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Parabolic Geodesics in Sasakian 3-Manifolds

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Abstract. We give explicit parametrizations for all parabolic geodesics in 3-dimensional Sasakian space forms.

1 Introduction

Let $M = (M, \varphi, \xi, \eta, g)$ be a 3-dimensional contact strongly pseudo-convex pseudo-Hermitian manifold with Tanaka–Webster connection $\widehat{\nabla}$.

A curve in M is said to be a *slant curve* if its tangent vector field makes constant angle with the Reeb vector field ξ of M [9].

In our previous paper [12], we proved that every $\widehat{\nabla}$-geodesic parametrized by arc length in a Sasakian 3-space form is a slant curve. Moreover, we showed that the acceleration vector field $\widehat{\nabla}_{\gamma'}\gamma'$ of a unit speed slant curve $\gamma(s)$ in a Sasakian 3-space form is orthogonal to ξ everywhere.

On the other hand, D. Jerison and J. M. Lee [14] introduced the notion of parabolic geodesics in contact strongly pseudo-convex pseudo-Hermitian manifolds.

According to Jerison and Lee, a curve $\gamma(s)$ in a contact strongly pseudo-convex pseudo-Hermitian manifold is said to be a *parabolic geodesic* if it satisfies $\widehat{\nabla}_{\gamma'}\gamma' = a\xi_{\gamma(s)}$ for some constant a and initial condition $\gamma'(0) \perp \xi_{\gamma(0)}$. Parabolic geodesics naturally induce *parabolic exponential maps*. The parabolic exponential map is a local diffeomorphism from a tangent space T_pM into M. Then any choice of orthonormal frame for the holomorphic subspace $\mathcal{H}_p$ of the complexified tangent space $T_p^{\mathbb{C}}M$ gives an identification of T_pM and the Heisenberg group Nil. Composing this identification with the parabolic exponential map yields *pseudo-Hermitian normal coordinates* around p. The pseudo-Hermitian normal coordinates allow us to considerably simplify the computation of Taylor series of the pseudo-Hermitian structure explicitly in terms of pseudo-Hermitian curvature and torsion.

The purpose of this paper is to give explicit parametric equations for all parabolic geodesics in Sasakian 3-space forms.

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2 Preliminaries

2.1 Contact Manifolds

We recall the fundamental ingredients of 3-dimensional contact Riemannian geometry. Our general references are D. E. Blair's lecture notes [4] and monograph [5].

Let M be a 3-dimensional manifold. A *contact form* is a one-form η such that $d\eta \wedge \eta \neq 0$ on M. A 3-manifold M together with a contact form η is called a *contact* 3*-manifold*. The *Reeb vector field* ξ is a unique vector field satisfying $\eta(\xi) = 1$, $d\eta(\xi, \cdot) = 0$.

On a contact 3-manifold (M, η), there exists a structure (φ, ξ, g) such that

$$\varphi^2 = -\mathrm{I} + \eta \otimes \xi, \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$
$$g(X, \varphi Y) = d\eta(X, Y), \quad X, Y \in \mathfrak{X}(M).$$

Here $\mathfrak{X}(M)$ denotes the Lie algebra of all smooth vector fields on M.

The structure (φ, ξ, η, g) is called the *contact Riemannian structure* of M associated with the contact form η. A contact 3-manifold together with its associated contact Riemannian structure is called a *contact Riemannian* 3*-manifold* and denoted by $(M, \varphi, \xi, \eta, g)$. A contact Riemannian 3-manifold M satisfies the following formula ([18]):

$$(\nabla_X \varphi)Y = g((\mathrm{I} + \mathrm{h})X, Y)\xi - \eta(Y)(\mathrm{I} + \mathrm{h})X, \quad X, Y \in \mathfrak{X}(M). \tag{2.1}$$

Here h is an endomorphism field defined by $\mathrm{h} = \pounds_\xi \varphi/2$. The formula (2.1) implies

$$\nabla_X \xi = -\varphi(\mathrm{I} + \mathrm{h})X, \quad X \in \mathfrak{X}(M). \tag{2.2}$$

One can see from (2.2) that ξ is a Killing vector field if and only if $\mathrm{h} = 0$.

A contact Riemannian 3-manifold $(M, \varphi, \xi, \eta, g)$ is called a *Sasakian manifold* if it satisfies

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X$$

for all $X, Y \in \mathfrak{X}(M)$.

The formulas (2.1) and (2.2) imply that a contact Riemannian 3-manifold is Sasakian if and only if its Reeb vector field ξ is a Killing vector field.

A plane section Π_p at a point p of a contact Riemannian 3-manifold is called a *holomorphic plane* if it is invariant under φ_p. The sectional curvature function of holomorphic planes is called the *holomorphic sectional curvature*. Sasakian 3-manifolds of constant holomorphic sectional curvature are called *Sasakian* 3*-space forms*.

2.2 Bianchi-Cartan-Vranceanu Spaces

To describe a parabolic geodesic in 3-dimensional Sasakian space forms explicitly, it is convenient to use the so-called Bianchi–Cartan–Vranceanu model spaces.

