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ABSTRACT

By use of a natural extension map and a power series method, we obtain a local stability theorem for p -Kähler structures with the $(p, p + 1)$ th mild $\partial\bar{\partial}$ -lemma under small differentiable deformations.

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1. Introduction

Local stabilities of complex structures are important topics in deformation theory of complex structures. We will prove local stabilities of p -Kähler structures with the $(p, p + 1)$ th mild $\partial\bar{\partial}$ -lemma by the power series method, initiated by Kodaira–Nirenberg–Spencer [KNS58] and Kuranishi [Kur64].

THEOREM 1.1. *For any positive integer $p \leq n - 1$, any small differentiable deformation X_t of an n -dimensional p -Kähler manifold X_0 satisfying the $(p, p + 1)$ th mild $\partial\bar{\partial}$ -lemma is still p -Kählerian.*

Here the $(p, p + 1)$ th mild $\partial\bar{\partial}$ -lemma for a complex manifold means that each $\bar{\partial}$ -closed ∂ -exact $(p, p + 1)$ -form on this manifold is $\partial\bar{\partial}$ -exact, which is a new notion generalizing the $(n - 1, n)$ th one first introduced in [RWZ16]. A complex manifold is p -Kählerian if it admits a p -Kähler form, i.e., a d -closed transverse (p, p) -form as in Definition 2.5.

Recall the fact that each n -dimensional complex manifold is n -Kählerian and the following two basic properties of p -Kählerian structures.

LEMMA 1.2 ([AA87, Proposition 1.15] and also [RWZ16, Corollary 4.6]). *A complex manifold M is 1-Kähler if and only if M is Kähler; an n -dimensional complex manifold M is $(n - 1)$ -Kähler if and only if M is balanced, i.e., it admits a real positive $(1, 1)$ -form ω , satisfying*

$$d(\omega^{n-1}) = 0.$$

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Thus, we obtain the following as a direct corollary of Theorem 1.1.

COROLLARY 1.3. *Let $\pi : \mathcal{X} \rightarrow B$ be a differentiable family of compact complex manifolds.*

- (i) [KS60, Theorem 15] *If a fiber $X_0 := \pi^{-1}(t_0)$ admits a Kähler metric, then, for a sufficiently small neighborhood U of t_0 on B , the fiber $X_t := \pi^{-1}(t)$ over any point $t \in U$ still admits a Kähler metric, which depends smoothly on t and coincides for $t = t_0$ with the given Kähler metric on X_0 .*
- (ii) [RWZ16, Theorem 1.5] *Let X_0 be a balanced manifold of complex dimension n , satisfying the $(n - 1, n)$ th mild $\partial\bar{\partial}$ -lemma. Then X_t also admits a balanced metric for t small.*

The first assertion of Corollary 1.3 is the fundamental Kodaira–Spencer’s local stability theorem of Kähler structure, and motivates the second assertion of Corollary 1.3 and many other related works on local stabilities of complex structures in [FY11, Voi02, Wu06, AU17, AU16]. The counter-example of Alessandrini and Bassanelli [AB90] tells us that the result in the second assertion of Corollary 1.3 does not necessarily hold without the $(n - 1, n)$ th mild $\partial\bar{\partial}$ -lemma assumption.

In §2, we will study the difference between the (p, q) th mild $\partial\bar{\partial}$ -lemma and other versions of $\partial\bar{\partial}$ -lemmata in the roles of Theorem 1.1, and the modification stability of the (p, q) th mild $\partial\bar{\partial}$ -lemma by Proposition 3.14, which provides us with more classes of complex manifolds to admit the (p, q) th mild $\partial\bar{\partial}$ -lemma. Here the (standard) $\partial\bar{\partial}$ -lemma refers to: for every pure-type d -closed form on a complex manifold, the properties of d -exactness, ∂ -exactness, $\bar{\partial}$ -exactness and $\partial\bar{\partial}$ -exactness are equivalent, while its variants are described by §3.1. Obviously, one has the implication hierarchy on a complex n -dimensional manifold for any positive integer $p \leq n - 1$:

$$\begin{aligned} & \text{the } \partial\bar{\partial}\text{-lemma} \\ \implies & \text{the } (p, p + 1)\text{th strong } \partial\bar{\partial}\text{-lemma} & (1) \\ \implies & \text{the } (p, p + 1)\text{th mild } \partial\bar{\partial}\text{-lemma} & (2) \\ \implies & \text{the } (p, p + 1)\text{th weak } \partial\bar{\partial}\text{-lemma.} & (3) \end{aligned}$$

For $p = n - 1$, the implication hierarchy is strict: [AU17, Example 4.10] is one example satisfying the strong $\partial\bar{\partial}$ -lemma but not the standard one; the nilmanifold endowed with a left-invariant abelian complex structure of dimension $2n$ or a left-invariant non-nilpotent balanced complex structure of complex dimension 3 by [RWZ16, Proposition 3.8 and Corollary 3.4] and [AU16, Proposition 2.9] distinguishes the mild and strong $\partial\bar{\partial}$ -lemmata; and the weak $\partial\bar{\partial}$ -lemma holds on the complex three-dimensional Iwasawa manifold [RWZ16, Example 3.7] but the mild one fails. Moreover, we construct a new ten-dimensional balanced nilmanifold in Example 3.8 for the strictness of implication 2, which satisfies the $(4, 5)$ th mild but not strong $\partial\bar{\partial}$ -lemma and also the deformation variance of the $(4, 4)$ th Bott–Chern numbers. Motivated by these, it is natural to ask the following question.

Question 1.4. Find an n -dimensional complex manifold or in particular a p -Kähler manifold such that one of implications (1), (2), (3) is strict for each positive integer $p < n - 1$.

Now let us describe our approach to proving local stability of p -Kähler structures. An application of Kuranishi’s completeness theorem [Kur64] reduces our power series proof to the Kuranishi family $\varpi : \mathcal{K} \rightarrow T$, that is, we will construct a natural p -Kähler extension $\tilde{\omega}_t$ of the p -Kähler form ω_0 on X_0 , such that $\tilde{\omega}_t$ is a p -Kähler form on the general fiber $\varpi^{-1}(t) = X_t$. More precisely, the extension is given by

$$e^{\iota_\varphi|_{\iota_{\bar{\varphi}}}} : A^{p,p}(X_0) \rightarrow A^{p,p}(X_t), \quad \omega_0 \rightarrow \tilde{\omega}_t := e^{\iota_\varphi|_{\iota_{\bar{\varphi}}}}(\omega(t)),$$

where $\omega(t)$ is a family of smooth (p, p) -forms to be constructed on X_0 , depending smoothly on t , and $\omega(0) = \omega_0$. Here φ is the family of Beltrami differentials induced by the Kuranishi family. The extension map $e^{t\varphi|_{\bar{\varphi}}}$ is first introduced in [ZR15, RZ18] and given in Definition 2.2.

This method is developed in [LSY09, Sun12, SY11, LRY15, ZR13, ZR15, RZ18, RWZ16, LRW17]. However, we have to solve many more equations in system (11) here than in the balanced case [RWZ16]; those in system (11) are much more difficult in essence. Fortunately, we are able to reduce this complicated system to that with only two equations as in (15) by comparing the types of the forms in the system and the orders in the induction simultaneously. This crucial consideration is also important in the solution of this system.

