} = R_{kji}^{h}, \qquad \tilde{R}_{kji}^{h'}, \qquad \tilde{R}_{kji'}^{h'} = -R_{kjh}^{i}, \qquad \tilde{R}_{k'ji}^{h'} = R_{hij}^{k}, \tag{3.1}$$

$R_{kji}^{h}$ being the components of the curvature tensor of $(M, D)$. Note that we omit the expression of $\tilde{R}_{kji}^{h'}$, since it is not necessary for our purposes in showing (3.2).

Next, let $\tilde{\pi} = \langle\{\tilde{X}, \tilde{Y}\}\rangle$ be an oriented, non-degenerate 2-plane on $T^*M$, with $\tilde{X} = \alpha_i\partial_i + \alpha_{i'}\partial_{i'}$ and $\tilde{Y} = \beta_i\partial_i + \beta_{i'}\partial_{i'}$ an orthonormal basis of $\tilde{\pi}$. Then it follows from (3.1) that the matrix of the skew-symmetric curvature operator $\tilde{\mathcal{R}}(\tilde{\pi})$ with respect to the basis $\{\partial_i, \partial_{i'}\}$ is of the form

$$\tilde{\mathcal{R}}(\tilde{\pi}) = \begin{pmatrix} \mathcal{R}(\pi) & 0 \\ * & -{}^t\mathcal{R}(\pi) \end{pmatrix}, \tag{3.2}$$

where $\mathcal{R}(\pi)$ is the matrix of the skew-symmetric $D$-curvature operator corresponding to $\pi = \langle\{X, Y\}\rangle$, with $X = \alpha_i\partial_i$ and $Y = \beta_i\partial_i$ on $M$, with respect to the basis $\{\partial_i\}$. Note that the characteristic polynomials $p_\lambda(\tilde{\mathcal{R}}(\tilde{\pi}))$ of $\tilde{\mathcal{R}}(\tilde{\pi})$ and $p_\lambda(\mathcal{R}(\pi))$ of $\mathcal{R}(\pi)$ are related by $p_\lambda(\tilde{\mathcal{R}}(\tilde{\pi})) = p_\lambda(\mathcal{R}(\pi)) \cdot p_\lambda(-\mathcal{R}(\pi))$.

Now, assume that $(T^*M, g_D + \pi^*\phi)$ is an IP space. If $\pi$ is a 2-plane on $M$, we can consider an oriented, non-degenerate 2-plane $\tilde{\pi}$ on $T^*M$, of a fixed signature, so that (3.2) holds for a suitable orthonormal basis. Since $p_\lambda(\tilde{\mathcal{R}}(\tilde{\pi}))$ must be constant for all planes $\tilde{\pi}$ of that fixed signature, (3.2) implies that $p_\lambda(\mathcal{R}(\pi))$ is independent of the 2-plane $\pi$ chosen. Then, if $\pi = \langle\{X, Y\}\rangle$ and $p_\lambda(\mathcal{R}(\pi)) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0$, for $\pi_\alpha = \langle\{\alpha X, \alpha Y\}\rangle$, $\alpha \neq 0$, we get

$$p_\lambda(\mathcal{R}(\pi_\alpha)) = \lambda^n + \alpha^2 a_{n-1}\lambda^{n-1} + \cdots + \alpha^{2n}a_0.$$

Hence, since $p_\lambda(\mathcal{R}(\pi_\alpha)) = p_\lambda(\mathcal{R}(\pi))$, it follows that $a_{n-1} = \cdots = a_0 = 0$, and therefore the skew-symmetric $D$-curvature operator of $M$ is necessarily nilpotent, thus showing that $(M, D)$ is affine IP.

Conversely, if the affine manifold $(M, D)$ is assumed to have nilpotent skew-symmetric $D$-curvature operator, then $\mathcal{R}(\pi)$ has zero eigenvalues for each $\pi$ on $M$. Therefore, it follows from (3.2) that the eigenvalues of $\tilde{\mathcal{R}}(\tilde{\pi})$ vanish for every oriented, non-degenerate 2-plane $\tilde{\pi}$ on $T^*M$. Thus $(T^*M, g_D + \pi^*\phi)$ is IP. ∎

## 4 Four-dimensional Riemann Extensions

First of all, it is worth noting that the Riemann extension $g_D + \pi^*\phi$ is necessarily a Walker metric. We recall that a *Walker manifold* is a triple $(\mathcal{M}, g, \mathcal{D})$, where $\mathcal{M}$ is an $n$-dimensional manifold, $g$ an indefinite metric, and $\mathcal{D}$ an $r$-dimensional parallel null distribution. Of special interest are those manifolds admitting a field of null planes of maximum dimension $r = \frac{n}{2}$. In this particular case, it is convenient to use special coordinate systems associated with any Walker metric. By a result of Walker [22], and using the notation of the cotangent bundle, there exist local coordinates $(x_1, \dots, x_{\frac{n}{2}}, x_{1'}, \dots, x_{(\frac{n}{2})'})$ around any point of $\mathcal{M}$ such that the matrix of $g$ in these coordinates has the following form

$$g_{(x_1,\dots,x_{\frac{n}{2}},x_{1'},\dots,x_{(\frac{n}{2})'})} = \begin{pmatrix} B & Id_{\frac{n}{2}} \\ Id_{\frac{n}{2}} & 0 \end{pmatrix},$$

where $B$ is a symmetric $\left(\frac{n}{2} \times \frac{n}{2}\right)$-matrix and $Id_{\frac{n}{2}}$ denotes the identity matrix.

From now on in this section we consider the 4-dimensional case. Hence, we choose suitable coordinates $(x_1, x_2, x_{1'}, x_{2'})$ where the metric expresses as

$$g_{(x_1,x_2,x_{1'},x_{2'})} = \begin{pmatrix} a & c & 1 & 0 \\ c & b & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{4.1}$$

for some functions $a$, $b$, and $c$ depending on the coordinates $(x_1, x_2, x_{1'}, x_{2'})$. Again unifying with the notation used in the cotangent bundle, we denote by $\{\partial_i, \partial_{i'}\}$ the coordinate vectors, $i = 1, 2$. Also, $h_{i\dots j'\dots}$ means partial derivatives $\frac{\partial h}{\partial x_i \dots \partial x_{j'} \dots}$ for any function $h(x_1, x_2, x_{1'}, x_{2'})$.

***Remark 4.1*** Self-dual Walker metrics have been previously investigated in [5], showing that a metric (4.1) is self-dual if and only if the functions $a$, $b$, $c$ have the form

$$\begin{aligned} a(x_1, x_2, x_{1'}, x_{2'}) &= x_{1'}^3 \mathcal{A} + x_{1'}^2 \mathcal{B} + x_{1'}^2 x_{2'} \mathcal{C} + x_{1'} x_{2'} \mathcal{D} + x_{1'} P + x_{2'} Q + \xi, \\ b(x_1, x_2, x_{1'}, x_{2'}) &= x_{2'}^3 \mathcal{C} + x_{2'}^2 \mathcal{E} + x_{1'} x_{2'}^2 \mathcal{A} + x_{1'} x_{2'} \mathcal{F} + x_{1'} S + x_{2'} T + \eta, \\ c(x_1, x_2, x_{1'}, x_{2'}) &= \tfrac{1}{2} x_{1'}^2 \mathcal{F} + \tfrac{1}{2} x_{2'}^2 \mathcal{D} + x_{1'}^2 x_{2'} \mathcal{A} + x_{1'} x_{2'}^2 \mathcal{C} + \tfrac{1}{2} x_{1'} x_{2'} (\mathcal{B} + \mathcal{E}) \\ &\quad + x_{1'} U + x_{2'} V + \gamma, \end{aligned} \tag{4.2}$$

where all capital, calligraphic, and Greek letters stand for arbitrary smooth functions depending only on the coordinates $(x_1, x_2)$.