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Let c be a real number and set

$$\mathcal{D} = \left\{ (x, y, z) \in \mathbb{R}^3(x, y, z) \;\middle|\; 1 + \frac{c}{2}(x^2 + y^2) > 0 \right\}.$$

Note that $\mathcal{D}$ is the whole $\mathbb{R}^3(x, y, z)$ for $c \geq 0$. We equip the region $\mathcal{D}$ with the following Riemannian metric:

$$g_c = \frac{dx^2 + dy^2}{\{1 + \frac{c}{2}(x^2 + y^2)\}^2} + \left(dz + \frac{ydx - xdy}{1 + \frac{c}{2}(x^2 + y^2)} \right)^2.$$

The one-parameter family of Riemannian 3-manifolds $\{(\mathcal{D}, g_c)\}_{c\in\mathbb{R}}$ was introduced by L. Bianchi [3], E. Cartan [8], and G. Vranceanu [21] (see also Kobayashi [15]).

Take the following orthonormal frame field on $(\mathcal{D}, g_c)$:

$$u_1 = \left\{ 1 + \frac{c}{2}(x^2 + y^2) \right\} \frac{\partial}{\partial x} - y\frac{\partial}{\partial z},\ u_2 = \left\{ 1 + \frac{c}{2}(x^2 + y^2) \right\} \frac{\partial}{\partial y} + x\frac{\partial}{\partial z},\ u_3 = \frac{\partial}{\partial z}.$$

Then the Levi–Civita connection ∇ of this Riemannian 3-manifold is described as

$$\begin{aligned}
&\nabla_{u_1} u_1 = c\, y u_2, && \nabla_{u_1} u_2 = -c\, y u_1 + u_3, && \nabla_{u_1} u_3 = -u_2,\\
&\nabla_{u_2} u_1 = -c\, x u_2 - u_3, && \nabla_{u_2} u_2 = c\, x u_1, && \nabla_{u_2} u_3 = u_1,\\
&\nabla_{u_3} u_1 = -u_2, && \nabla_{u_3} u_2 = u_1, && \nabla_{u_3} u_3 = 0.
\end{aligned}$$

$$[u_1, u_2] = -c\, y u_1 + c\, x u_2 + 2u_3, \quad [u_2, u_3] = [u_3, u_1] = 0.$$

Define the endomorphism field φ by

$$\varphi u_1 = u_2, \quad \varphi u_2 = -u_1, \quad \varphi u_3 = 0.$$

The dual one-form η of the vector field $\xi = u_3$ is a contact form on $\mathcal{D}$ and satisfies

$$d\eta(X, Y) = g(X, \varphi Y), \quad X, Y \in \mathfrak{X}(\mathcal{D}).$$

Moreover, the structure (φ, ξ, η, g) is Sasakian, and $(\mathcal{D}, g_c)$ is of constant holomorphic sectional curvature $H = -3 + 2c$ (*cf.* [2, 16]). Hereafter we denote this model $(\mathcal{D}, g_c)$ of Sasakian space form by $\mathcal{M}^3(H)$. The model $\mathcal{M}^3(H)$ of Sasakian 3-space form is called the *Bianchi–Cartan–Vranceanu model* of Sasakian 3-space forms.

The Reeb flows are the translations in the z-directions. Hence the orbit space $\overline{\mathcal{M}^2}(H + 3) = \mathcal{M}^3(H)/\xi$ under the Reeb flow is given explicitly by

$$\overline{\mathcal{M}^2} = \left(\left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; 1 + \frac{c}{2}(x^2 + y^2) > 0 \right\}, \frac{dx^2 + dy^2}{\{1 + \frac{c}{2}(x^2 + y^2)\}^2} \right).$$

The natural projection $\pi\colon \mathcal{M}^3(H) \to \overline{\mathcal{M}^2}(H + 3)$ is given by $\pi(x, y, z) = (x, y)$. Note that the orbit space is of constant curvature $H + 3$.

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Example 2.1 (Heisenberg group) The Sasakian space form $\mathcal{M}^3(-3)$ of constant holomorphic sectional curvature -3 is isomorphic to the *Heisenberg group* Nil_3. The Heisenberg group $\mathrm{Nil}_3 = \mathcal{M}^3(-3)$ is realized as $\mathbb{R}^3(x, y, z)$ with Sasakian metric

$$g_0 = dx^2 + dy^2 + (dz + ydx - xdy)^2$$

and group structure

$$(x, y, z) \cdot (\tilde{x}, \tilde{y}, \tilde{z}) := (x + \tilde{x}, y + \tilde{y}, z + \tilde{z} + x\tilde{y} - \tilde{x}y) \tag{2.3}$$

for (x, y, z), $(\tilde{x}, \tilde{y}, \tilde{z}) \in \mathbb{R}^3(x, y, z)$. The Riemannian metric g_0 is invariant under left translations with respect to the group structure (2.3). Note that Nil_3 is the model space of nilgeometry in the sense of W. M. Thurston [20].

Example 2.2 ($H > -3$) The Sasakian space form $\mathcal{M}^3(1)$ of constant holomorphic sectional curvature 1 is of constant curvature 1. Hence $\mathcal{M}^3(1)$ is an open portion of the unit 3-sphere S^3 equipped with canonical Sasakian structure. The Sasakian space form $\mathcal{M}^3(H)$ with $H > -3$ and $H \neq 1$ is an open portion of the Berger sphere [1].