In this approach, we will use the following observation crucially.

PROPOSITION 1.5 [RWZ16, Proposition 4.12]. *Let $\pi : \mathcal{X} \rightarrow B$ be a differentiable family of compact complex n -dimensional manifolds and Ω_t a family of real (p, p) -forms with $p < n$, depending smoothly on t . Assume that Ω_0 is a transverse (p, p) -form on X_0 . Then Ω_t is also transverse on X_t for small t .*

This proposition actually shows that any smooth real extension of a transverse (p, p) -form is still transverse. So the obstruction to extend a d -closed transverse (p, p) -form on a compact complex manifold lies in the d -closedness, to be resolved in Theorem 1.6 in a more general setting. The detailed proof of main Theorem 1.1 is given in § 4.

THEOREM 1.6 (= Theorem 4.1). *If X_0 satisfies the $(p, q + 1)$ th and $(q, p + 1)$ th mild $\partial\bar{\partial}$ -lemmata, then there is a d -closed (p, q) -form $\Omega(t)$ on X_t depending smoothly on t with $\Omega(0) = \Omega_0$ for any d -closed $\Omega_0 \in A^{p,q}(X_0)$.*

Remark 1.7. The case $p = q = n - 1$ of Theorem 1.6 implies that the dimension of the space of d -closed left-invariant $(n - 1, n - 1)$ -forms on a $2n$ -dimensional nilmanifold endowed with a left-invariant abelian complex structure is deformation invariant, where the $(n - 1, n)$ th mild $\partial\bar{\partial}$ -lemma holds from [RWZ16, Corollary 3.4].

In § 5, inspired by [RZ18], we will use Theorem 1.6 to prove a result on deformation invariance of Bott–Chern numbers in Theorem 5.1.

This paper will follow the notation in [LRY15, RZ18, RWZ16]. All manifolds in this paper are assumed to be compact complex n -dimensional manifolds. The symbol $A^{p,q}(X, E)$ stands for the space of the holomorphic vector bundle E -valued (p, q) -forms on a complex manifold X . We will always consider the differentiable family $\pi : \mathcal{X} \rightarrow B$ of compact complex n -dimensional manifolds over a sufficiently small domain in \mathbb{R}^k with the reference fiber $X_0 := \pi^{-1}(0)$ for the reference point 0 and the general fibers $X_t := \pi^{-1}(t)$.

2. Deformation and p -Kähler structure

This section is to state some basics of analytic deformation theory of complex structures and the notion of p -Kähler structure.

2.1 Deformation theory

For the holomorphic family of compact complex manifolds, we adopt the definition [Kod86, Definition 2.8]; while for the differentiable one, we adopt the following definition.

DEFINITION 2.1 [Kod86, Definition 4.1]. Let \mathcal{X} be a differentiable manifold, B a domain of \mathbb{R}^k and π a smooth map of \mathcal{X} onto B . By a *differentiable family of n -dimensional compact complex manifolds* we mean the triple $\pi : \mathcal{X} \rightarrow B$ satisfying the following conditions.

- (i) The rank of the Jacobian matrix of π is equal to k at every point of \mathcal{X} .
- (ii) For each point $t \in B$, $\pi^{-1}(t)$ is a compact connected subset of \mathcal{X} .
- (iii) The fiber $\pi^{-1}(t)$ is the underlying differentiable manifold of the n -dimensional compact complex manifold X_t associated to each $t \in B$.
- (iv) There is a locally finite open covering $\{\mathcal{U}_j \mid j = 1, 2, \dots\}$ of \mathcal{X} and complex-valued smooth functions $\zeta_j^1(p), \dots, \zeta_j^n(p)$, defined on \mathcal{U}_j such that for each t ,

$$\{p \rightarrow (\zeta_j^1(p), \dots, \zeta_j^n(p)) \mid \mathcal{U}_j \cap \pi^{-1}(t) \neq \emptyset\}$$

form a system of local holomorphic coordinates of X_t .

Beltrami differentials play an important role in deformation theory. A *Beltrami differential* on X , generally denoted by ϕ , is an element in $A^{0,1}(X, T_X^{1,0})$, where $T_X^{1,0}$ is the holomorphic tangent bundle of X . Then ι_ϕ or $\phi \lrcorner$ denotes the contraction operator with respect to $\phi \in A^{0,1}(X, T_X^{1,0})$ or other analogous vector-valued complex differential forms alternatively if there is no confusion. We also use the convention

$$e^\spadesuit = \sum_{k=0}^\infty \frac{1}{k!} \spadesuit^k, \tag{4}$$

where \spadesuit^k denotes k -time action of the operator \spadesuit . As the dimension of X is finite, the summation in the above formulation is always finite.

We will always consider the differentiable family $\pi : \mathcal{X} \rightarrow B$ of compact complex n -dimensional manifolds over a sufficiently small domain in \mathbb{R}^k with the reference fiber $X_0 := \pi^{-1}(0)$ and the general fibers $X_t := \pi^{-1}(t)$. For simplicity we set $k = 1$. Denote by $\zeta := (\zeta_j^\alpha(z, t))$ the holomorphic coordinates of X_t induced by the family with the holomorphic coordinates $z := (z^i)$ of X_0 , under a coordinate covering $\{\mathcal{U}_j\}$ of \mathcal{X} , when t is assumed to be fixed, as the standard notions in deformation theory described at the beginning of [MK71, ch. 4]. This family induces a canonical differentiable family of integrable Beltrami differentials on X_0 , denoted by $\varphi(z, t)$, $\varphi(t)$ and φ interchangeably.

In [ZR15, ZR13], the first and third authors introduced an extension map

$$e^{\iota_{\varphi(t)} \overline{\iota_{\varphi(t)}}} : A^{p,q}(X_0) \rightarrow A^{p,q}(X_t),$$

to play an important role in this paper.

DEFINITION 2.2. For $s \in A^{p,q}(X_0)$, we define

$$e^{\iota_{\varphi(t)} \overline{\iota_{\varphi(t)}}}(s) = s_{i_1 \dots i_p j_1 \dots j_q}(z(\zeta))(e^{\iota_{\varphi(t)}}(dz^{i_1} \wedge \dots \wedge dz^{i_p})) \wedge (e^{\overline{\iota_{\varphi(t)}}}(d\bar{z}^{j_1} \wedge \dots \wedge d\bar{z}^{j_q})),$$

where s is locally written as

$$s = s_{i_1 \dots i_p j_1 \dots j_q}(z) dz^{i_1} \wedge \dots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \dots \wedge d\bar{z}^{j_q}$$

and the operators $e^{\iota_{\varphi(t)}}$, $e^{\overline{\iota_{\varphi(t)}}}$ follow convention (4). It is easy to check that this map is a real linear isomorphism as in [RZ18, Lemma 2.8].

The following proposition is crucial in this paper.