As a direct consequence of the previous remark, we see that *any Riemann extension is necessarily a self-dual Walker manifold*. Next we investigate some particular cases where the converse also holds.

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***Theorem 4.2*** *A four-dimensional* IP *self-dual Walker metric is necessarily a Riemann extension.*

**Proof** We start with a technical remark about the IP condition. For a general 4-dimensional pseudo-Riemannian manifold $(\mathcal{M}, g)$ with metric of neutral signature, it is easy to check that the characteristic polynomial $p_\lambda(\mathcal{R}(\pi))$ of $\mathcal{R}(\pi)$ is given by

$$p_\lambda(\mathcal{R}(\pi)) = \lambda^4 - \tfrac{1}{2}\operatorname{Tr}(\mathcal{R}(\pi)^2)\ \lambda^2 + \det(\mathcal{R}(\pi)),$$

and therefore the manifold is IP if and only if $\det(\mathcal{R}(\pi))$ and $\operatorname{Tr}(\mathcal{R}(\pi)^2)$ do not depend on the oriented, non-degenerate, spacelike (respectively, mixed or timelike) 2-plane $\pi$ (see [3]). In the particular case of a Walker metric (4.1), a straightforward calculation shows that the skew-symmetric curvature operator $\mathcal{R}(\pi)$ associated with any non-degenerate 2-plane $\pi$, when expressed with respect to the coordinate vectors $\{\partial_i, \partial_{i'}\}$, $i = 1, 2$, has the matrix form

$$\mathcal{R}(\pi) = \begin{pmatrix} F(\pi) & 0 \\ G(\pi) & -{}^tF(\pi) \end{pmatrix}$$

for certain $(2 \times 2)$-matrices $F(\pi)$ and $G(\pi)$. Hence, the determinant of $\mathcal{R}(\pi)$ and the trace of $\mathcal{R}(\pi)^2$ are determined by those of $F(\pi)$ and $F(\pi)^2$, respectively. Indeed, $\det(\mathcal{R}(\pi)) = (\det(F(\pi)))^2$, while $\operatorname{Tr}(\mathcal{R}(\pi)^2) = 2\operatorname{Tr}(F(\pi)^2)$. Therefore, the above characterization of IP $(2,2)$-metrics means that a Walker 4-metric (4.1) is IP if and only if $\det(F(\pi))$ and $\operatorname{Tr}(F(\pi)^2)$ do not depend on the oriented, non-degenerate spacelike (respectively, mixed or timelike) 2-plane $\pi$. This characterization will be repeatedly used in the rest of the proof; as a matter of notation, let

$$F(\pi) = \begin{pmatrix} f_{11}(\pi) & f_{12}(\pi) \\ f_{21}(\pi) & f_{22}(\pi) \end{pmatrix}.$$

After this technical observation, and in view of Remark 4.1, assume that $g$ is given by (4.1)–(4.2) and that is IP. We start our analysis considering the non-degenerate 2-plane $\pi_1 = \langle\{\partial_1, \partial_{1'} + \lambda\partial_{2'}\}\rangle$. We obtain:

$$\begin{aligned}
f_{11}(\pi_1) &= -x_{1'}(\lambda\mathcal{C} + 3\mathcal{A}) - x_{2'}\mathcal{C} - \tfrac{1}{2}(\lambda\mathcal{D} + 2\mathcal{B}),\\
f_{12}(\pi_1) &= -x_{1'}\lambda\mathcal{A} - x_{2'}(\lambda\mathcal{C} + \mathcal{A}) - \tfrac{1}{4}(\lambda(\mathcal{B} + \mathcal{E}) + 2\mathcal{F}),\\
f_{21}(\pi_1) &= -x_{1'}\mathcal{C} - \tfrac{1}{2}\mathcal{D},\\
f_{22}(\pi_1) &= -x_{1'}(\lambda\mathcal{C} + \mathcal{A}) - x_{2'}\mathcal{C} - \tfrac{1}{4}(2\lambda\mathcal{D} + \mathcal{B} + \mathcal{E}).
\end{aligned}$$

As a consequence, it follows that $\partial_{1'}\partial_{1'}(\det(F(\pi_1)) = 2\lambda^2\mathcal{C}^2 + 6\lambda\mathcal{A}\mathcal{C} + 6\mathcal{A}^2$, so $\mathcal{A} = \mathcal{C} = 0$ and one gets $\det(F(\pi_1)) = \frac{1}{4}\lambda^2\mathcal{D}^2 + \frac{1}{2}\lambda\mathcal{B}\mathcal{D} + \frac{1}{4}(\mathcal{B}^2 + \mathcal{B}\mathcal{E} - \mathcal{D}\mathcal{F})$, from where $\mathcal{D} = 0$. Hence, after this first analysis, (4.2) reduces to

$$\begin{aligned}
a &= x_{1'}^2\mathcal{B} + x_{1'}P + x_{2'}Q + \xi,\\
b &= x_{2'}^2\mathcal{E} + x_{1'}x_{2'}\mathcal{F} + x_{1'}S + x_{2'}T + \eta,\\
c &= \tfrac{1}{2}x_{1'}^2\mathcal{F} + \tfrac{1}{2}x_{1'}x_{2'}(\mathcal{B} + \mathcal{E}) + x_{1'}U + x_{2'}V + \gamma.
\end{aligned} \tag{4.3}$$

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Now, we consider the non-degenerate 2-plane $\pi_2 = \langle\{\partial_1 + \lambda\partial_2, \partial_{1'}\}\rangle$. In this case,

$$f_{11}(\pi_2) = -\tfrac{1}{2}(\lambda\mathcal{F} + 2\mathcal{B}), \qquad f_{12}(\pi_2) = -\tfrac{1}{2}\mathcal{F},$$
$$f_{21}(\pi_2) = -\tfrac{1}{4}\lambda(\mathcal{B} + \mathcal{E}), \qquad f_{22}(\pi_2) = -\tfrac{1}{4}(2\lambda\mathcal{F} + \mathcal{B} + \mathcal{E}),$$

from where $\det(F(\pi_2)) = \frac{1}{4}\lambda^2\mathcal{F}^2 + \frac{1}{2}\lambda\mathcal{B}\mathcal{F} + \frac{1}{4}\mathcal{B}(\mathcal{B} + \mathcal{E})$. Hence, $\mathcal{F} = 0$ and (4.3) reduces to

$$\begin{aligned} a &= x_{1'}^2\mathcal{B} + x_{1'}P + x_{2'}Q + \xi, \\ b &= x_{2'}^2\mathcal{E} + x_{1'}S + x_{2'}T + \eta, \\ c &= \tfrac{1}{2}x_{1'}x_{2'}(\mathcal{B} + \mathcal{E}) + x_{1'}U + x_{2'}V + \gamma \end{aligned} \tag{4.4}$$

with

$$\det(F(\pi_2)) = \tfrac{1}{4}\mathcal{B}(\mathcal{B} + \mathcal{E}), \qquad \operatorname{Tr}(F(\pi_2)^2) = \mathcal{B}^2 + \tfrac{1}{16}(\mathcal{B} + \mathcal{E})^2. \tag{4.5}$$