Example 2.3 The Sasakian space form $\mathcal{M}^3(H)$ with $H < -3$ is the universal covering of the special linear group $\mathrm{SL}_2\mathbb{R}$ equipped with canonical Sasakian structure.

3 Parabolic Geodesics

3.1 Pseudo-Hermitian Structures

For a contact Riemannian 3-manifold $M = (M, \eta; \xi, \varphi, g)$, the tangent space T_pM of M at a point $p \in M$ can be decomposed as the direct sum $T_pM = D_p \oplus \mathbb{R}\xi_p$, with $D_p = \{v \in T_pM \mid \eta(v) = 0\}$. Then the correspondence $D\colon p \longmapsto D_p$ defines a 2-dimensional distribution orthogonal to ξ, called the *contact distribution*. We see that the restriction $J = \varphi|_D$ of φ to D defines an almost complex structure on D. Denote by $T^{\mathbb{C}}M$ the *complexified tangent bundle* of M. The holomorphic subbundle

$$\mathcal{H} = \{X - \sqrt{-1}JX \mid X \in D\}$$

is called the *almost CR-structure* of M associated with the contact Riemannian structure (φ, ξ, η, g). We can see that each fiber $\mathcal{H}_p$ is of complex dimension 1, $\mathcal{H} \cap \overline{\mathcal{H}} = \{0\}$, and $D^{\mathbb{C}} = \mathcal{H} \oplus \overline{\mathcal{H}}$. Furthermore, the associated almost CR-structure is always *integrable*, that is, the space $\Gamma(\mathcal{H})$ of all smooth sections of $\mathcal{H}$ satisfies the *integrability condition* $[\Gamma(\mathcal{H}), \Gamma(\mathcal{H})] \subset \Gamma(\mathcal{H})$. The *Levi form* L associated with $\mathcal{H}$ is defined by

$$L\colon \Gamma(D) \times \Gamma(D) \to \mathfrak{F}(M), \quad L(X, Y) = -d\eta(X, JY),$$

where $\mathfrak{F}(M)$ denotes the algebra of all smooth functions on M. It is easy to check that the Levi form is Hermitian and positive definite. We call the pair (η, L) a *contact strongly pseudo-convex pseudo-Hermitian structure* on M.

3.2 Tanaka–Webster Connection

Now, we review the *Tanaka–Webster connection* ([17, 22]) on a contact strongly pseudo-convex pseudo-Hermitian manifold $M = (M; \eta, L)$ with the associated contact Riemannian structure (φ, ξ, η, g). The Tanaka–Webster connection $\widehat{\nabla}$ is defined by

$$\widehat{\nabla}_X Y = \nabla_X Y + \eta(X)\varphi Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi$$

for all vector fields X,Y on M. Together with (2.1), $\widehat{\nabla}$ may be rewritten as

$$\widehat{\nabla}_X Y = \nabla_X Y + A(X, Y),$$

where we have put

$$A(X, Y) = \eta(X)\varphi Y + \eta(Y)(\varphi(\mathrm{I} + \mathrm{h})X) - g(\varphi(\mathrm{I} + \mathrm{h})X, Y)\xi.$$

We see that the Tanaka-Webster connection $\widehat{\nabla}$ has the torsion

$$\widehat{T}(X, Y) = 2g(X, \varphi Y)\xi + \eta(Y)\varphi \mathrm{h} X - \eta(X)\varphi \mathrm{h} Y.$$

In particular, for a Sasakian manifold M, the difference tensor A and the torsion tensor $\widehat{T}$ have simpler forms:

$$A(X, Y) = \eta(X)\varphi Y + \eta(Y)\varphi X - g(\varphi X, Y)\xi,$$
$$\widehat{T}(X, Y) = 2g(X, \varphi Y)\xi.$$

Furthermore, the following was proved in [19].

Proposition 3.1 *The Tanaka–Webster connection $\widehat{\nabla}$ on a contact Riemannian 3-manifold $(M, \varphi, \xi, \eta, g)$ is the unique linear connection satisfying the following conditions:*

- $\widehat{\nabla}\eta = 0$, $\widehat{\nabla}\xi = 0$;
- $\widehat{\nabla}g = 0$, $\widehat{\nabla}\varphi = 0$;
- $\widehat{T}(X, Y) = -\eta([X, Y])\xi$, $X, Y \in \Gamma(D)$;
- $\widehat{T}(\xi, \varphi Y) = -\varphi\widehat{T}(\xi, Y)$, $Y \in \Gamma(D)$.

The Tanaka–Webster connection $\widehat{\nabla}$ of the Bianchi–Cartan–Vranceanu model space is described as

$$\widehat{\nabla}_{u_1} u_1 = c\, y u_2, \quad \widehat{\nabla}_{u_1} u_2 = -c\, y u_1, \quad \widehat{\nabla}_{u_2} u_1 = -c\, x u_2, \quad \widehat{\nabla}_{u_2} u_2 = c\, x u_1;$$

all others are zero.

Here we recall the notion of parabolic geodesic in the sense of Jerison and Lee.