PROPOSITION 2.3 [LRY15, Theorem 3.4], [RZ18, Proposition 2.2]. *Let $\phi \in A^{0,1}(X, T_X^{1,0})$ on a complex manifold X . Then on the space $A^{*,*}(X)$,*

$$d \circ e^{\iota\phi} = e^{\iota\phi}(d + \partial \circ \iota_\phi - \iota_\phi \circ \partial - \iota_{\bar{\partial}\phi - (1/2)[\phi, \phi]}). \tag{5}$$

From the proof of Proposition 2.3, we see that (5) is a natural generalization of the Tian–Todorov lemma [Tia87, Tod89], whose variants appeared in [Fri91, BK98, Li05, LSY09, Cle05] and also [LR12, LRY15] for vector bundle valued forms.

LEMMA 2.4. *For $\phi, \psi \in A^{0,1}(X, T_X^{1,0})$ and $\alpha \in A^{*,*}(X)$ on an n -dimensional complex manifold X ,*

$$[\phi, \psi] \lrcorner \alpha = -\partial(\psi \lrcorner (\phi \lrcorner \alpha)) - \psi \lrcorner (\phi \lrcorner \partial \alpha) + \phi \lrcorner \partial(\psi \lrcorner \alpha) + \psi \lrcorner \partial(\phi \lrcorner \alpha),$$

where

$$[\phi, \psi] := \sum_{i,j=1}^n (\phi^i \wedge \partial_i \psi^j + \psi^i \wedge \partial_i \phi^j) \otimes \partial_j$$

for $\varphi = \sum_i \varphi^i \otimes \partial_i$ and $\psi = \sum_i \psi^i \otimes \partial_i$.

2.2 The p -Kähler structures

Let V be a complex n -dimensional vector space with its dual space V^* , i.e., the space of complex linear functionals over V . Denote the complexified space of the exterior m -vectors of V^* by $\bigwedge_{\mathbb{C}}^m V^*$, which admits a natural direct sum decomposition

$$\bigwedge_{\mathbb{C}}^m V^* = \sum_{r+s=m} \bigwedge^{r,s} V^*,$$

where $\bigwedge^{r,s} V^*$ denotes the complex vector space of (r, s) -forms on V^* . The case $m = 1$ exactly reads

$$\bigwedge_{\mathbb{C}}^1 V^* = V^* \oplus \overline{V^*},$$

where the natural isomorphism $V^* \cong \bigwedge^{1,0} V^*$ is used. Let $q \in \{1, \dots, n\}$ and $p = n - q$. Clearly, the complex dimension N of $\bigwedge^{q,0} V^*$ is equal to the combination number C_n^q . After a basis $\{\beta_i\}_{i=1}^N$ of the complex vector space $\bigwedge^{q,0} V^*$ is fixed, the canonical Plücker embedding as in [GH78, p. 209] is given by

$$\begin{aligned} \rho: G(q, n) &\hookrightarrow \mathbb{P}\left(\bigwedge^{q,0} V^*\right) \\ \Lambda &\mapsto [\dots, \Lambda_i, \dots]. \end{aligned}$$

Here $G(q, n)$ denotes the Grassmannian of q -planes in the vector space V^* and $\mathbb{P}(\bigwedge^{q,0} V^*)$ is the projectivization of $\bigwedge^{q,0} V^*$. A q -plane in V^* can be represented by a decomposable $(q, 0)$ -form $\Lambda \in \bigwedge^{q,0} V^*$ up to a nonzero complex number, and $\{\Lambda_i\}_{i=1}^N$ are exactly the coordinates of Λ under the fixed basis $\{\beta_i\}_{i=1}^N$. *Decomposable $(q, 0)$ -forms* are those forms in $\bigwedge^{q,0} V^*$ that can be expressed as $\gamma_1 \wedge \dots \wedge \gamma_q$ with $\gamma_i \in V^* \cong \bigwedge^{1,0} V^*$ for $1 \leq i \leq q$. Set

$$k = (N - 1) - pq$$

to be the codimension of $\rho(G(q, n))$ in $\mathbb{P}(\bigwedge^{q,0} V^*)$, whose locus characterizes the decomposable $(q, 0)$ -forms in $\mathbb{P}(\bigwedge^{q,0} V^*)$.

Now we list notation for several types of positivity and refer the readers to [HK74, Har75, Dem12] for more details. A (q, q) -form Θ in $\bigwedge^{q,q} V^*$ is defined to be *strictly positive* (respectively, *positive*) if

$$\Theta = \sigma_q \sum_{i,j=1}^N \Theta_{i\bar{j}} \beta_i \wedge \bar{\beta}_j,$$

where Θ_{ij} is a positive (respectively, semi-positive) hermitian matrix of size $N \times N$ with $N = C_n^q$ under the basis $\{\beta_i\}_{i=1}^N$ of the complex vector space $\bigwedge^{q,0} V^*$ and σ_q is defined to be the constant $2^{-q}(\sqrt{-1})^{q^2}$. According to this definition, the fundamental form of a hermitian metric on a complex manifold is actually a strictly positive $(1, 1)$ -form everywhere. A (p, p) -form $\Gamma \in \bigwedge^{p,p} V^*$ is called *weakly positive* if the volume form

$$\Gamma \wedge \sigma_q \tau \wedge \bar{\tau}$$

is positive for every nonzero decomposable $(q, 0)$ -form τ of V^* , while a (q, q) -form $\Upsilon \in \bigwedge^{q,q} V^*$ is said to be *strongly positive* if Υ is a convex combination

$$\Upsilon = \sum_s \gamma_s \sqrt{-1} \alpha_{s,1} \wedge \bar{\alpha}_{s,1} \wedge \cdots \wedge \sqrt{-1} \alpha_{s,q} \wedge \bar{\alpha}_{s,q},$$

where $\alpha_{s,i} \in V^*$ and $\gamma_s \geq 0$. As shown in [Dem12, ch. III.§1.A], the sets of weakly positive and strongly positive forms are closed convex cones, and by definition, the weakly positive cone is dual to the strongly positive cone via the pairing

$$\bigwedge^{p,p} V^* \times \bigwedge^{q,q} V^* \longrightarrow \mathbb{C}.$$

Then all weakly positive forms are real. An element Ξ in $\bigwedge^{p,p} V^*$ is called *transverse*, if the volume form

$$\Xi \wedge \sigma_q \tau \wedge \bar{\tau}$$

is strictly positive for every nonzero decomposable $(q, 0)$ -form τ of V^* . There exist many names for this terminology and we refer to [AB91, Appendix] for a list.

This positivity notation on complex vector spaces can be extended pointwise to complex differential forms on a complex manifold. Let M be an n -dimensional complex manifold. Then we have the following definition.

DEFINITION 2.5 ([AA87, Definition 1.11], for example). Let p be an integer, $1 \leq p \leq n$. Then M is called a *p-Kähler manifold* if there exists a *p-Kähler form*, that is a d -closed transverse (p, p) -form on M .

The readers are referred to [Sul76] for more related concepts (such as differential form transversal to the cone structure on a real differentiable manifold) to p -Kähler structures.

3. Relevance to mild $\partial\bar{\partial}$ -lemma and modification

We will introduce the so-called (p, q) th mild $\partial\bar{\partial}$ -lemma and its relevance, and also present its modification stability on compact complex manifolds.