Next we work with the non-degenerate 2-plane $\pi_3 = \langle\{\partial_{1'} - \partial_{2'}, \partial_{2'} - \partial_1 + \partial_2\}\rangle$, for which

$$f_{11}(\pi_3) = -\tfrac{1}{8}(5\mathcal{B} + \mathcal{E}), \quad f_{12}(\pi_3) = f_{21}(\pi_3) = \tfrac{1}{8}(\mathcal{B} + \mathcal{E}), \quad f_{22}(\pi_3) = -\tfrac{1}{8}(\mathcal{B} + 5\mathcal{E})$$

and therefore

$$\det(F(\pi_3)) = \tfrac{1}{16}(\mathcal{B}^2 + \mathcal{E}^2 + 6\mathcal{B}\mathcal{E}), \quad \operatorname{Tr}(F(\pi_3)^2) = \tfrac{1}{16}(7\mathcal{B}^2 + 7\mathcal{E}^2 + 6\mathcal{B}\mathcal{E}). \tag{4.6}$$

Comparing (4.5) and (4.6), we conclude that $\mathcal{E} = \mathcal{B}$. Hence, $\det(F(\pi_3)) = \frac{1}{2}\mathcal{B}^2$, which implies $\mathcal{B} = \kappa$, and thus (4.4) transforms into

$$\begin{aligned} a &= x_{1'}^2\kappa + x_{1'}P + x_{2'}Q + \xi, \\ b &= x_{2'}^2\kappa + x_{1'}S + x_{2'}T + \eta, \\ c &= x_{1'}x_{2'}\kappa + x_{1'}U + x_{2'}V + \gamma. \end{aligned} \tag{4.7}$$

In the last step, we will show that $\kappa = 0$. To do this, take the non-degenerate 2-plane $\pi_4 = \langle\{\partial_2 - c\partial_{1'} - \frac{1+b}{2}\partial_{2'}, \partial_1 + \frac{1-a}{2}\partial_{1'}\}\rangle$. A long but straightforward calculation shows that

$$f_{11}(\pi_4) = -\tfrac{1}{4}(x_{1'}x_{2'}3\kappa^2 + (x_{1'}U + x_{2'}V)3\kappa + QS - UV + 4\kappa\gamma - 2P_2 + 2U_1),$$
$$f_{12}(\pi_4) = -\tfrac{1}{4}(\kappa + \kappa\eta + S(V - P) - TU + U^2 + 2S_1 - 2U_2),$$
$$f_{21}(\pi_4) = -\tfrac{1}{4}(\kappa - \kappa\xi + Q(T - U) + PV - V^2 - 2Q_2 + 2V_1),$$
$$f_{22}(\pi_4) = -\tfrac{1}{4}(x_{1'}x_{2'}3\kappa^2 + (x_{1'}U + x_{2'}V)3\kappa - QS + UV + 2\kappa\gamma + 2T_1 - 2V_2),$$

from where we check that $\partial_{1'}\partial_{1'}\partial_{2'}\partial_{2'}(\det(F(\pi_4))) = \frac{9}{4}\kappa^4$, which shows that $\kappa = 0$. This reduces (4.7) to

$$a = x_{1'}P + x_{2'}Q + \xi, \quad b = x_{1'}S + x_{2'}T + \eta, \quad c = x_{1'}U + x_{2'}V + \gamma,$$

and therefore the metric corresponds with a Riemann extension. ∎

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***Theorem 4.3*** *A four-dimensional Ricci flat self-dual Walker metric is necessarily a Riemann extension.*

**Proof** Assume the Ricci flat metric $g$ is given by (4.1) and (4.2). Then a straightforward calculation shows that

$$\rho_{11'} = \tfrac{1}{4}(16x_{1'}\mathcal{A} + 8x_{2'}\mathcal{C} + 5\mathcal{B} + \mathcal{E}), \qquad \rho_{12'} = 2x_{1'}\mathcal{C} + \mathcal{D},$$
$$\rho_{22'} = \tfrac{1}{4}(8x_{1'}\mathcal{A} + 16x_{2'}\mathcal{C} + \mathcal{B} + 5\mathcal{E}), \qquad \rho_{21'} = 2x_{2'}\mathcal{A} + \mathcal{F},$$

and therefore all the calligraphic letters vanish in (4.2), showing the result. ■

***Remark 4.4*** Ricci flat self-dual Walker metrics have been investigated in [9], showing that they correspond to Riemann extensions of torsion-free connections with skew-symmetric Ricci tensor (see also [4, 17]).

On the other hand, Walker metrics with nilpotent Ricci operator are not necessarily Riemann extensions. For instance, a Walker metric of the type

$$a = 0, \qquad b = x_{1'}x_{2'}\mathcal{A}(x_1, x_2), \qquad c = \tfrac{1}{2}x_{1'}^2\mathcal{A}(x_1, x_2),$$

has two-step nilpotent Ricci operator and it does not correspond to a Riemann extension.

## 5 Affine Surfaces with Nilpotent Skew-symmetric Curvature Operator

The curvature of an affine surface is encoded by its Ricci tensor. Hence, it is natural to investigate affine surfaces whose Ricci tensor shares some kind of pseudo-Riemannian property (*i.e.,* it is symmetric). In such a case (equiaffine geometry), the Ricci tensor defines a pseudo-Riemannian metric whenever it is non-degenerate and thus a special situation occurs for those affine connections whose Ricci tensor is symmetric and degenerate. This is our case of study.

***Theorem 5.1*** *Let $(\Sigma, D)$ be an affine surface. Then $(\Sigma, D)$ is affine* IP *if and only if the Ricci tensor is symmetric and degenerate.*

**Proof** Fixing coordinates $(x_1, x_2)$ and taking a 2-plane $\pi = \langle\{X, Y\}\rangle$ on $\Sigma$, with $X = a_1\partial_1 + a_2\partial_2$, $Y = b_1\partial_1 + b_2\partial_2$, it is easy to check that the skew-symmetric $D$-curvature operator $\mathcal{R}(\pi)$ expresses, with respect to the basis $\{\partial_1, \partial_2\}$, as

$$\mathcal{R}(\pi) = (a_1b_2 - a_2b_1)\begin{pmatrix} -\rho_{21} & -\rho_{22} \\ \rho_{11} & \rho_{12} \end{pmatrix},$$

where $\rho_{ij} = \rho(\partial_i, \partial_j)$. Hence, the characteristic polynomial of $\mathcal{R}(\pi)$ is given by

$$p_\lambda(\mathcal{R}(\pi)) = \lambda^2 + \lambda(a_1b_2 - a_2b_1)(\rho_{21} - \rho_{12}) + (a_1b_2 - a_2b_1)^2 \det\rho,$$

which implies that the skew-symmetric $D$-curvature operator is nilpotent if and only if $\rho_{12} = \rho_{21}$ and $\det\rho = 0$, showing the result. ■

Recall that a tensor field $K$ is said to be *recurrent* if there exists a 1-form $\sigma$ such that $D_X K = \sigma(X)K$ for each vector field $X$. Further, an affine surface $(\Sigma, D)$ is said to be recurrent if its Ricci tensor is recurrent.