Definition 3.2 ([14]) A regular curve $\gamma \colon I \to M$, defined on some open interval I containing the origin, is a *parabolic geodesic* of a contact strongly pseudo-convex pseudo-Hermitian 3-manifold M if

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(i) $\gamma(0) = p \in M$ and $\gamma'(0) \in D_p$, and
(ii) there is a constant $a \in \mathbb{R}$ so that $\widehat{\nabla}_{\gamma'}\gamma' = 2a\,\xi_{\gamma(t)}$ for any $t \in I$.

Take a tangent vector $W \in T_pM$ orthogonal to ξ_p and define a curve $\sigma(s)$ in T_pM by $\sigma_{W,a}(s) = sW + as^2\xi_p$. Let $\gamma_{W,a}(s)$ be the parabolic geodesic in M with initial condition $\gamma_{W,a}(0) = p$ and $\gamma'_{W,a}(0) = W$. Then the *parabolic exponential map* $\exp_p^D : T_pM \to M$ is defined by

$$\exp_p^D(W + a\xi) = \gamma_{W,a}(1).$$

Jerison and Lee [14] showed that $\exp_p^D$ maps a neighborhood of 0 in T_pM diffeomorphically to a neighborhood of p in M and maps $\sigma_{W,a}$ to $\gamma_{W,a}$. By means of the parabolic exponential map, Jerison and Lee [14] defined a family of natural charts near p called the *pseudo-Hermitian normal coordinates*. Note that the pseudo-Hermitian normal coordinates are normal coordinates in the sense of Folland and Stein [13].

3.3 Parabolic Geodesic Equations

To obtain explicit parametrizations of parabolic geodesics, we use the Bianchi–Cartan–Vranceanu model space $\mathcal{M}^3(H)$. Let $\gamma(s) = (x(s), y(s), z(s))$ be a curve in $\mathcal{M}^3(H)$. Then by using a local orthonormal frame field $\{u_1, u_2, u_3 = \xi\}$ in Section 2.2, we can write

$$\gamma'(s) = T(s) = T_1(s)u_1 + T_2(s)u_2 + T_3(s)u_3.$$

Now we have the parabolic geodesic equation for γ:

$$\widehat{\nabla}_T T = \{T_1' - T_2(cyT_1 - cxT_2)\}u_1 + \{T_2' + T_1(cyT_1 - cxT_2)\}u_2 + T_3'u_3 = 2a\xi.$$

Hence, γ is a parabolic geodesic if and only if

$$\text{(3.1)} \qquad \begin{cases} T_1' - T_2(cyT_1 - cxT_2) = 0, \\ T_2' + T_1(cyT_1 - cxT_2) = 0, \\ T_3' = 2a. \end{cases}$$

From the third equation of (3.1) and the initial condition, it follows that $T_3(s) = 2as$. Let us multiply the first equation in (3.1) by $T_1(s)$ and the second equation by $T_2(s)$, then we get

$$\begin{cases} T_1T_1' - T_1T_2(cyT_1 - cxT_2) = 0, \\ T_2T_2' + T_1T_2(cyT_1 - cxT_2) = 0. \end{cases}$$

Adding these equations, we obtain $T_1T_1' + T_2T_2' = 0$, which is equivalent to $\frac{d}{ds}(T_1(s)^2 + T_2(s)^2) = 0$. This implies that $T_1(s)^2 + T_2(s)^2$ is a non-negative constant, say $b^2 \in \mathbb{R}$. Thus the tangent vector field $T(s) = \gamma'(s)$ has the form

$$\text{(3.2)} \qquad T(s) = b\{\cos\beta(s)u_1 + \sin\beta(s)u_2\} + (2as)u_3,$$

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since $\gamma'(0) \in D_p$.

Inserting (3.2) into the first equation of (3.1), we have

$$b \sin \beta(s) \{ \beta'(s) + bc(y(s) \cos \beta(s) - x(s) \sin \beta(s)) \} = 0. \tag{3.3}$$

Next, inserting (3.2) into the second equation of (3.1), we have

$$b \cos \beta(s) \{ \beta'(s) + bc(y(s) \cos \beta(s) - x(s) \sin \beta(s)) \} = 0. \tag{3.4}$$

Equations (3.3) and (3.4) imply that

$$b \{ \beta'(s) + bc(y(s) \cos \beta(s) - x(s) \sin \beta(s)) \} = 0. \tag{3.5}$$

Hence $b = 0$ or

$$\beta'(s) + bc(y(s) \cos \beta(s) - x(s) \sin \beta(s)) = 0. \tag{3.6}$$

On the other hand, tangent vector field T of γ is also represented as:

$$T = \left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds} \right) = \frac{dx}{ds} \frac{\partial}{\partial x} + \frac{dy}{ds} \frac{\partial}{\partial y} + \frac{dz}{ds} \frac{\partial}{\partial z}.$$

Using the relations:

$$\frac{\partial}{\partial x} = \frac{1}{1 + \frac{c}{2}(x^2 + y^2)}(u_1 + yu_3), \ \frac{\partial}{\partial y} = \frac{1}{1 + \frac{c}{2}(x^2 + y^2)}(u_2 - xu_3), \ \frac{\partial}{\partial z} = u_3,$$

we get

$$\frac{dx}{ds} = \left\{ 1 + \frac{c}{2}(x^2 + y^2) \right\} T_1, \quad \frac{dy}{ds} = \left\{ 1 + \frac{c}{2}(x^2 + y^2) \right\} T_2,$$
$$\frac{dz}{ds} = T_3 - \frac{1}{1 + \frac{c}{2}(x^2 + y^2)} \left(\frac{dx}{ds} y - x \frac{dy}{ds} \right).$$

Hence we obtain the following.