3.1 The (p, q) th mild $\partial\bar{\partial}$ -lemma and its relevance

This subsection is to study various $\partial\bar{\partial}$ -lemmata related to local stabilities of complex structures, their properties, differences and roles there in some special case. More details can be found in [RWZ16, § 3.1] and the references therein.

Now we introduce a new notion.

DEFINITION 3.1. We say a complex manifold X satisfies the (p, q) th mild $\partial\bar{\partial}$ -lemma if for any complex differential $(p - 1, q)$ -form ξ with $\partial\bar{\partial}\xi = 0$ on X , there exists a $(p - 1, q - 1)$ -form θ such that $\partial\bar{\partial}\theta = \partial\xi$.

So we can state our main theorem.

THEOREM 3.2. For any positive integer $p \leq n - 1$, any small differentiable deformation X_t of a p -Kähler manifold X_0 satisfying the $(p, p + 1)$ th mild $\partial\bar{\partial}$ -lemma is still p -Kählerian.

According to Lemma 1.2, Theorem 3.2 unifies the local stabilities of Kähler structures [KS60, Theorem 15] and balanced structures under the $(n - 1, n)$ th mild $\partial\bar{\partial}$ -lemma [RWZ16, Theorem 1.5], which is an obvious generalization of Wu’s result [Wu06, Theorem 5.13] that the balanced structure is stable under small deformation when the $\partial\bar{\partial}$ -lemma holds, to p -Kähler mild $\partial\bar{\partial}$ -structures for $1 \leq p \leq n - 1$.

Let X be a compact complex manifold of complex dimension n with the following commutative diagram.

$$\begin{array}{ccccc}
 & & H_{\partial}^{p,q}(X) & & \\
 & \nearrow \iota_{BC,\partial}^{p,q} & & \searrow \iota_{\partial,A}^{p,q} & \\
 H_{BC}^{p,q}(X) & \xrightarrow{\iota_{BC,dR}^{p,q}} & H_{dR}^{p+q}(X) & \xrightarrow{\iota_{dR,A}^{p,q}} & H_A^{p,q}(X) \\
 & \searrow \iota_{BC,\bar{\partial}}^{p,q} & & \nearrow \iota_{\bar{\partial},A}^{p,q} & \\
 & & H_{\bar{\partial}}^{p,q}(X) & &
 \end{array} \tag{6}$$

Recall that Dolbeault cohomology groups $H_{\bar{\partial}}^{\bullet,\bullet}(X)$ of X are defined by

$$H_{\bar{\partial}}^{\bullet,\bullet}(X) := \frac{\ker \bar{\partial}}{\text{im } \bar{\partial}},$$

with $H_{\partial}^{\bullet,\bullet}(X)$ similarly defined, while Bott–Chern and Aeppli cohomology groups are defined as

$$H_{BC}^{\bullet,\bullet}(X) := \frac{\ker \partial \cap \ker \bar{\partial}}{\text{im } \partial\bar{\partial}} \quad \text{and} \quad H_A^{\bullet,\bullet}(X) := \frac{\ker \partial\bar{\partial}}{\text{im } \partial + \text{im } \bar{\partial}},$$

respectively. The dimensions of $H_{dR}^{p+q}(X)$, $H_{\bar{\partial}}^{p,q}(X)$, $H_{BC}^{p,q}(X)$, $H_A^{p,q}(X)$ and $H_{\partial}^{p,q}(X)$ over \mathbb{C} are denoted by $b_{p+q}(X)$, $h_{\bar{\partial}}^{p,q}(X)$, $h_{BC}^{p,q}(X)$, $h_A^{p,q}(X)$ and $h_{\partial}^{p,q}(X)$, respectively, and the first four of them are usually called $(p + q)$ th Betti numbers, (p, q) -Hodge numbers, Bott–Chern numbers and Aeppli numbers, respectively. So the (standard) $\partial\bar{\partial}$ -lemma is equivalent to the injectivities of the mappings

$$\iota_{BC,dR}^{p,q} : H_{BC}^{p,q}(X) \rightarrow H_{dR}^{p+q}(X)$$

for all p, q , or to the isomorphisms of all the maps in diagram (6) by [DGMS75, Remark 5.16].

Notice that the $(1, 2)$ th mild $\partial\bar{\partial}$ -lemma is different from the $\partial\bar{\partial}$ -lemma on a complex manifold. It is easy to see that the $(1, 2)$ th mild $\partial\bar{\partial}$ -lemma amounts to the injectivity of the mapping

$$\iota_{\text{BC},\partial}^{1,2} : H_{\text{BC}}^{1,2}(X) \rightarrow H_{\partial}^{1,2}(X).$$

Then by [AK17, Tables 5 and 6 in Appendix A] and [Kas13, the case B in Example 1], we have the following example.

Example 3.3 [RWZ16, Example 1.7]. Let X be the manifold in case (ii) of the completely-solvable Nakamura manifold as given in [AK17, Example 3.1]. Then the manifold X satisfies the $(1, 2)$ th mild $\partial\bar{\partial}$ -lemma, but not the $\partial\bar{\partial}$ -lemma.

There are another three similar conditions relating to the local stabilities of complex structures. The $(p, p + 1)$ th weak $\partial\bar{\partial}$ -lemma on a compact complex manifold X , first introduced by Fu and Yau [FY11] for $(p, p + 1) = (n - 1, n)$, says that for any real (p, p) -form ψ such that $\bar{\partial}\psi$ is ∂ -exact, there is a $(p - 1, p)$ -form θ , satisfying

$$\partial\bar{\partial}\theta = \bar{\partial}\psi.$$

And the (p, q) th strong $\partial\bar{\partial}$ -lemma on X , first proposed by Angella–Ugarte [AU17] in the case $(p, q) = (n - 1, n)$, states that the induced mapping $\iota_{\text{BC},A}^{p,q} : H_{\text{BC}}^{p,q}(X) \rightarrow H_A^{p,q}(X)$ by the identity map is injective, which is equivalent to the statement that for any d -closed (p, q) -form Γ of the type $\Gamma = \partial\xi + \bar{\partial}\psi$, there exists a $(p - 1, q - 1)$ -form θ such that

$$\partial\bar{\partial}\theta = \Gamma.$$

Angella–Ugarte [AU17, Proposition 4.8] showed the deformation openness of the $(n - 1, n)$ th strong $\partial\bar{\partial}$ -lemma. Besides, the condition that the induced mapping $\iota_{\text{BC},\bar{\partial}}^{p,q} : H_{\text{BC}}^{p,q}(X) \rightarrow H_{\bar{\partial}}^{p,q}(X)$ by the identity map is injective, is first presented by Angella–Ugarte [AU16] in the case $(p, q) = (n - 1, n)$ to study local conformal balanced structures and global ones, which we may call the (p, q) th dual mild $\partial\bar{\partial}$ -lemma.

After a simple check, we have the following observation.

Observation 3.4. The compact complex manifold X satisfies the (p, q) th strong $\partial\bar{\partial}$ -lemma if and only if both of the mild and dual mild ones hold on X .