***Theorem 5.2*** *Let $(\Sigma, D)$ be an affine* IP *surface. Then $(\Sigma, D)$ is recurrent if and only if around each point there exists a coordinate system $(x_1, x_2)$ in which the non-zero component of $D$ is given by*

$$D_{\partial_1}\partial_1 = a(x_1, x_2)\partial_2,$$

*for some function $a(x_1, x_2)$. Moreover, $(\Sigma, D)$ is locally symmetric if and only if $a(x_1, x_2) = \alpha\, x_2 + \xi(x_1)$, with $\alpha \in \mathbb{R}$ and $\xi$ a smooth function depending on $x_1$ and $(\Sigma, D)$ is flat if and only if $\partial_2 a(x_1, x_2) = 0$.*

**Proof** Decomposing $\rho = \rho_s + \rho_a$ into the symmetric and the anti-symmetric parts, Theorem 5.1 implies that $(\Sigma, D)$ is affine IP if and only if $\rho_a = 0$ and $\det \rho_s = 0$. Now it follows from [24] that the only possibility for a non-flat recurrent affine surface is the one in which around each point there exists a coordinate system $(x_1, x_2)$ with the non-zero component of $D$ given by

$$D_{\partial_1}\partial_1 = a(x_1, x_2)\partial_2,$$

with $\partial_2 a(x_1, x_2) \neq 0$. Now, one easily checks that the only non-vanishing component of the Ricci tensor is $\rho_{11} = \partial_2 a(x_1, x_2)$, and therefore it follows that the local symmetry of $(\Sigma, D)$ is equivalent to $a(x_1, x_2) = \alpha\, x_2 + \xi(x_1)$. ∎

***Remark 5.3*** Observe that all locally symmetric connections at Theorem 5.2 are projectively flat.

### 5.1 Homogeneous Affine Connections

In [18] it is shown that if an affine surface $(\Sigma, D)$ is locally homogeneous, then either $D$ is a Levi-Civita connection of constant curvature, or around each point there exists a coordinate system $(x_1, x_2)$ and constants $a$, $b$, $c$, $d$ $e$, $f$ such that $D$ is expressed by one of the following:

$$(5.1)\quad D_{\partial_1}\partial_1 = a\partial_1 + b\partial_2, \quad D_{\partial_1}\partial_2 = c\partial_1 + d\partial_2, \quad D_{\partial_2}\partial_2 = e\partial_1 + f\partial_2, \quad \text{or}$$

$$(5.2)\quad D_{\partial_1}\partial_1 = \tfrac{a}{x_1}\partial_1 + \tfrac{b}{x_1}\partial_2, \quad D_{\partial_1}\partial_2 = \tfrac{c}{x_1}\partial_1 + \tfrac{d}{x_1}\partial_2, \quad D_{\partial_2}\partial_2 = \tfrac{e}{x_1}\partial_1 + \tfrac{f}{x_1}\partial_2.$$

Following the terminology in [18], from now on we refer to the two cases above as *Type A* and *Type B* locally homogeneous affine connections, respectively. Also, recall that an affine surface $(\Sigma, D)$ is said to be equiaffine if around each point there exists a parallel volume 2-form.

### 5.1.1 Type A Locally Homogeneous Affine Connections

The Ricci tensor of a Type A locally homogeneous affine surface is given by

$$\begin{aligned} \rho_{11} &= -d^2 + ad + (f-c)b, \\ \rho_{12} &= \rho_{21} = cd - eb, \\ \rho_{22} &= -c^2 + fc + (a-d)e, \end{aligned} \tag{5.3}$$

which shows that it is symmetric. An immediate consequence of (5.3) is the following.

***Theorem 5.4*** *Let* $(\Sigma, D)$ *be a Type A locally homogeneous affine surface. Then either the Ricci tensor defines a flat metric on* $\Sigma$*, or* $(\Sigma, D)$ *is affine* IP.

A straightforward calculation from (5.3) shows that

$$\begin{aligned} \tfrac{1}{2}\rho_{11;1} &= -da^2 + (d^2 - bf + cb)a + (be - cd)b, \\ \tfrac{1}{2}\rho_{11;2} &= \tfrac{1}{2}\rho_{12;1} = -acd + (c^2 - fc + de)b, \\ \tfrac{1}{2}\rho_{12;2} &= \tfrac{1}{2}\rho_{22;1} = bce - (ae + cf - de)d, \\ \tfrac{1}{2}\rho_{22;2} &= fc^2 - (de + f^2)c - (af - be - df)e, \end{aligned} \tag{5.4}$$

with $\rho_{ij;k} = (D_{\partial_k}\rho)(\partial_i, \partial_j)$.

***Proposition 5.5*** *Any Type A locally homogeneous affine surface is projectively flat.*

**Proof** Note that any Type A locally homogeneous affine surface $(\Sigma, D)$ is equiaffine and around each point there exists a coordinate system $(x_1, x_2)$ such that

$$\rho_{22;1} = \rho_{12;2} = 2(bce - (ae + cf - de)d),$$

$$\rho_{11;2} = \rho_{12;1} = 2(-acd + (c^2 - fc + de)b).$$

Hence, it follows that $(\Sigma, D)$ is projectively flat. ∎

***Theorem 5.6*** *Let* $(\Sigma, D)$ *be a Type A locally homogeneous affine surface. Then* $(\Sigma, D)$ *is affine* IP *if and only if it is recurrent.*

**Proof** Assume that a Type A locally homogeneous affine surface $(\Sigma, D)$ is affine IP, *i.e.*, the Ricci operator is symmetric and degenerate (see Theorem 5.1). First note that (5.3) implies that the Ricci tensor $\rho$ is always symmetric, and it is degenerate if and only if

$$\begin{aligned} b^2e^2 - \left\{ d^3 - 2ad^2 + (a^2 + 3bc - bf)d + (f-c)ab \right\} e \\ + \left\{ fd^2 + a(c-f)d - b(c-f)^2 \right\} c = 0. \end{aligned} \tag{5.5}$$

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Next we investigate the solutions of the above equation and, combined with (5.4), we show that all the cases lead to a recurrent affine connection. Starting with the case $b = 0$, (5.5) reduces to

$$d \cdot \left\{ ac^2 - (a-d)fc - (a-d)^2 e \right\} = 0, \tag{5.6}$$

and therefore one of the following cases occurs:

(A.1) $d = 0$ and, in such a case, $D\rho = \omega \otimes \rho$, with $\omega = (-2f)dx^2$.
(A.2) $d \neq 0$ and $a = 0$. In this case, $e = d^{-1}cf$ and $D\rho = 0$.
(A.3) $d \neq 0 \neq a$. Hence, (5.6) gives

$$ac^2 - (a-d)fc - (a-d)^2 e = 0$$

and therefore $c = (2a)^{-1}(a-d)(f + \varepsilon(f^2 + 4ae)^{\frac{1}{2}})$, with $\varepsilon \in \{-1, 1\}$ ($c$ being real). Hence, if $d = a$, $D\rho = 0$ is obtained, while for $d \neq a$ a straightforward calculation shows that $D\rho = \omega \otimes \rho$, with $\omega = (-2a)dx^1 - (f + \varepsilon(f^2 + 4ae)^{\frac{1}{2}})dx^2$.