Lemma 3.3 *Let $\gamma \colon I \to M$ be a parabolic geodesic in a Sasakian space form $\mathcal{M}^3(H)$. Then the system of differential equations for γ is as follows:*

$$\frac{dx}{ds}(s) = b \cos \beta(s) \left\{ 1 + \frac{c}{2} \big(x(s)^2 + y(s)^2 \big) \right\}, \tag{3.7}$$

$$\frac{dy}{ds}(s) = b \sin \beta(s) \left\{ 1 + \frac{c}{2} \big(x(s)^2 + y(s)^2 \big) \right\}, \tag{3.8}$$

$$\frac{dz}{ds}(s) = 2as + b \{ x(s) \sin \beta(s) - y(s) \cos \beta(s) \}. \tag{3.9}$$

Here $\beta(s)$ is a solution to (3.5).

Now we determine the parametric equation of a parabolic geodesic.

3.3.1 $b = 0$

In this case, we have $T = (2as)u_3$. The parabolic geodesic $\gamma(s)$ with initial condition $\gamma(0) = (x_0, y_0, z_0) = p$ is given explicitly by $\gamma(s) = (x_0, y_0, as^2 + z_0)$. Note that the initial velocity if γ is $\gamma'(0) = 0 \in D_p$.

3.3.2 $b \neq 0$ and $c = 0$

In this case $\mathcal{M}^3(H)$ is the Heisenberg group Nil_3. Equation (3.6) is simplified as $\beta' = 0$. Namely, β is a constant, say β_0. Thus, from (3.7) and (3.8), the parabolic geodesic starting at $\gamma(0) = (x_0, y_0, z_0) = p$ is given by

$$\begin{aligned} x(s) &= (b\cos\beta_0)s + x_0,\\ y(s) &= (b\sin\beta_0)s + y_0,\\ z(s) &= as^2 + b(x_0\sin\beta_0 - y_0\cos\beta_0)s + z_0. \end{aligned} \tag{3.10}$$

The initial velocity of this parabolic geodesic is $\gamma'(0) = b(\cos\beta_0 u_1 + \sin\beta_0 u_2) \in D_p$. Note that by choosing $b = 0$ in (3.10), we obtain parabolic geodesics discussed in Subsection 3.3.1 for $c = 0$.

3.3.3 $b \neq 0$ and $c \neq 0$

We solve the parabolic geodesic equation under the initial condition

$$(x(0), y(0), z(0)) = (x_0, y_0, z_0).$$

Then together with (3.6), we see that the equation (3.9) becomes

$$\frac{dz}{ds}(s) = 2as + \frac{1}{c}\beta'(s).$$

Thus we have

$$z(s) = as^2 + \frac{1}{c}\beta(s) + \tilde{z}_0,$$

where $\tilde{z}_0$ is a constant defined by $\tilde{z}_0 = z_0 - \beta(0)/c$. We now compute the x- and y-coordinates. We put $h(s) := 1 + \frac{c}{2}\{x(s)^2 + y(s)^2\}$. Then (3.7) and (3.8) become

$$\frac{dx}{ds}(s) = b\cos\beta(s)h(s), \quad \text{and} \quad \frac{dy}{ds}(s) = b\sin\beta(s)h(s), \tag{3.11}$$

respectively.

(i) Subcase-1: $d\beta/ds = 0$.
In this case β is a constant, say β_0. Moreover, the projected curve $(x(s), y(s))$ is a line in the orbit space $\overline{\mathcal{M}}^2$. The z-coordinate is $z(s) = as^2 + z_0$. We have two possibilities.

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- $\cos\beta_0 \neq 0$: In this case, from (3.6) we have $y(s) = \tan\beta_0\, x(s)$, and hence $x(s)$ is a solution to

$$\frac{dx}{ds}(s) = b\cos\beta_0 \left[1 + \frac{c}{2}(\sec^2\beta_0)x(s)^2\right].$$

Thus $x(s)$ is given explictly as follows:

$$x(s) = \sqrt{\frac{2}{c}}\cos\beta_0 \tan\Big(\frac{b\sqrt{c}s}{\sqrt{2}}\Big) + x_0, \quad c > 0,$$

$$x(s) = \sqrt{\frac{2}{-c}}\cos\beta_0 \tanh\Big(\frac{b\sqrt{-c}s}{\sqrt{2}}\Big) + x_0, \quad c < 0.$$

The angle β_0 satisfies $y_0 = \tan\beta_0 x_0$ because of (3.6).
In particular, if $\sin\beta_0 = 0$, then $y = y_0 = 0$ from (3.6) (or (3.1)). The x-coordinate is given by

$$x(s) = \pm\sqrt{\frac{2}{c}}\tan\Big(\frac{b\sqrt{c}s}{\sqrt{2}}\Big) + x_0, \quad c > 0,$$

$$x(s) = \pm\sqrt{\frac{2}{-c}}\tanh\Big(\frac{b\sqrt{-c}s}{\sqrt{2}}\Big) + x_0, \quad c < 0.$$

Note that if we choose $\sin\beta = 0$ in (3.2), then (3.1) implies that $y = 0$.