All these four ‘ $\partial\bar{\partial}$ -lemmata’ hold if the compact complex manifold X satisfies the standard $\partial\bar{\partial}$ -lemma. And either the $(p, p + 1)$ th mild or dual mild $\partial\bar{\partial}$ -lemma implies the weak one, while [RWZ16, Corollary 3.9] implies that the $(n - 1, n)$ th mild $\partial\bar{\partial}$ -lemma and the dual mild one are unrelated.

By [AB90], a small deformation of the Iwasawa manifold, which satisfies the $(2, 3)$ th weak $\partial\bar{\partial}$ -lemma but does not satisfy the mild one from Example 3.5, may not be balanced. Thus, the condition ‘ $(n - 1, n)$ th mild $\partial\bar{\partial}$ -lemma’ in Corollary 1.3(ii) cannot be replaced by the weak one.

Example 3.5 [RWZ16, Example 3.7]. The complex structure in category (i) of [UV15, Proposition 2.3], i.e., the complex parallelizable case of complex dimension 3, satisfies the $(2, 3)$ th weak $\partial\bar{\partial}$ -lemma and the dual mild one, but does not satisfy the mild one. The Iwasawa manifold belongs to category (i).

The next example shows that neither the $(n - 1, n)$ th weak $\partial\bar{\partial}$ -lemma nor the mild one is deformation open. And it shows that the condition in [FY11, Theorem 6] is not a necessary one for the deformation openness of balanced structures as mentioned in [UV15, the discussion ahead of Example 3.7]. Recall that [FY11, Theorem 6] says that the balanced structure is deformation open, if the $(n - 1, n)$ th weak $\partial\bar{\partial}$ -lemma holds on the general fibers X_t for $t \neq 0$. Fortunately, Corollary 1.3(ii) can be applied to this example. See also [AU17, Remark 4.7], where Corollary 1.3(ii) can also be applied.

Example 3.6 [UV15, Example 3.7]. Ugarte–Villacampa constructed an explicit family of nilmanifolds with left-invariant complex structures I_λ for $\lambda \in [0, 1)$ (of complex dimension 3), with the fixed underlying manifold the Iwasawa manifold. The complex structure of the reference fiber I_0 is abelian and admits a left-invariant balanced metric, satisfying the $(2, 3)$ th mild $\partial\bar{\partial}$ -lemma by [RWZ16, Proposition 3.8]. The complex structures of I_λ for $\lambda \neq 0$ are nilpotent from [CFP06, Corollary 2], but neither complex-parallelizable nor abelian. Thus, they do not satisfy the $(2, 3)$ th weak $\partial\bar{\partial}$ -lemma by [UV15, Proposition 3.6]. However, the nilmanifolds I_λ for $\lambda \neq 0$ admit balanced metrics.

Meanwhile, a $2n$ -dimensional nilmanifold endowed with a left-invariant abelian complex structure satisfies the $(n - 1, n)$ th mild $\partial\bar{\partial}$ -lemma but never satisfies the $(n - 1, n)$ th dual mild $\partial\bar{\partial}$ -lemma. It shows that the deformation openness of balanced structures with the reference fiber a nilmanifold endowed with a left-invariant abelian balanced Hermitian structure is easily obtained by Corollary 1.3(ii), but not from [AU17, Theorem 4.9], which says that if X_0 admits a locally conformal balanced metric and satisfies the $(n - 1, n)$ th strong $\partial\bar{\partial}$ -lemma, then X_t is balanced for small t .

Moreover, the deformation invariance of the $(n - 1, n - 1)$ th Bott–Chern numbers $h_{BC}^{n-1, n-1}(X_t)$ can assure the deformation openness of balanced structures as shown in [AU17, Proposition 4.1]. Inspired by Wu’s result [Wu06, Theorem 5.13], we have the following generalization.

THEOREM 3.7 [RWZ16, Theorem 1.9]. *For any positive integer $p \leq n - 1$, any small differentiable deformation X_t of a p -Kähler manifold X_0 satisfying the deformation invariance of (p, p) -Bott–Chern numbers is still p -Kählerian.*

Nevertheless, Corollary 1.3(ii) may be applied to some cases with deformation variance of the $(n - 1, n - 1)$ th Bott–Chern numbers. The newly constructed example that follows is one case among nilmanifolds, while the manifold in [AU17, Example 4.10], satisfying the $(2, 3)$ th strong $\partial\bar{\partial}$ -lemma, is a solvable manifold but not a nilmanifold by [RWZ16, Corollary 3.9].

Example 3.8. Let G be the simply connected nilpotent Lie group determined by a ten-dimensional 3-step nilpotent Lie algebra \mathfrak{g} endowed with a left-invariant abelian complex structure J , satisfying the structure equation

$$\begin{cases} d\gamma^1 = d\gamma^2 = d\gamma^3 = 0, \\ d\gamma^4 = \gamma^{1\bar{3}}, \\ d\gamma^5 = \gamma^{3\bar{4}}, \end{cases}$$

where the natural decomposition with respect to J yields

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g}_J^{1,0} \oplus \mathfrak{g}_J^{0,1}; \quad \mathfrak{g}_{\mathbb{C}}^* = \mathfrak{g}^* \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g}_J^{*(1,0)} \oplus \mathfrak{g}_J^{*(0,1)},$$

$\{\gamma^i\}_{i=1}^5$ is the basis of $\mathfrak{g}^{*(1,0)}$ and the convention $\gamma^{1\bar{3}} = \gamma^1 \wedge \overline{\gamma^3}$ is used here and afterwards. Define a lattice Γ in G , determined by the rational span of $\{\gamma^i, \bar{\gamma}^i\}_{i=1}^5$. Then $M := \Gamma \backslash G$ is a compact

nilmanifold with the abelian complex structure J given above. It is easy to check that

$$\Omega = \gamma^{1234\overline{1234}} + \gamma^{1235\overline{1235}} + \gamma^{1245\overline{1245}} + \gamma^{1345\overline{1345}} + \gamma^{2345\overline{2345}}$$

is a left-invariant balanced metric on (\mathfrak{g}, J) , descending to M . Denote the basis of $\mathfrak{g}^{1,0}$ dual to $\{\gamma^i\}_{i=1}^5$ by $\{\theta_i\}_{i=1}^5$. The equation $d\omega(\theta, \theta') = -\omega([\theta, \theta'])$ for $\omega \in \mathfrak{g}^*$ and $\theta, \theta' \in \mathfrak{g}_{\mathbb{C}}$, establishes the equalities

$$[\bar{\theta}_3, \theta_1] = \theta_4, \quad [\bar{\theta}_4, \theta_3] = \theta_5.$$