Now, for $b \neq 0$, viewing (5.5) as a quadratic equation for $e$ we have the following last case:

(A.4) Note that, in this case,

$$e = \tfrac{1}{2b^2}\left\{ d^3 - 2ad^2 + (a^2 + 3bc - bf)d + (f-c)ab + \varepsilon(d^2 - ad + (c-f)b)\zeta^{\frac{1}{2}} \right\},$$

with $\zeta = (a-d)^2 + 4bc$ and $\varepsilon \in \{-1, 1\}$ ($e$ being real). Now, a long but straightforward calculation shows that if $c = b^{-1}(-d^2 + ad + bf)$, then $D\rho = 0$ is obtained. Otherwise, $D\rho = \omega \otimes \rho$, with

$$\omega = (-a - d - \varepsilon\zeta^{\frac{1}{2}})dx^1 + b^{-1}(-d^2 + ad - 2bc - \varepsilon d\zeta^{\frac{1}{2}})dx^2.$$

So, in any case, any Type A locally homogeneous affine IP connection is recurrent.

Conversely, a recurrent Type A locally homogeneous affine connection has nilpotent skew-symmetric $D$-curvature operator, since

$$\det \rho = -\tfrac{1}{2}e(\rho_{11;1} - \omega_1\rho_{11}) + \tfrac{1}{2}(c-f)(\rho_{12;1} - \omega_1\rho_{12}) + \tfrac{1}{2}d(\rho_{22;1} - \omega_1\rho_{22}),$$

where $D\rho = \omega \otimes \rho$, with $\omega = \omega_1 dx^1 + \omega_2 dx^2$. ∎

***Remark 5.7*** Type A locally homogeneous affine connections need not be recurrent, as it can be easily checked in (5.1) by considering $b = c = 0$. In such a case, $\rho_{12} = 0$, but $\rho_{12;2} = 2de(d-a)$, which shows that the connection is not recurrent in general.

### 5.1.2 Type B Locally Homogeneous Affine Connections

A direct calculation shows that the Ricci tensor of a Type B locally homogeneous affine surface is given by

$$
\begin{aligned}
\rho_{11} &= \tfrac{1}{x_1^2}\{(a-d+1)d+(f-c)b\},\\
\rho_{12} &= \tfrac{1}{x_1^2}\{cd-be+f\},\\
\rho_{21} &= \tfrac{1}{x_1^2}\{cd-be-c\},\\
\rho_{22} &= \tfrac{1}{x_1^2}\{(a-d-1)e+(f-c)c\}.
\end{aligned}
\tag{5.7}
$$

Note that a Type B locally homogeneous affine connection need not be equiaffine. Indeed, the equiaffine condition is equivalent to $f=-c$.

***Theorem 5.8*** *Let $(\Sigma, D)$ be an equiaffine Type B locally homogeneous affine surface. Then, either the Ricci tensor defines a metric of constant Gauss curvature on $\Sigma$, or otherwise $(\Sigma, D)$ is affine* IP.

**Proof** A straightforward calculation shows that if the Ricci tensor defines a metric on $\Sigma$, then its Gauss curvature satisfies

$$
\begin{aligned}
K &= \tfrac{-1}{x_1^2}\tfrac{\rho_{22}}{\det(\rho)}\\
&= \frac{2c^2+e(d-a+1)}{4bc^3+(d^2-2ad-1)c^2+2(2d-a)bec+e\big(((a-d)^2-1)d-b^2e\big)},
\end{aligned}
$$

from where the result follows. ∎

Next we show that, contrary to the Type A case, Type B locally homogeneous affine surfaces are not projectively flat in general.

***Theorem 5.9*** *Let $(\Sigma, D)$ be a Type B locally homogeneous affine surface. If $(\Sigma, D)$ is projectively flat, then around each point there exists a coordinate system $(x_1, x_2)$ such that $D$ is expressed by one of the following:*

(i) $e=f=c=0$, *or*
(ii) $e\neq 0$, $f=-c$, $a=\frac{3c^2+2de+e}{e}$, $b=\frac{-c^3-ce}{e^2}$.

**Proof** From (5.7) we determine the covariant derivative of the Ricci tensor of an equiaffine Type B locally homogeneous affine connection, which satisfies

$$
\begin{aligned}
\tfrac{x_1^3}{2}\rho_{11;1} &= (a+1)(d-a-1)d+(2a-d+3)bc+b^2e,\\
\tfrac{x_1^3}{2}\rho_{11;2} &= 2bc^2-adc+bde,\\
x_1^3\rho_{12;1} &= (a+4bc-2(a+1)d+2)c+(2d+3)be,\\
\tfrac{x_1^3}{2}\rho_{12;2} &= c^2d+bec+(d-a)de,\\
\tfrac{x_1^3}{2}\rho_{22;1} &= (d+1)(d-a+1)e+(d+3)c^2+bce,\\
\tfrac{x_1^3}{2}\rho_{22;2} &= -2c^3+be^2+(a-2d)ce,
\end{aligned}
\tag{5.8}
$$

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with $\rho_{ij;k} = (D_{\partial_k}\rho)(\partial_i, \partial_j)$.

Now, recall that $(\Sigma, D)$ is projectively flat if and only if $(T^*\Sigma, g_D)$ is locally conformally flat, which is equivalent to the vanishing of its Weyl conformal curvature tensor, $W$. It follows that if a Type B locally homogeneous affine surface is projectively flat, then its Ricci tensor $\rho$ is necessarily symmetric, since $W(\partial_1, \partial_2, \partial_1, \partial_{1'}) = \frac{c+f}{2x_1^2}$, which shows that $f = -c$, which is exactly the necessary and sufficient condition for $\rho$ to be symmetric. Assuming this condition, $f = -c$, (5.8) implies that $(\Sigma, D)$ is projectively flat if and only if

$$\rho_{21;1} - \rho_{11;2} = \frac{c(a-2d+2)+3be}{x_1^3} = 0,$$

$$\rho_{22;1} - \rho_{12;2} = \frac{2(3c^2 - ae + 2de + e)}{x_1^3} = 0.$$

Finally, (i) and (ii) are obtained solving the previous equations. ■

In the last part of this subsection we classify Type B locally homogeneous affine surfaces with nilpotent skew-symmetric $D$-curvature operator. Contrary to the Type A case, the next result shows that those surfaces need not be recurrent in general.

***Theorem 5.10*** *Let $(\Sigma, D)$ be a Type B locally homogeneous affine surface. Then $(\Sigma, D)$ is affine* IP *if and only if either $(\Sigma, D)$ is recurrent, or otherwise around each point there exists a coordinate system $(x_1, x_2)$ in which $D$ is expressed by*

$$D_{\partial_1}\partial_1 = \tfrac{a}{x_1}\partial_1 + \tfrac{b}{x_1}\partial_2, \quad D_{\partial_1}\partial_2 = \tfrac{c}{x_1}\partial_1 + \tfrac{d}{x_1}\partial_2, \quad D_{\partial_2}\partial_2 = \tfrac{e}{x_1}\partial_1 - \tfrac{c}{x_1}\partial_2,$$

*for real constants a, b, c, d, and e satisfying one of the following:*

(i) $b = 0$ *and*

(i.1) $d \neq 0$, $e = 0$, $c \neq 0$, $a = \frac{d^2-1}{2d}$, *or*
(i.2) $d \cdot e \neq 0$, $c = 0$, $a = d \pm 1$, *or*
(i.3) $d \cdot e \neq 0$, $c \neq 0$, $e \neq -\frac{c^2}{d}$, $a = \frac{d(c^2+de)\pm\zeta^{\frac{1}{2}}}{de}$, *with* $\zeta = d(c^2d+e)(c^2+de) \geq 0$, *or*