- $\cos\beta_0 = 0$: In this case we have $T(s) = \pm bu_2 + (2as)u_3$. Then from (3.1), we have $x(s) = x_0 = 0$, and $y(s)$ is a solution to

$$\frac{dy}{ds}(s) = \pm b\Big(1 + \frac{c}{2}y(s)^2\Big).$$

Hence $y(s)$ is given by

$$y(s) = \pm\sqrt{\frac{2}{c}}\tan\Big(\frac{b\sqrt{c}s}{\sqrt{2}}\Big) + y_0, \quad c > 0,$$

$$y(s) = \mp\sqrt{\frac{2}{-c}}\tanh\Big(\frac{b\sqrt{-c}s}{\sqrt{2}}\Big) + y_0, \quad c < 0,$$

The z-coordinate is given by $z(s) = as^2 + z_0$.

(ii) Subcase-2: $d\beta/ds \neq 0$.
Next, we assume that $d\beta/ds(s_0) \neq 0$ for some $s = s_0$. Then we see that $d\beta/ds \neq 0$ nearby $s = s_0$. We note that the function $h(s)$ satisfies the following ordinary differential equation:

$$\frac{d}{ds}\log h(s) = bc\{x(s)\cos\beta(s) + y(s)\sin\beta(s)\}. \tag{3.12}$$

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Differentiating (3.6) and using (3.12), we have

$$\frac{d^2}{ds^2}\beta(s) = \frac{d\beta}{ds}(s)\frac{d}{ds}\log h(s). \tag{3.13}$$

Since $\frac{d\beta}{ds} \neq 0$, from (3.13) we obtain

$$\frac{d\beta}{ds}(s) = rh(s), \quad r \in \mathbb{R}^\times = \mathbb{R} \setminus \{0\}. \tag{3.14}$$

Using (3.11) and (3.14) we obtain

$$\frac{dx}{ds}(s) = \frac{b}{r}\cos\beta(s)\beta'(s), \quad \frac{dy}{ds}(s) = \frac{b}{r}\sin\beta(s)\beta'(s).$$

Hence, the parabolic geodesic $\gamma(s)$ starting at $\gamma(0) = (x_0, y_0, z_0)$ is given by

$$\begin{cases} x(s) = \frac{b}{r}\sin\beta(s) + x_0, \\ y(s) = -\frac{b}{r}\cos\beta(s) + \frac{1}{r} + y_0, \\ z(s) = as^2 + \frac{1}{c}\beta(s) + z_0, \end{cases} \tag{3.15}$$

where $\beta(s)$ is a solution to (3.6) with $\beta(0) = 0$. Inserting (3.15) into (3.14), we get

$$\frac{d\beta}{ds} = \frac{b^2c}{r}(1 - \cos\beta) + bc(x_0\sin\beta - y_0\cos\beta). \tag{3.16}$$

On the other hand, from (3.15), we have

(3.17)
$$\begin{aligned} rh(s) &= r\Big[1 + \frac{c}{2}\big\{x(s)^2 + y(s)^2\big\}\Big] \\ &= r + \frac{b^2c}{r}(1 - \cos\beta) + bc(x_0\sin\beta - y_0\cos\beta) + \frac{rc}{2}\Big\{x_0^2 + y_0^2 + \frac{2b}{r}y_0\Big\}. \end{aligned}$$

Comparing (3.14), (3.16), and (3.17), we obtain the following relation for the initial data (x_0, y_0):

$$1 + \frac{c}{2}\Big\{x_0^2 + \frac{y_0}{r}(2b + ry_0)\Big\} = 0. \tag{3.18}$$

Now we integrate the ordinary differential equation (3.16). The ordinary differential equation (3.16) is rewritten as

$$\int \frac{d\beta}{\frac{b}{r} - \left(\frac{b}{r} + y_0\right)\cos\beta + x_0\sin\beta} = bcs. \tag{3.19}$$

Put $t := \tan(\beta/2)$. Then (3.19) becomes

$$\int \frac{2dt}{\left(\frac{2b+ry_0}{r}\right)t^2 + 2x_0t - y_0} = bcs. \tag{3.20}$$

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- $2b + ry_0 = 0$: In this case (3.18) implies $x_0^2 = -2/c$. Hence $c < 0$. Moreover (3.20) reduces to

$$2\int \frac{dt}{2x_0 t - y_0} = bcs.$$

Hence we obtain

$$\log\left| t - \frac{y_0}{2x_0} \right| = x_0 bcs + C, \quad C \in \mathbb{R}.$$