According to [CFP06, Theorem 3.6], the linear operator $\bar{\partial}$ on $\mathfrak{g}^{1,0}$, defined in [CFP06, § 3.2], amounts to

$$\bar{\partial} : \mathfrak{g}^{1,0} \rightarrow \mathfrak{g}^{*(0,1)} \otimes \mathfrak{g}^{1,0} : \bar{\partial}V = \bar{\gamma}^i \otimes [\bar{\theta}_i, V]^{1,0} \quad \text{for } V \in \mathfrak{g}^{1,0},$$

which induces an isomorphism $H^1(M, T_M^{1,0}) \cong H_{\bar{\partial}}^1(\mathfrak{g}^{1,0})$. Therefore, from Kodaira–Spencer’s deformation theory, an analytic deformation M_t of M can be constructed by use of the integrable left-invariant Beltrami differential

$$\varphi(t) = (t_1\bar{\gamma}^4 + t_2\bar{\gamma}^5) \otimes \theta_2 + (t_3\bar{\gamma}^4 + t_4\bar{\gamma}^5) \otimes \theta_5$$

for $t = (t_1, t_2, t_3, t_4)$ and $|t_4| < 1$, which satisfies $\bar{\partial}\varphi(t) = \frac{1}{2}[\varphi(t), \varphi(t)]$ and the so-called Schouten–Nijenhuis bracket $[\cdot, \cdot]$ (cf. [CFP06, Formula (4.1)]) works as

$$[\bar{\gamma} \otimes \theta, \bar{\gamma}' \otimes \theta'] = \bar{\gamma}' \wedge \iota_{\theta'} d\bar{\gamma} \otimes \theta + \bar{\gamma} \wedge \iota_{\theta} d\bar{\gamma}' \otimes \theta' \quad \text{for } \gamma, \gamma' \in \mathfrak{g}^{*(1,0)}, \theta, \theta' \in \mathfrak{g}^{1,0}.$$

Then the general fibers M_t are still nilmanifolds, determined by the Lie algebra \mathfrak{g} with respect to the decompositions

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g}_{\varphi(t)}^{1,0} \oplus \mathfrak{g}_{\varphi(t)}^{0,1}; \quad \mathfrak{g}_{\mathbb{C}}^* = \mathfrak{g}^* \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g}_{\varphi(t)}^{*(1,0)} \oplus \mathfrak{g}_{\varphi(t)}^{*(0,1)},$$

where the basis of $\mathfrak{g}_{\varphi(t)}^{*(1,0)}$ is given by $\gamma^i(t) = e^{t\varphi(t)}(\gamma^i) = (\mathbb{1} + \varphi(t)) \lrcorner \gamma^i$ for $1 \leq i \leq 5$. Hence, the structure equation of $\{\gamma^i(t)\}_{i=1}^5$ is

$$\begin{cases} d\gamma^1(t) = d\gamma^3(t) = 0, \\ d\gamma^2(t) = -t_1\gamma^{3\bar{1}}(t) - t_2\gamma^{4\bar{3}}(t), \\ d\gamma^4(t) = \gamma^{1\bar{3}}(t), \\ d\gamma^5(t) = \gamma^{3\bar{4}}(t) - t_3\gamma^{3\bar{1}}(t) - t_4\gamma^{4\bar{3}}(t), \end{cases}$$

where $\gamma^{3\bar{1}}(t)$ denotes $\gamma^3(t) \wedge \overline{\gamma^1(t)}$, similarly for others. It is well known from [CF01, Rol09, Ang13] that the Bott–Chern cohomologies of nilmanifolds with abelian complex structures and their small deformation can be calculated via left-invariant differential forms. Remark 1.7 tells us that the dimension of the space of the d -closed left-invariant $(4, 4)$ -forms is invariant along the deformation M_t , which is equal to 21. And one can calculate the $\partial\bar{\partial}$ -exact terms directly by use of the structure equation

$$\begin{aligned} & \partial_t \bar{\partial}_t (\mathfrak{g}_{\varphi(t)}^{*(3,3)}) \\ &= \langle -(1 + |t_4|^2)\gamma^{1234\overline{1234}}(t) - |t_2|^2\gamma^{1345\overline{1345}} + t_2\bar{t}_4\gamma^{1345\overline{1234}}(t) + t_4\bar{t}_2\gamma^{1234\overline{1345}}(t), \\ & \quad -\gamma^{1235\overline{1235}}(t) - |t_1|^2\gamma^{1345\overline{1345}}(t) - |t_3|^2\gamma^{1234\overline{1234}}(t) + t_1\bar{t}_3\gamma^{1345\overline{1234}}(t) + t_3\bar{t}_1\gamma^{1234\overline{1345}}(t), \\ & \quad -t_1\gamma^{1345\overline{1234}}(t) + t_2\gamma^{1345\overline{1235}}(t) + t_3\gamma^{1234\overline{1234}}(t) - t_4\gamma^{1234\overline{1235}}(t), \\ & \quad -\bar{t}_1\gamma^{1234\overline{1345}}(t) + \bar{t}_2\gamma^{1235\overline{1345}}(t) + \bar{t}_3\gamma^{1234\overline{1234}}(t) - \bar{t}_4\gamma^{1235\overline{1234}}(t) \rangle. \end{aligned}$$

It is clear that $\dim \partial\bar{\partial}(\mathfrak{g}_J^{*(3,3)}) = 2$ and $\dim \partial_t \bar{\partial}_t (\mathfrak{g}_{\varphi(t)}^{*(3,3)}) = 4$ for general t . Therefore, the Bott–Chern number $h_{\text{BC}}^{4,4}(M_t)$ varies from 19 to 17 along the deformation M_t .

It may not be difficult to find an example of a non-Kähler p -Kähler manifold in the Fujiki class for $1 < p < n - 1$ in the literature, and thus it satisfies the $(p, p + 1)$ th mild $\partial\bar{\partial}$ -lemma, for example [AB92, § 4]. However, motivated by Example 3.8, we ask the following question.

Question 3.9. Is it possible to find an n -dimensional nilmanifold with a left-invariant complex structure of complex dimension n , which admits a left-invariant p -Kähler metric for $1 < p < n - 1$ and satisfies the $(p, p + 1)$ th mild $\partial\bar{\partial}$ -lemma, but the (p, p) -Bott–Chern number varies along some deformation? This example would not satisfy the standard $\partial\bar{\partial}$ -lemma.

Finally, from the perspective of Corollary 1.3(ii), we may have a clear picture of Angella–Ugarte’s result [AU17, Theorem 4.9]. Actually, Observation 3.4 tells us that the $(n - 1, n)$ th strong $\partial\bar{\partial}$ -lemma decomposes into the mild one and the dual mild one. A locally conformal balanced metric can be transformed into a balanced one by the $(n - 1, n)$ th dual mild $\partial\bar{\partial}$ -lemma, from [AU16, Theorem 2.5]. Then the $(n - 1, n)$ th mild $\partial\bar{\partial}$ -lemma assures the deformation openness of balanced structures originally from the transformed balanced metric on the reference fiber, thanks to Corollary 1.3(ii).

3.2 Modification stabilities

We consider the modification on a compact complex manifold defined as follows and refer the reader to [Uen75, § 2] for its general definition on complex spaces.

DEFINITION 3.10. A *modification* of an n -dimensional compact complex manifold M is a holomorphic map

$$\mu : \tilde{M} \rightarrow M$$

so that:

- (i) \tilde{M} is also an n -dimensional compact complex manifold;
- (ii) there exists an analytic subset $S \subseteq M$ of codimension greater than or equal to 1 such that $\mu|_{\tilde{M} \setminus \mu^{-1}(S)} : \tilde{M} \setminus \mu^{-1}(S) \rightarrow M \setminus S$ is a biholomorphism.

