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A SINGULAR INTEGRAL ON $L^2(\mathbb{R}^n)$

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ABSTRACT. We consider a convolution singular integral operator T_b^Ω associated to a kernel $K(x) = b(x)\Omega(x)|x|^{-n}$, and prove that if $b \in L^\infty(\mathbb{R}^n)$ is a radial function and $\Omega \in H(\Sigma_{n-1})$ with mean zero condition (1), then T_b^Ω is a bounded linear operator in the space $L^2(\mathbb{R}^n)$.

Let $\mathbb{R}^n$ be the n-dimensional real Euclidean space. The unit ball B_n is the set $\{z \in \mathbb{R}^n, |z| \leq 1\}$, and the unit sphere Σ_{n-1} is the boundary of B_n. Let $d\sigma(x')$ be the element of Lebesgue measure on Σ_{n-1} so that the measure of Σ_{n-1} is 1. Let $L^p(\mathbb{R}^n)$ and $L^p(\Sigma_{n-1})$ be the spaces of Lebesgue L^p-integrable functions on $\mathbb{R}^n$ and Σ_{n-1}, respectively. Besides considering these L^p spaces, we are also interested in the Hardy spaces on both $\mathbb{R}^n$ and Σ_{n-1}.

The Poisson kernel $P_t(x)$ on $\mathbb{R}^n$ is defined by

$$P_t(x) = C_n t(t^2 + |x|^2)^{-(n+1)/2}, \quad C_n = \Gamma\big((n+1)/2\big)\pi^{-(n+1)/2}.$$

For any $f \in s'(\mathbb{R}^n)$, we define the radial maximal function P^*f by

$$P^*f(x) = \sup_{t>0} |P_t * f(x)|,$$

where $s'(\mathbb{R}^n)$ is the space of Schwartz distributions on $\mathbb{R}^n$.

The Hardy space $H(\mathbb{R}^n)$ is the linear space of distributions f with the finite norm $\|f\|_{H(\mathbb{R}^n)} = \|P^*f\|_{L'(\mathbb{R}^n)} < \infty$. More details about the Hardy space on $\mathbb{R}^n$ can be found in [8].

The Poisson kernel on Σ_{n-1} is defined by

$$P_{ry'}(x') = (1 - r^2)/|ry' - x'|^n,$$

where $0 \leq r < 1$ and $x', y' \in \Sigma_{n-1}$. For any $f \in s'(\Sigma_{n-1})$, we define the radial maximal function $P^+f(x')$ by

$$P^+f(x') = \sup_{0\leq r<1} \left| \int_{\Sigma_{n-1}} f(y')P_{rx'}(y')\, d\sigma(y') \right|,$$

where $s'(\Sigma_{n-1})$ is the space of Schwartz distributions on Σ_{n-1}. The Hardy space $H(\Sigma_{n-1})$ is the linear space of distributions $f \in s'(\Sigma_{n-1})$ with the finite norm $\|f\|_{H(\Sigma_{n-1})} = \|P^+f\|_{L'(\Sigma_{n-1})} < \infty$. Various properties of Hardy space on Σ_{n-1} were studied in [5]. In particular, a well-known result is $L'(\Sigma_{n-1}) \supseteq H(\Sigma_{n-1}) \supseteq L^q(\Sigma_{n-1})$ for any $q > 1$.

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Suppose Ω is a homogeneous function of degree zero and satisfies

$$\int_{\Sigma_{n-1}} \Omega(x')\, d\sigma(x') = 0. \tag{1}$$

Let b be a bounded and radial function. We define a kernel K by

$$K(x) = b(x)\Omega(x)|x|^{-n} \tag{2}$$

and consider the singular integral $(T_b^\Omega f)(x) = p.v.(K * f)(x)$. This operator was first studied by Calderón and Zygmund in their pioneer papers [1] and [2] for the case $b(x) \equiv 1$. in [2], Calderón and Zygmund proved that if $b(x) = 1$ and Ω satisfies (1), then this operator T_1^Ω is $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$ bounded provided $\Omega \in L^q(\Sigma_{n-1})$ for some $q > 1$. Coifman and Weiss [4] improved Calderón and Zygmund's result under a weaker condition $\Omega \in H(\Sigma_{n-1})$.