(ii) $b \neq 0$ *and*

(ii.1) $d = 0$, $c \neq 0$, $e = \frac{-abc\pm\left(b^2c^2(a^2+4bc-1)\right)^{\frac{1}{2}}}{b^2}$, *with* $a^2 + 4bc - 1 \geq 0$, *or*
(ii.2) $d \neq 0$, $a \neq \pm(d-1)$, $c = \frac{(a-d+1)d}{2b}$, $e = \frac{(a-d+1)(d-1)d}{2b^2}$, *or*
(ii.3) $d \neq 0$, $c \notin \{0, \frac{ad}{b}, \frac{(a-d+1)d}{2b}\}$, $e = \frac{((d-a)^2+4bc-1)d-2abc\pm\zeta^{\frac{1}{2}}}{2b^2}$, *with* $\zeta = ((d-a+1)d+2bc)((d-a-1)d+2bc)((d-a)^2+4bc-1) \geq 0$, *or*
(ii.4) $d \neq 0$, $c = 0$, $a \neq d-1$, $e = \frac{d^3-2ad^2+(a^2-1)d\pm|d||(a-d)^2-1|}{2b^2} \neq 0$, *or*
(ii.5) $d \neq 0$, $c = \frac{ad}{b}$, $a \neq 1-d$, $e = \frac{d^3+2ad^2-(a^2+1)d\pm|d||(a+d)^2-1|}{2b^2} \neq -\frac{a^2d}{b^2}$.

**Proof** First, using (5.7), we get that $\rho$ is symmetric if and only if $f = -c$, which we assume from now on, and hence $\rho$ is degenerate if and only if

$$(5.9)\quad b^2e^2 - \left\{d^3 - 2ad^2 + (a^2+4bc-1)d - 2abc\right\}e - \left\{d^2 - 2ad + 4bc - 1\right\}c^2 = 0.$$

Next we analyze the solutions of this equation proceeding as in Theorem 5.6 and using (5.8). First, if $b = 0$, equation (5.9) reduces to

$$(5.10)\qquad dea^2 - 2d(c^2+de)a + (d^2-1)(c^2+de) = 0,$$

so we have the following possible cases:

(B.1) $d = 0$. In such a case, $c = 0$, and it follows that $D\rho = \omega \otimes \rho$, with $\omega = -\frac{2}{x_1}dx^1$.

(B.2) $d \neq 0$, $e = 0$, $c = 0$. In this case, $D\rho = \omega \otimes \rho$, with $\omega = -\frac{2+2a}{x_1}dx^1$.

(B.3) $d \neq 0$, $e = 0$, $c \neq 0$. For this case, necessarily $a = (2d)^{-1}(d^2-1)$, and it follows that the affine connection is never recurrent (case (i.1)). Indeed, writing $D\rho = \omega \otimes \rho$, with $\omega = \omega_1 dx^1 + \omega_2 dx^2$, first we get

$$\rho_{22;2} - \omega_2\rho_{22} = \frac{2c^2(x_1\omega_2 - 2c)}{x_1^3},$$

so $\omega_2 = \frac{2c}{x_1}$ and, under this condition, $\rho_{12;2} - \omega_2\rho_{12} = \frac{2c^2}{x_1^3}$, which does not vanish.

(B.4) $d \neq 0 \neq e$. Note that for $d \neq 0 \neq e$, (5.10) can be viewed as a quadratic equation for $a$; therefore, $a = (de)^{-1}(d(c^2+de) + \varepsilon\zeta^{\frac{1}{2}})$, with $\varepsilon \in \{-1,1\}$ and $\zeta = d(c^2d+e)(c^2+de) \geq 0$. Now, a straightforward calculation shows that if such an affine connection is not recurrent, then case (i.2) or (i.3) holds. Indeed, write $D\rho = \omega \otimes \rho$, with $\omega = \omega_1 dx^1 + \omega_2 dx^2$. Taking $c = 0$, we see that $\zeta = d^2e^2 \geq 0$ always holds, while $a = d \pm 1$, and we compute $\rho_{12;2} - \omega_2\rho_{12} = \frac{-2\varepsilon|d||e|}{x_1^3}$, which is always not null (case (i.2)). Now, for $c \neq 0$, we get

$$(5.11)\quad \begin{aligned} x_1^3(\rho_{12;1} - \omega_1\rho_{12}) &= \frac{c^3d(1-2d) - cde(2d^2+d-2) - \varepsilon c(2d-1)\zeta^{\frac{1}{2}}}{de} \\ &\quad - c(d-1)x_1\omega_1, \\ x_1^3(\rho_{12;2} - \omega_2\rho_{12}) &= -2\varepsilon\zeta^{\frac{1}{2}} - c(d-1)x_1\omega_2. \end{aligned}$$

For $d = 1$, the second expression above reduces to $\rho_{12;2} - \omega_2\rho_{12} = \frac{-2\varepsilon|c^2+e|}{x_1^3}$; hence, if $e \neq -c^2$, the affine connection is not recurrent (case $(i.3)$ with $d = 1$), while for $e = -c^2$, one checks that $D\rho = 0$. Now, for $d \neq 1$, $\omega_1$ and $\omega_2$ are determined by (5.11), and we get

$$\begin{aligned} dx_1^3(\rho_{22;1} - \omega_1\rho_{22}) &= c^2 + de - \varepsilon\zeta^{\frac{1}{2}}, \\ \tfrac{1}{2}c(d-1)ex_1^3(\rho_{11;2} - \omega_2\rho_{11}) &= (c^2+de)(d(c^2+e) + \varepsilon\zeta^{\frac{1}{2}}), \\ \tfrac{1}{2}c(d-1)dx_1^3(\rho_{22;2} - \omega_2\rho_{22}) &= (c^2+de)(d(c^2+e) - \varepsilon\zeta^{\frac{1}{2}}). \end{aligned}$$

Note that the three expressions above do not vanish simultaneously for $e \neq -\frac{c^2}{d}$ and hence, in such a case the affine connection is not recurrent (case $(i.3)$ with $d \neq 1$); for $e = -\frac{c^2}{d}$ we have $\omega = -\frac{2}{x_1}dx^1$ and a straightforward calculation shows that $D\rho = \omega \otimes \rho$ holds.

Finally, for $b \neq 0$, we have the following last case:

(B.5) Viewing (5.9) as a quadratic equation for $e$ we can clear $e$ up. More precisely,

$$e = \frac{1}{2b^2}\left\{ ((d-a)^2 + 4bc - 1)d - 2abc + \varepsilon\zeta^{\frac{1}{2}} \right\},$$

with $\zeta = ((d-a+1)d + 2bc)((d-a-1)d + 2bc)((d-a)^2 + 4bc - 1) \geq 0$ and $\varepsilon \in \{-1, 1\}$. Write $D\rho = \omega \otimes \rho$, with $\omega = \omega_1 dx^1 + \omega_2 dx^2$. First, for $d = 0$, we have

$$\frac{1}{2b}x_1^3(\rho_{11;2} - \omega_2\rho_{11}) = c(2c + x_1\omega_2).$$

Hence, for $c \neq 0$, it follows that $\omega_2 = -2c/x_1$ and, under this condition, we get $\rho_{12;2} - \omega_2\rho_{12} = -2c^2/x_1^3$, which does not vanish (case $(ii.1)$); in case of $c = 0$, $D\rho = 0$ is obtained.