It is a classical result, [Par66] or [DGMS75, Theorem 5.22], that if the modification of a complex manifold is a $\partial\bar{\partial}$ -manifold, then so is this manifold. Therefore, each compact complex manifold in the *Fujiki class* \mathcal{C} (i.e. admitting a Kähler modification) is a $\partial\bar{\partial}$ -manifold. The converse is an open question as in [Ale17, Introduction]: Is the modification of a $\partial\bar{\partial}$ -manifold still a $\partial\bar{\partial}$ -manifold? A recent result [YY17, Theorem 1.3] of S. Yang and X. Yang, by means of a blow-up formula for Bott–Chern cohomologies and the characterizations by Angella and Tomassini [AT13] and Angella and Tardini [AT17] of $\partial\bar{\partial}$ -manifolds, and also [RYY17, Main Theorem 1.1] by S. Yang, X. Yang and the first author confirm this question in dimension three, that is, the modification of a $\partial\bar{\partial}$ -threefold is still a $\partial\bar{\partial}$ -threefold. See also, more recently, [ASTT17, Theorem 2.1]. These results provide us with more classes of complex manifolds satisfying mild $\partial\bar{\partial}$ -lemmata. Moreover, it is natural to ask the following analogous question.

Question 3.11. Does the modification of a complex manifold satisfying the mild (p, q) th $\partial\bar{\partial}$ -lemma still satisfy the mild (p, q) th $\partial\bar{\partial}$ -lemma for each p, q ?

Now we present a modification stability of (p, q) th mild $\partial\bar{\partial}$ -lemma on a compact complex manifold. Let M be a complex manifold. One has the \mathbb{K} -valued de Rham complex $(A^\bullet(M)_{\mathbb{K}}, d)$

for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Fixing a d -closed 1-form $\phi \in A^1(M)_{\mathbb{K}}$, we consider another complex. Namely, define

$$d_\phi = d + L_\phi,$$

where $L_\phi := \phi \wedge \bullet$. The cochain complex

$$(A^\bullet(M)_{\mathbb{K}}, d_\phi)$$

can be regarded as the de Rham complex with values in the topologically trivial flat bundle $M \times \mathbb{K}$ with the connection form ϕ . We study cohomologies and Hodge theory for general complex manifolds with twisted differentials. More precisely, for $\theta_1, \theta_2 \in H_{\text{BC}}^{1,0}(M)$, consider the bi-differential \mathbb{Z} -graded complex

$$(A^\bullet(M)_{\mathbb{C}}, \partial_{(\theta_1, \theta_2)}, \bar{\partial}_{(\theta_1, \theta_2)}),$$

where

$$\begin{aligned} \partial_{(\theta_1, \theta_2)} &:= \partial + L_{\theta_2} + L_{\bar{\theta}_1}, \\ \bar{\partial}_{(\theta_1, \theta_2)} &:= \bar{\partial} - L_{\bar{\theta}_2} + L_{\theta_1}. \end{aligned}$$

It is easy to check that

$$\partial_{(\theta_1, \theta_2)} \bar{\partial}_{(\theta_1, \theta_2)} = \bar{\partial}_{(\theta_1, \theta_2)} \partial_{(\theta_1, \theta_2)} = \partial_{(\theta_1, \theta_2)} \bar{\partial}_{(\theta_1, \theta_2)} + \bar{\partial}_{(\theta_1, \theta_2)} \partial_{(\theta_1, \theta_2)} = 0.$$

Angella and Kasuya [AK14] investigated cohomological properties of this bi-differential complex and considered (more than) two cohomologies:

$$H^\bullet(A^\bullet(M)_{\mathbb{C}}; \partial_{(\theta_1, \theta_2)}, \bar{\partial}_{(\theta_1, \theta_2)}; \partial_{(\theta_1, \theta_2)} \bar{\partial}_{(\theta_1, \theta_2)}) := \frac{\ker \partial_{(\theta_1, \theta_2)} \cap \ker \bar{\partial}_{(\theta_1, \theta_2)}}{\text{im } \partial_{(\theta_1, \theta_2)} \bar{\partial}_{(\theta_1, \theta_2)}}$$

and

$$H^\bullet(A^\bullet(M)_{\mathbb{C}}; \partial_{(\theta_1, \theta_2)}) := \frac{\ker \partial_{(\theta_1, \theta_2)}}{\text{im } \partial_{(\theta_1, \theta_2)}},$$

which are simply denoted by $H_{\text{BC}}^\bullet(M, \theta_1, \theta_2)$ and $H_{\partial}^\bullet(M, \theta_1, \theta_2)$, respectively. If one sets $\theta_1 = \theta_2 = 0$, $H_{\text{BC}}^\bullet(M, \theta_1, \theta_2)$ and $H_{\partial}^\bullet(M, \theta_1, \theta_2)$ are just the ordinary Bott–Chern cohomology $H_{\text{BC}}^{\bullet, \bullet}(M)$ and $H_{\partial}^{\bullet, \bullet}(M)$ of M , respectively.

Following Wells in [Wel74, Theorem 3.1], Angella and Kasuya proved the following proposition.

PROPOSITION 3.12 [AK14, Theorem 2.4]. *Let $\mu : \tilde{M} \rightarrow M$ be a modification of a compact complex manifold M . Then the induced maps*

$$\begin{aligned} \mu_{\text{BC}}^* &: H_{\text{BC}}^\bullet(M, \theta_1, \theta_2) \rightarrow H_{\text{BC}}^\bullet(\tilde{M}, \mu^* \theta_1, \mu^* \theta_2), \\ \mu_{\partial}^* &: H_{\partial}^\bullet(M, \theta_1, \theta_2) \rightarrow H_{\partial}^\bullet(\tilde{M}, \mu^* \theta_1, \mu^* \theta_2) \end{aligned}$$

are injective.

We reformulate Definition 3.1 for the (p, q) th mild $\partial\bar{\partial}$ -lemma as follows.

DEFINITION 3.13. For any positive integers $p, q \leq n$, an n -dimensional complex manifold X satisfies the (p, q) th mild $\partial\bar{\partial}$ -lemma if the induced map $\iota_{\text{BC}, \partial}^{p, q} : H_{\text{BC}}^{p, q}(X) \rightarrow H_{\partial}^{p, q}(X)$ by the identity map is injective.