In [6], R. Fefferman generalized this singular operator by considering any L^∞ function b. In this Ph.D. thesis, (see [3]) K. Chen proved that T_b^Ω is a linear bounded operator in the space $L^p(R^n)$ $(n > 2, 1 < p < \infty)$ if $\Omega \in L^q(\Sigma_{n-1})$ for some $q > 1$ and satisfies the condition (1). The first step of Chen's proof is to prove the L^2 boundedness of T_b^Ω; then the L^p result follows by an interpolation argument. Recently, Jian and Lu (see [7], p. 140) improved Chen's L^2 result by using a weaker assumption on Ω. They proved that if Ω satisfies the mean zero condition (1) and Ω is in the block space $B_q(\Sigma_{n-1})$, then T_b^Ω is a bounded operator in $L^2(\mathbb{R}^n)$ $(n > 2)$.

Comparing the early result [4] of Coifman and Weiss, we find a more natural condition on Ω should be $\Omega \in H(\Sigma_{n-1})$. The following theorem then is the main purpose of this short note.

THEOREM. *Suppose that Ω is a homogeneous function of degree zero, and satisfies (1). If b is a bounded radial function and $\Omega \in H(\Sigma_{n-1})$, $n \geq 2$, then the operator T_b^Ω is bounded in $L^2(\mathbb{R}^n)$ and its operator norm is bounded by $C\|b\|_\infty \|\Omega\|_{H(\Sigma_{n-1})}$, where C is a constant independent of function $b(x)$ and $\Omega(x)$.*

PROOF. By the Plancherel theorem, we need only to prove that

$$\|\hat{K}\|_\infty \leq C\|b\|_\infty \|\Omega\|_H. \tag{3}$$

In fact,

$$|\widehat{K(x)}| \leq C\|b\|_\infty \int_0^1 \left| \int_{\Sigma_{n-1}} \Omega(\xi')(e^{it\langle x',\xi'\rangle} - 1)\, d\sigma(\xi') \right| t^{-1}\, dt + C\|b\|_\infty \int_1^\infty \left| \int_{\Sigma_{n-1}} \Omega(\xi') e^{it\langle x',\xi'\rangle}\, d\sigma(\xi') \right| t^{-1}\, dt.$$

It is easy to see that the first term above is bounded by $\|b\|_\infty \|\Omega\|_{L'} \leq C\|b\|_\infty \|\Omega\|_H$. Thus now we only have to prove that

$$\int_1^\infty \left| \int_{\Sigma_{n-1}} \Omega(\xi') e^{it\langle x',\xi'\rangle}\, d\sigma(\xi') \right| t^{-1}\, dt \leq C\|\Omega\|_{H(\Sigma_{n-1})}. \tag{4}$$

In order to prove (4), we need introduce the atomic decomposition of $H(\Sigma_{n-1})$. An exceptional atom is an L^∞ function $a(x)$ satisfying $\|a\|_\infty \leq 1$. A $(1,\infty)$ atom is an L^∞ function $a(x)$ which satisfies

(i) $\operatorname{supp}(a) \subset \{x' \in \Sigma_{n-1}, |x' - x'_0| < \rho$ for some $x'_0 \in \Sigma_{n-1}$ and $\rho > 0\}$,

(ii) $\int_{\Sigma_{n-1}} a(\xi')\, d\sigma(\xi') = 0$,

(iii) $\|a\|_\infty \leq \rho^{-n+1}$.

By [5], we know that any $\Omega \in H(\Sigma_{n-1})$ has an atomic decomposition $\Omega(\xi') = \sum \lambda_j a_j(\xi')$, where the a_j's are either exceptional atoms or $(1,\infty)$ atoms and $\sum |\lambda_j| \leq C\|\Omega\|_{H(\Sigma_{n-1})}$. Therefore it remains to prove that for all atoms $a(\xi')$,

$$L_a(x') = \int_1^\infty \left| \int_{\Sigma_{n-1}} a(\xi') e^{it\langle x',\xi'\rangle}\, d\sigma(\xi') \right| t^{-1}\, dt \leq C \tag{5}$$

with a constant C independent of $a(\xi')$ and $x' \in \Sigma_{n-1}$. We will prove (5) in the two different cases $n > 2$ and $n = 2$, respectively.