Finally, we examine the case $d \neq 0$. In this case, we compute

$$\begin{aligned}
x_1^3(\rho_{11;1} - \omega_1\rho_{11}) &= (a + d + 3)((d-a-1)d + 2bc) + \varepsilon\zeta^{\frac{1}{2}} \\
&\quad + ((d-a-1)d + 2bc)x_1\omega_1, \\
x_1^3(\rho_{11;2} - \omega_2\rho_{11}) &= b^{-1}\left\{ ((d-a-1)d + 2bc)((d-a+1)d + 2bc) + d\varepsilon\zeta^{\frac{1}{2}} \right\} \\
&\quad + ((d-a-1)d + 2bc)x_1\omega_2.
\end{aligned}$$

Now, if $(d-a-1)d + 2bc = 0$, i.e, $c = \frac{(a-d+1)d}{2b}$, the expressions above vanish and, moreover,

$$\rho_{12;1} - \omega_1\rho_{12} = \frac{(a^2 - (d-1)^2)d}{2bx_1^3}.$$

It follows that for $a \neq \pm(d-1)$, the affine connection is not recurrent (case (ii.2)), while for $a = \pm(d-1)$, we get $D\rho = 0$. On the other hand, if $c \neq \frac{(a-d+1)d}{2b}$, then $\omega_1$ and $\omega_2$ are determined by the expressions above and a very long but straightforward calculation leads to

$$2b^{-1}d(\rho_{12;1} - \omega_1\rho_{12}) - (\rho_{12;2} - \omega_2\rho_{12}) = \frac{2c(bc - ad)}{bx_1^3}.$$

Therefore, if $c(bc - ad) \neq 0$, then the affine connection is not recurrent (case (ii.3)). If $c = 0$, then $c \neq \frac{(a-d+1)d}{2b}$ means $a \neq d - 1$ and $D\rho = \omega \otimes \rho$ if and only if $e = 0$ (case (ii.4)). If $bc - ad = 0$, then $c \neq \frac{(a-d+1)d}{2b}$ means $a \neq 1 - d$, and $D\rho = \omega \otimes \rho$ if and only if $e = -\frac{a^2 d}{b^2}$ (case (ii.5)). ∎

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***Remark 5.11*** Projectively flat Type B locally homogeneous connections are determined in Theorem 5.9 as follows:

(i) $e = f = c = 0$, or

(ii) $e \neq 0$, $f = -c$, $a = \frac{3c^2+2de+e}{e}$, $b = \frac{-c^3-ce}{e^2}$.

Note that, in case (i), the Ricci tensor is always degenerate and connections are recurrent (Theorem 5.10). In case (ii) with $c = 0$, the Ricci tensor is degenerate for $d = 0$ or $d = -2$; $d = 0$ implies that the connection is flat, while for $d = -2$, the connection is not recurrent (Theorem 5.10(i.2)). Next, we analyze case (ii) with $c \neq 0$. In this case, the Ricci tensor is degenerate if and only if $d = -c^2/e$ or $d = -c^2/e - 2$. For $d = -c^2/e$ the connection is flat. If $d = -c^2/e - 2$ and $b = 0$ we have the conditions $a = -4$, $b = 0$, $c \neq 0$, $d = -1$ and $e = -c^2 \neq 0$, so Theorem 5.10(i.3) holds, and the connection is not recurrent. Finally, if $d = -\frac{c^2}{e} - 2$ and $b \neq 0$, a long but straightforward calculation shows that, for $d = 0$, Theorem 5.10(ii.1) holds, while for $d \neq 0$, (ii.5) or (ii.3) in Theorem 5.10 holds, depending on whether $c$ equals $\frac{ad}{b}$ or not; in any case, again the connection is not recurrent.

### 5.1.3 Non-equivalent Locally Homogeneous Affine Connections

As an application of previous results, we answer a question posed by O. Kowalski on whether or not types A and B are affinely inequivalent classes, by showing that non-flat types A and B are completely inequivalent except the case corresponding to

$$D_{\partial_1}\partial_1 = \frac{1}{x_1}(a\partial_1 + b\partial_2), \qquad D_{\partial_1}\partial_2 = \frac{1}{x_1}d\partial_2, \qquad D_{\partial_2}\partial_2 = 0. \tag{5.12}$$

First of all, recall that any Type A locally homogeneous affine connection is projectively flat (cf. Proposition 5.5). Moreover, the Ricci tensor is always symmetric and defines a flat metric on the surface or, otherwise it is degenerate (cf. Theorem 5.4). Further note that in the later case, the Ricci tensor is always recurrent as shown in Theorem 5.6.

In what follows, we use the properties above to study, and distinguish when possible, types A and B. Observe that projectively flat Type B locally homogeneous connections are listed in Theorem 5.9 as follows:

(i) $e = f = c = 0$, or

(ii) $e \neq 0$, $f = -c$, $a = \frac{3c^2+2de+e}{e}$, $b = \frac{-c^3-ce}{e^2}$.

In case (ii), if the Ricci tensor is non-degenerate, then it defines a metric of non-zero curvature. Moreover, if the Ricci tensor is degenerate and the connection is supposed to be non-flat, then Remark 5.11 shows that such a connection is not both projectively flat and recurrent, and hence they cannot be affinely equivalent to any Type A connection.

Next we show that (i) is affinely equivalent to a Type A connection. In doing so, we recall the discussion at [2,18]. Any Type A locally homogeneous affine connection admits a pair of linearly independent affine-Killing vector fields such that $[X, Y] = 0$ (just put $X = \partial_1$, $Y = \partial_2$). Conversely, if there exist two linearly independent commuting affine-Killing vector fields $X$, $Y$, then there is a local coordinate system

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$(x_1, x_2)$ such that $X = \partial_1$, $Y = \partial_2$ and all the Christoffel symbols of the connection are constant (*i.e.*, of Type A).

A vector field $X = A(x_1, x_2)\partial_1 + B(x_1, x_2)\partial_2$ on the coordinate domain $\mathcal{U}(x_1, x_2)$ of a connection (i) is affine-Killing (*i.e.*, $[X, D_Y Z] - D_Y[X, Z] - D_{[X,Y]}Z = 0$ for all vector fields $Y, Z$) if and only if (cf. [18, Equation (6)])

$$A_{11} + \frac{a}{x_1}A_1 - \frac{b}{x_1}A_2 - \frac{a}{x_1^2}A = 0, \quad A_{12} + \frac{a-d}{x_1}A_2 = 0, \quad A_{22} = 0,$$

$$B_{11} + \frac{2b}{x_1}A_1 + \frac{2d-a}{x_1}B_1 - \frac{b}{x_1}B_2 - \frac{b}{x_1^2}A = 0,$$

$$B_{12} + \frac{d}{x_1}A_1 + \frac{b}{x_1}A_2 - \frac{d}{x_1^2}A = 0, \quad B_{22} + \frac{2d}{x_1}A_2 = 0.$$

Now, a direct calculation shows that

$$X = x_1\partial_2, \qquad Y = x_1\partial_1 + (x_1 + x_2)\partial_2, \qquad a - 2d = 0,$$

and

$$X = \partial_2, \qquad Y = x_1\partial_1 + \frac{b}{a-2d}x_1\partial_2, \qquad a - 2d \neq 0,$$

are linearly independent, commuting, affine-Killing vector fields, and hence the connection is of Type A.