This formula is rewritten as

$$t = \frac{y_0}{2x_0} + A\exp(x_0 bcs).$$

Here we put $A = \pm e^C$. By the initial condition $\beta(0) = 0$, $A = -\frac{y_0}{2x_0}$. Since $(x_0, y_0) = (\pm\sqrt{2}/\sqrt{-c}, -2b/r)$, we get

$$t = \mp\frac{b\sqrt{-c}}{\sqrt{2}r}\Big\{ 1 - \exp\,(\mp b\sqrt{-2c}\, s)\Big\}.$$

Now we obtain the following formula for $\beta(s)$:

$$\beta(s) = \mp 2\tan^{-1}\left[\frac{b\sqrt{-c}}{\sqrt{2}r}\big\{ 1 - \exp(\mp b\sqrt{-2cs})\big\}\right].$$

- $2b + ry_0 \neq 0$: In this case, (3.20) is computed as

$$\frac{2r}{2b + ry_0}\int \frac{dt}{\left(t + \frac{rx_0}{2b+yr_0}\right)^2 - \frac{r^2(x_0^2+y_0^2)+2bry_0}{(2b+ry_0)^2}} = bcs.$$

By (3.18),

$$\frac{r^2(x_0^2 + y_0^2) + 2bry_0}{(2b + ry_0)^2} = -\frac{2r^2}{c(2b + ry_0)^2}.$$

Thus we have

(3.21) $$\frac{2r}{2b + ry_0}\int \frac{dt}{\left(t + \frac{rx_0}{2b+yr_0}\right)^2 + \frac{2r^2}{c(2b+ry_0)^2}} = bcs.$$

First we consider the case $c > 0$. When $c < 0$, (3.21) is rewritten as

$$\frac{2r}{2b + ry_0}\int \frac{dt}{\left(t + \frac{rx_0}{2b+yr_0}\right)^2 + \left(\frac{\sqrt{2}r}{\sqrt{c}(2b+ry_0)}\right)^2} = bcs.$$

Solving this ODE, we obtain

$$t = -\frac{rx_0}{2b + ry_0} + \frac{\sqrt{2}r}{\sqrt{c}(2b + ry_0)}\tan\left\{\frac{b\sqrt{c}s}{\sqrt{2}} + C\right\}$$

for some constant C. By the initial condition $\beta(0) = 0$, the constant C is determined as $C = \tan^{-1}(\sqrt{c}x_0/\sqrt{2})$. Hence the function $\beta(s)$ is given explicitly by

$$\beta(s) = 2\tan^{-1}\left[-\frac{rx_0}{2b+ry_0} + \frac{\sqrt{2}r}{\sqrt{c}(2b+ry_0)}\tan\left\{\frac{b\sqrt{c}s}{\sqrt{2}} + \tan^{-1}\left(\frac{\sqrt{c}x_0}{\sqrt{2}}\right)\right\}\right].$$

For the case $c < 0$, one can show that $\beta(s)$ is given explictly by

$$\beta(s) = 2\tan^{-1}\left[-\frac{rx_0}{2b+ry_0} + \frac{\sqrt{2}r}{\sqrt{-c}(2b+ry_0)}\tanh\left\{\frac{b\sqrt{-c}s}{\sqrt{2}} + \tanh^{-1}\left(\frac{\sqrt{-c}x_0}{\sqrt{2}}\right)\right\}\right].$$

Remark 3.4 If we look for parabolic geodesics starting at $(0, -b/r, 0)$, we have

$$\begin{cases} x(s) = \frac{b}{r}\sin\beta(s), \\ y(s) = -\frac{b}{r}\cos\beta(s), \\ z(s) = as^2 + \frac{1}{c}\beta(s). \end{cases}$$

with $\beta(0) = 0$. In this case, we get $h(s) = 1 + \frac{b^2c}{2r^2}$ and $\beta' = b^2c/r$. From (3.18), we have $b^2c = 2r^2$. Hence we obtain

$$\beta(s) = \frac{b^2c}{r}s = 2rs.$$

Now we arrive at our main theorems.

Theorem 3.5 *The parametric equations of all parabolic geodesics in the Heisenberg group $\mathcal{M}^3(-3)$ with initial condition $(x(0), y(0), z(0)) = (x_0, y_0, z_0)$ are given by*

$$\begin{cases} x(s) = (b\cos\beta_0)s + x_0, \\ y(s) = (b\sin\beta_0)s + y_0, \\ z(s) = as^2 + b(x_0\sin\beta_0 - y_0\cos\beta_0)s + z_0, \end{cases}$$

where b and β_0 are constants.

Here we give a geometric interpretation of this result. To this end, we recall the group structure (2.3) of the Heisenberg group. Let us define a curve $\gamma_0(s)$ by

$$\gamma_0(s) = (b\cos\beta_0\, s, b\sin\beta_0\, s, as^2).$$

Then γ_0 is a parabolic geodesic starting at the origin $(0, 0, 0)$. Take a point $p = (x_0, y_0, z_0) \in \mathrm{Nil}_3$. Then Theorem 3.5 implies that the parabolic geodesic $\gamma(s)$ starting at p is given by $\gamma(s) = p \cdot \gamma_0(s)$. Namely, $\gamma(s)$ is a left translation of $\gamma_0(s)$ by p.