CASE $n > 2$. If a is exceptional, by [3], we have $\|L_a\|_\infty \leq C\|a\|_2 \leq C$. Suppose a is a $(1,\infty)$ atom; without loss of generality, we may assume that $\operatorname{supp}(a)$ is contained in the ball $B(\mathbf{1},\rho)$, where $\mathbf{1} = (1,0,\ldots,0)$. By a rotation we also can assume $x' = \mathbf{1}$. Under these assumptions, we let

$$\xi' = (s, \xi_1, \xi_2, \ldots, \xi_n).$$

Then

$$L_a(x') \leq C \int_0^\infty t^{-1} |\widehat{F(t)}|\, dt,$$

where $F(s) = (1-s^2)^{(n-3)/2} \chi_{(-1,1)}(s) \int_{\Sigma_{n-2}} a\big(s, (1-s^2)^{1/2} y'\big)\, d\sigma(y')$ and $\widehat{F(t)}$ is the Fourier transform of $F(s)$. Now we easily see that $\int_{\mathbb{R}} F(s)\, ds = 0$ and $\operatorname{supp}(F) \subseteq (1-\rho, 1)$. Furthermore, we have $\|F\|_\infty \leq \|a\|_\infty \int_{\Sigma_{n-2} \cap B(\mathbf{1},\rho)} d\sigma(y') \leq C\rho^{-1}$. These imply that, up to a constant independent of atom $a(\xi')$, F is a $(1,\infty)$ atom on $\mathbb{R}$. Thus using the Hardy inequality (see [8]), we obtain that $L_a(x') \leq C$. Thus the case $n > 2$ of (5) is now proved.

CASE $n = 2$. In this case $\Sigma_1 = \mathbb{T}$, the one-dimension torus. We will first prove (5) for any $(1,\infty)$ atom $a(\theta)$. As before, we may assume that $\operatorname{supp}(a) \subseteq (-\rho, \rho)$. Let $x' = e^{i\alpha}$; then

$$L_a(x') = \left\{ \int_{\rho^{-2}}^\infty + \int_1^{\rho^{-2}} \right\} t^{-1} \left| \int_{-\pi}^{\pi} a(\theta) e^{it\cos(\theta-\alpha)}\, d\theta \right| dt \leq J_1 + J_2.$$

We will only estimate J_1 and J_2 for the case $\cos\alpha \neq 0$; the estimate of the case $\cos\alpha = 0$ is easier than the prior case.

$$\begin{aligned} J_2 &= \int_1^{\rho^{-2}} t^{-1} \left| \int_{-\pi}^{\pi} a(\theta) e^{it\cos\theta\cos\alpha} e^{it\sin\theta\sin\alpha}\, d\theta \right| dt \\ &= \int_{\cos\alpha}^{\rho^{-2}\cos\alpha} \left| t^{-1} \int_{-\pi}^{\pi} a(\theta) e^{it(\cos\theta - 1)} e^{it\tan\alpha\sin\theta}\, d\theta \right| dt \\ &\leq \int_{\cos\alpha}^{\rho^{-2}\cos\alpha} t^{-1} \left| \int_{-\pi}^{\pi} a(\theta) e^{it\tan\alpha\sin\theta}\, d\theta \right| dt \\ &\quad + \int_0^{\rho^{-2}} \int_{-\pi}^{\pi} |a(\theta)(\cos\theta - 1)|\, d\theta\, dt = I + II. \end{aligned}$$

It is easy to see that $II \leq C$.