### 5.1.4 Recurrent and Projectively Flat Affine Connections with Degenerate Ricci Tensor

Observe from Theorems 5.9 and 5.10 that locally homogeneous connections given by (5.12) are projectively flat and recurrent with symmetric and degenerate Ricci tensor. However, not every projectively flat and recurrent locally homogeneous affine connection with symmetric and degenerate Ricci tensor is necessarily of Type B. In what follows, we complete the analysis above by giving a complete description of all locally homogeneous projectively flat and recurrent affine connections with symmetric and degenerate Ricci tensor.

***Theorem 5.12*** *Let $(\Sigma, D)$ be an affine surface with symmetric and degenerate Ricci tensor, which is recurrent and projectively flat. Then $(\Sigma, D)$ is locally homogeneous if and only if around each point there exists a coordinate system $(x_1, x_2)$ in which the non-zero component of $D$ is given by*

$$D_{\partial_1}\partial_1 = x_2\frac{\mu}{(\alpha + \kappa x_1)^2}\partial_2, \tag{5.13}$$

*for some constants $\mu$, $\alpha$, and $\kappa$. Moreover, any such a connection is locally homogeneous of Type A, and it is also of Type B if and only if $\kappa^2 - 4\mu \geq 0$.*

**Proof** It follows from Theorem 5.2 that a recurrent affine connection with symmetric and degenerate Ricci tensor expresses, in suitable coordinates $(x_1, x_2)$, as

$$D_{\partial_1}\partial_1 = a(x_1, x_2)\partial_2, \tag{5.14}$$

for some function $a(x_1, x_2)$. Now, a straightforward calculation shows that (5.14) is projectively flat if and only if $a(x_1, x_2) = x_2\theta(x_1) + \gamma(x_1)$, for some functions $\theta$ and $\gamma$. Note that we may assume, without loss of generality, that $\gamma(x_1) \equiv 0$, just by using the equivalence theorem at [16, Theorem 7.2] since both the Ricci tensor and its covariant derivatives are independent of $\gamma$.

Next, a vector field $X = A(x_1, x_2)\partial_1 + B(x_1, x_2)\partial_2$ is affine-Killing if and only if

$$\begin{aligned} &A_{12} = 0, \qquad A_{22} = 0, \qquad B_{22} = 0, \\ &A_{11} - x_2\theta(x_1)A_2 = 0, \qquad B_{12} + x_2\theta(x_1)A_2 = 0, \\ &B_{11} + 2x_2\theta(x_1)A_1 - x_2\theta(x_1)B_2 + x_2\theta'(x_1)A + \theta(x_1)B = 0. \end{aligned} \tag{5.15}$$

We start by showing that the connection must be of the form (5.13). Integration of the above equations shows that any affine-Killing vector field must be of the form

$$X(x_1, x_2) = (x_1\kappa + \alpha)\partial_1 + (x_2\beta + b(x_1))\partial_2$$

for some constants $\kappa$, $\alpha$, $\beta$ and a function $b(x_1)$ which is a solution of

$$b''(x_1) + b(x_1)\theta(x_1) + 2x_2\kappa\theta(x_1) + x_2(\alpha + x_1\kappa)\theta'(x_1) = 0. \tag{5.16}$$

Now, taking the derivative in (5.16) with respect to $x_2$, we obtain

$$(x_1\kappa + \alpha)\theta'(x_1) + 2\kappa\theta(x_1) = 0,$$

which shows that

$$\theta(x_1) = \frac{\mu}{(x_1\kappa + \alpha)^2}$$

for some constants $\mu$, $\kappa$ and $\alpha$, where $\kappa$ and $\alpha$ are not simultaneously zero, thus showing (5.13). Note that for $\kappa = \alpha = 0$ any affine-Killing vector field must be of the form $X(x_1, x_2) = (x_2\beta + b(x_1))\partial_2$, and hence there are no two-linearly independent affine-Killing vector fields, in contradiction with local homogeneity.

In what follows, take a connection given by (5.13). Observe that such a connection is of Type A, since a direct calculation shows that

$$X(x_1, x_2) = (x_1\kappa + \alpha)\partial_1, \qquad Y(x_1, x_2) = x_2\partial_2$$

are linearly independent commuting affine-Killing vector fields.

Finally, we will show that a connection (5.13) is of Type B if and only if $\kappa^2 - 4\mu \geq 0$. From (5.15), and proceeding as above, an affine-Killing vector field must be of the form

$$X(x_1, x_2) = (x_1\lambda + \nu)\partial_1 + (x_2\beta + b(x_1))\partial_2$$

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for constants $\lambda$, $\nu$, and $\beta$, and a function $b(x_1)$, such that

$$\lambda\alpha - \nu\kappa = 0, \qquad \mu b(x_1) + (x_1\kappa + \alpha)^2 b''(x_1) = 0.$$

As a consequence, assuming $\kappa \neq 0$ in (5.13), if two affine-Killing vector fields $X, Y$ satisfy $[X, Y] = X$, then

$$X(x_1, x_2) = C(x_1\kappa + \alpha)^{\frac{\beta-1}{\lambda}} \partial_2,$$
$$Y(x_1, x_2) = \lambda(x_1 + \frac{\alpha}{\kappa})\partial_1 + (x_2\beta + b(x_1))\partial_2$$

for some constants $C$, $\lambda$, $\beta$ and some function $b(x_1)$, such that

$$(\beta - 1)\kappa^2(\beta - \lambda - 1) + \lambda^2\mu = 0, \qquad \mu b(x_1) + (x_1\kappa + \alpha)^2 b''(x_1) = 0. \tag{5.17}$$

Moreover, the first equation in (5.17) has real solutions if and only if $\kappa^2 - 4\mu \geq 0$. In this case, a choice of $X$ and $Y$ as follows

$$X(x_1, x_2) = (x_1\kappa + \alpha)^{\frac{\kappa - \sqrt{\kappa^2 - 4\mu}}{2\kappa}} \partial_2,$$
$$Y(x_1, x_2) = (x_1\kappa + \alpha)\partial_1 + x_2\Big(1 + \frac{\kappa - \sqrt{\kappa^2 - 4\mu}}{2}\Big)\partial_2,$$

shows that $D$ is locally homogeneous of Type B.

Next, put $\kappa = 0$ in (5.13). If two affine-Killing vector fields $X, Y$ satisfy $[X, Y] = X$, then

$$X(x_1, x_2) = Ce^{\frac{x_1(\beta-1)}{\nu}} \partial_2,$$
$$Y(x_1, x_2) = \nu\partial_1 + (x_2\beta + b(x_1))\partial_2$$

for some constants $C$, $\nu$, $\beta$ and some function $b(x_1)$ such that

$$\alpha^2(\beta - 1)^2 + \nu^2\mu = 0, \qquad \mu b(x_1) + \alpha^2 b''(x_1) = 0. \tag{5.18}$$

Now, the first equation in (5.18) has real solutions if and only if $\mu \leq 0$. Moreover, in such a case,

$$X(x_1, x_2) = e^{-\frac{x_1\sqrt{-\mu}}{\alpha}} \partial_2,$$
$$Y(x_1, x_2) = \alpha\partial_1 + x_2(1 - \sqrt{-\mu})\partial_2$$

are affine-Killing vector fields, with $[X, Y] = X$, which finishes the proof. ■

***Remark 5.13*** A connection (5.13) is flat if and only if $\mu = 0$ and locally symmetric if and only if $\kappa = 0$. Hence, a projectively flat locally symmetric connection with symmetric and degenerate Ricci tensor is of Type B if and only if the only non-zero Christoffel symbol is $\Gamma_{11}^2 = Kx_2$, with $K \leq 0$.

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