Corollary 3.6 *Every parabolic geodesic in* Nil_3 *is obtained as a left translation of a parabolic geodesic starting at the origin.*

Remark 3.7 R. Caddeo, C. Oniciuc, and P. Piu [7] classified all unit speed curves in Nil_3 which are *biharmonic* with respect to the metric g_0. In particular, they showed that every proper biharmonic curve in Nil_3 is a helix. Moreover, every proper biharmonic helix starting at p is obtained from a proper biharmonic helix starting at the origin by means of a left translation. For the classification of proper biharmonic curves in Sasakian 3-space forms, we refer to [6, 10].

Theorem 3.8 *Let* $\mathcal{M}^3(H)$ *be the Bianchi–Cartan–Vranceanu model space of constant holomorphic sectional curvature* $H = -3 + 2c$ *with* $c \neq 0$. *Then the parametric equations of all parabolic geodesics in* $\mathcal{M}^3(H)$ *starting at* (x_0, y_0, z_0) *are one of the following types:*

(i) *A vertical line through* (x_0, y_0, z_0); $\gamma(s) = (x_0, y_0, as^2 + z_0)$.

(ii) $\gamma(s) = (x(s), \tan\beta_0 x(s), as^2 + z_0)$, *where* β_0 *is a constant such that* $\cos\beta_0 \neq 0$. *The x-coordinate is given by*

$$x(s) = \sqrt{\frac{2}{c}}\cos\beta_0 \tan\Big(\frac{b\sqrt{c}s}{\sqrt{2}}\Big) + x_0, \quad c > 0,$$

$$x(s) = \sqrt{\frac{2}{-c}}\cos\beta_0 \tanh\Big(\frac{b\sqrt{-c}s}{\sqrt{2}}\Big) + x_0, \quad c < 0.$$

The constant β_0 *satisfies* $y_0 = \tan\beta_0 x_0$.

(iii) $x_0 = 0$ *and* $\gamma(s) = (0, y(s), as^2 + z_0)$, *where*

$$y(s) = \pm\sqrt{\frac{2}{c}}\tan\Big(\frac{b\sqrt{c}s}{\sqrt{2}}\Big) + y_0, \quad c > 0,$$

$$y(s) = \mp\sqrt{\frac{2}{-c}}\tanh\Big(\frac{b\sqrt{-c}s}{\sqrt{2}}\Big) + y_0, \quad c < 0,$$

(iv) $\gamma(s) = \big(\frac{1}{r}\sin\beta(s) + x_0, -\frac{1}{r}\cos\beta(s) + \frac{1}{r} + y_0, as^2 + \frac{1}{c}\beta(s) + z_0\big)$, *where* $\beta(s)$ *is one of the following functions:*

- $y_0 \neq -2b/r$ *and* $c > 0$:

$$\beta(s) = 2\tan^{-1}\Bigg[-\frac{rx_0}{2b + ry_0} + \frac{\sqrt{2}r}{\sqrt{c}(2b + ry_0)}\tan\Big\{\frac{b\sqrt{c}s}{\sqrt{2}} + \tan^{-1}\Big(\frac{\sqrt{c}x_0}{\sqrt{2}}\Big)\Big\}\Bigg],$$

- $y_0 \neq -2b/r$ *and* $c < 0$:

$$\beta(s) = 2\tan^{-1}\Bigg[-\frac{rx_0}{2b + ry_0} + \frac{\sqrt{2}r}{\sqrt{-c}(2b + ry_0)}\tanh\Big\{\frac{b\sqrt{-c}s}{\sqrt{2}} + \tanh^{-1}\Big(\frac{\sqrt{-c}x_0}{\sqrt{2}}\Big)\Big\}\Bigg],$$

- $y_0 = -2b/r$:

$$\beta(s) = \mp 2\tan^{-1}\Bigg[\frac{b\sqrt{-c}}{\sqrt{2}r}\Big\{1 - \exp\big(\mp b\sqrt{-2c}s\big)\Big\}\Bigg].$$

In case $y_0 = -2b/r$, x_0 *is given by* $x_0 = \pm\sqrt{2}/\sqrt{-c}$ *and* $c < 0$.

Remark 3.9 Let $\gamma(s)$ be a regular curve in a 3-dimensional contact strongly pseudo-convex pseudo-Hermitian manifold. The *contact angle* $\alpha(s)$ is the angle function between the Reeb vector field ξ and the tangent vector field $\gamma'(s)$ of $\gamma(s)$. Namely, $\alpha(s)$ is defined by the formula:

$$\cos\alpha(s) = \frac{\eta(\gamma'(s))}{|\gamma'(s)|^2}.$$

A regular curve $\gamma(s)$ is said to be a *slant curve* if its contact angle is constant ([9]). In our previous work [12], the following result was obtained.

Proposition 3.10 *Let* $\gamma\colon I \to M$ *be a unit speed slant curve in a Sasakian* 3*-space form. Then the acceleration vector field* $\widehat{\nabla}_{\gamma'}\gamma'$ *with respect to the Tanaka–Webster connection is orthogonal to* ξ *everywhere.*

Note that every $\widehat{\nabla}$-geodesic in a Sasakian 3-space form is a slant curve. Moreover, one can see that every biharmonic unit speed curve in a Sasakian 3-space form, and $\mathcal{M}^3(H)$ is a slant helix (see [7, 9, 10]).

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