$$\begin{aligned} I &\leq \int_{\cos\alpha}^{\rho^{-2}\cos\alpha} t^{-1}\left|\int_{-\pi}^{\pi} a(\theta)e^{it\tan\alpha(\sin\theta-\theta)}e^{it\tan\alpha\theta}\,d\theta\right| dt \\ &\leq \int_0^{\infty} t^{-1}\left|\int_{-\pi}^{\pi} a(\theta)e^{it\theta}\,d\theta\right| dt \\ &\quad + \int_{\cos\alpha}^{\rho^{-2}\cos\alpha} |\tan\alpha| \int_{-\pi}^{\pi} |a(\theta)(\sin\theta-\theta)|\,d\theta\,dt \\ &\leq \left(1+\int_0^{\infty} |\widehat{a(t)}|t^{-1}\,dt\right) \leq C \quad \text{(by the Hardy inequality).} \end{aligned}$$

Next we estimate

$$J_1 = \int_{\rho^{-2}}^{\infty} t^{-1}\left|\int_{-\pi}^{\pi} a(\theta+\alpha)e^{it\cos\theta}\,d\theta\right| dt.$$

By Hölder's inequality, we have

$$\begin{aligned} J_1 &\leq 2\rho^{2/q}\left(\int_{\rho^{-2}}^{\infty}\left|\int_0^{\pi}+\int_{-\pi}^{0} a(\theta+\alpha)e^{it\cos\theta}\,d\theta\right|^q dt\right)^{1/q} \\ &\leq 2\rho^{2/q}(J_{1,1}+J_{1,2}), \quad \text{where } 1<p=q/(q-1)\leq 3/2. \end{aligned}$$

We will only estimate the above term

$$J_{1,1} = \left(\int_{\rho^{-2}}^{\infty}\left|\int_0^{\pi} a(\theta+\alpha)e^{it\cos\theta}\,d\theta\right|^q dt\right)^{1/q};$$

the estimate of $J_{1,2}$ is exactly same. After change variable $u=\cos\theta$, we know that

$$J_{1,1} = \left(\int_{\rho^{-2}}^{\infty}\left|\int_{\cos\rho}^{1} a(\alpha+\cos^{-1}u)(1-u^2)^{-1/2}e^{itu}\,du\right|^q dt\right)^{1/q}.$$

Thus by the Hausdorff-Young inequality, we have

$$\begin{aligned} J_{1,1} &\leq \left(\int_{\mathbb{R}} \chi_{(\cos\rho,1)}(t)|a(\alpha+\cos^{-1}t)(1-t^2)^{-1/2}|^p\,dt\right)^{1/p} \\ &\leq \rho^{-1}\left(\int_{\cos\rho}^{1}(1-t^2)^{-p/2}\,dt\right)^{1/p} \leq \rho^{-1}\left(\int_0^{\rho}|\sin^{1-p}(\theta)|\,d\theta\right)^{1/p} \\ &\leq \rho^{-2}\rho^{2/p} = \rho^{-2/q}. \end{aligned}$$

This shows that $J_1 \leq C$. Therefore we complete the proof of (5) for any $(1,\infty)$ atom. Finally we need to prove (5) for an exceptional atom $a(\theta)$. But this case easily follows by mimicking the estimate of J_1 in the above argument. Now the theorem is proved.

REFERENCES

1. A. P. Calderón and A. Zygmund, *On the existence of certain singular integrals*, Acta. Math. **88**(1952), 85–139.

2. ———, *On singular integrals*, Amer. J. Math. **18**(1956), 289–309.

3. K. Chen, *On a singular integral*, Studia Math. **LXXXV**(1987), 61–72.

4. R. Coifman and G. Weiss, *Extension of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. **83**(1977), 569–645.

5. L. Colzani, *Hardy spaces on Sphere*, Ph.D. thesis, Washington University, St. Louis, 1982.

6. R. Fefferman, *A note on singular integrals*, Proc. Amer. Math. Soc. **74**(1979), 266–270.
7. S. Lu, M. Taibleson and G. Weiss, *Space Generated by Blocks*, Beijing Normal University Mathematics Series, Beijian Normal University Press, 1989.
8. A. Torchinsky, *Real-variable Methods in Harmonic Analysis*, Academic Press, 1986.

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