PROPOSITION 3.14. *With the notation in Proposition 3.12, if \tilde{M} satisfies the (p, q) th mild $\partial\bar{\partial}$ -lemma, then so does M .*

Proof. Taking $\theta_1 = \theta_2 = 0$, we have the following commutative diagram.

$$\begin{array}{ccc} H_{\text{BC}}^{p,q}(M) & \xrightarrow{\mu_{\text{BC}}^*} & H_{\text{BC}}^{p,q}(\tilde{M}) \\ \downarrow \iota_{\text{BC},\partial}^{p,q} & & \downarrow \iota_{\text{BC},\partial}^{p,q} \\ H_{\partial}^{p,q}(M) & \xrightarrow{\mu_{\partial}^*} & H_{\partial}^{p,q}(\tilde{M}) \end{array}$$

By the (p, q) th mild $\partial\bar{\partial}$ -lemma assumption on \tilde{M} , the map $\iota_{\text{BC},\partial}^{p,q}$ for \tilde{M} is injective and so are $\mu_{\text{BC}}^*, \mu_{\partial}^*$ by Proposition 3.12. So the map $\iota_{\text{BC},\partial}^{p,q}$ for M is injective, i.e., M satisfies the (p, q) th mild $\partial\bar{\partial}$ -lemma. □

4. Power series proof of main result

This section is used to prove main Theorem 3.2. Let us sketch Kodaira–Spencer’s proof of the local stability theorem [KS60]. Let F_t be the orthogonal projection to the kernel \mathbb{F}_t of the *first 4th order Kodaira–Spencer operator* (also often called *Bott–Chern Laplacian*)

$$\square_{\text{BC},t} = \partial_t \bar{\partial}_t \bar{\partial}_t^* \partial_t^* + \bar{\partial}_t^* \partial_t^* \partial_t \bar{\partial}_t + \bar{\partial}_t^* \partial_t \partial_t^* \bar{\partial}_t + \partial_t^* \bar{\partial}_t \bar{\partial}_t^* \partial_t + \bar{\partial}_t^* \bar{\partial}_t + \partial_t^* \partial_t \tag{7}$$

and \mathbb{G}_t the corresponding Green’s operator with respect to α_t on X_t . Here

$$\alpha_t = \sqrt{-1} g_{i\bar{j}}(\zeta, t) d\zeta^i \wedge d\bar{\zeta}^{\bar{j}}$$

is a hermitian metric on X_t depending differentiably on t with α_0 being a Kähler metric on X_0 , and $\bar{\partial}_t^*$ (respectively, ∂_t^*) is the dual of $\bar{\partial}_t$ (respectively, ∂_t) with respect to α_t . By a cohomological argument with the upper semi-continuity theorem, they prove that F_t and \mathbb{G}_t depend differentiably on t . Then they can construct the desired Kähler metric on X_t as

$$\tilde{\alpha}_t = \frac{1}{2}(F_t \alpha_t + \overline{F_t \alpha_t}).$$

See also [Voi02, §9.3].

Our proof is quite different. As explained in §1, to prove Theorem 3.2, it suffices to prove the special case $p = q$ of the following theorem.

THEOREM 4.1. *If X_0 satisfies the $(p, q + 1)$ th and $(q, p + 1)$ th mild $\partial\bar{\partial}$ -lemmata, then there is a d -closed (p, q) -form $\Omega(t)$ on X_t depending smoothly on t with $\Omega(0) = \Omega_0$ for any d -closed $\Omega_0 \in A^{p,q}(X_0)$.*

We first reduce the local stability Theorem 4.1 to the Kuranishi family since the family of Beltrami differentials induced by this Kuranishi family plays an important role in the construction of the family of d -closed (p, q) -forms $\Omega(t)$.

4.1 Kuranishi family and Beltrami differentials

We introduce some basics on the Kuranishi family of complex structures in this subsection, which is extracted from [RZ18, RWZ16] and obviously originally from [Kur64].

By (the proof of) Kuranishi’s completeness theorem [Kur64], for any compact complex manifold X_0 , there exists a complete holomorphic family $\varpi : \mathcal{K} \rightarrow T$ of complex manifolds at the reference point $0 \in T$ in the sense that for any differentiable family $\pi : \mathcal{X} \rightarrow B$ with $\pi^{-1}(s_0) = \varpi^{-1}(0) = X_0$, there exist a sufficiently small neighborhood $E \subseteq B$ of s_0 , and smooth maps $\Phi : \mathcal{X}_E \rightarrow \mathcal{K}$, $\tau : E \rightarrow T$ with $\tau(s_0) = 0$ such that the diagram

$$\begin{array}{ccc} \mathcal{X}_E & \xrightarrow{\Phi} & \mathcal{K} \\ \pi \downarrow & & \downarrow \varpi \\ (E, s_0) & \xrightarrow{\tau} & (T, 0) \end{array}$$

commutes, Φ maps $\pi^{-1}(s)$ biholomorphically onto $\varpi^{-1}(\tau(s))$ for each $s \in E$, and

$$\Phi : \pi^{-1}(s_0) = X_0 \rightarrow \varpi^{-1}(0) = X_0$$

is the identity map. This family is called *Kuranishi family* and is constructed as follows. Let $\{\eta_\nu\}_{\nu=1}^m$ be a base for $\mathbb{H}^{0,1}(X_0, T_{X_0}^{1,0})$, where some suitable hermitian metric is fixed on X_0 and $m \geq 1$. Otherwise the complex manifold X_0 would be *rigid*, i.e., for any differentiable family $\kappa : \mathcal{M} \rightarrow P$ with $s_0 \in P$ and $\kappa^{-1}(s_0) = X_0$, there is a neighborhood $V \subseteq P$ of s_0 such that $\kappa : \kappa^{-1}(V) \rightarrow V$ is trivial. Then one can construct a holomorphic family

$$\varphi(t) = \sum_{|I|=1}^{\infty} \varphi_I t^I := \sum_{j=1}^{\infty} \varphi_j(t), \quad I = (i_1, \dots, i_m), t = (t_1, \dots, t_m) \in \mathbb{C}^m,$$

for $|t| < \rho$ a small positive constant, of Beltrami differentials as follows:

$$\varphi_1(t) = \sum_{\nu=1}^m t_\nu \eta_\nu$$

and for $|I| \geq 2$,

$$\varphi_I = \frac{1}{2} \bar{\partial}^* \mathbb{G} \sum_{J+L=I} [\varphi_J, \varphi_L].$$

It is clear that $\varphi(t)$ satisfies the equation

$$\varphi(t) = \varphi_1 + \frac{1}{2} \bar{\partial}^* \mathbb{G} [\varphi(t), \varphi(t)].$$

Let

$$T = \{t \mid \mathbb{H}[\varphi(t), \varphi(t)] = 0\}.$$

So for each $t \in T$, $\varphi(t)$ satisfies

$$\bar{\partial} \varphi(t) = \frac{1}{2} [\varphi(t), \varphi(t)],$$

and determines a complex structure X_t on the underlying differentiable manifold of X_0 . More importantly, $\varphi(t)$ represents the complete holomorphic family $\varpi : \mathcal{K} \rightarrow T$ of complex manifolds. Roughly speaking, the Kuranishi family $\varpi : \mathcal{K} \rightarrow T$ contains all sufficiently small differentiable deformations of X_0 .

By means of these, one can reduce the local stability Theorem 4.1 to the Kuranishi family by shrinking E if necessary, that is, it suffices to construct a p -Kähler metric on each X_t . From now on, we use $\varphi(t)$ and φ interchangeably to denote this holomorphic family of integrable Beltrami differentials, and assume $m = 1$ for simplicity.

4.2 Obstruction equation and construction of power series

Now we begin to prove the d -closed smooth extension of (p, q) -forms as in Theorem 4.1 by using the power series method.

As both $e^{\iota(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}$ and $e^{\iota\varphi}$ are invertible operators when t is sufficiently small, it follows that for any $\Omega \in A^{p,q}(X_0)$,

$$e^{\iota\varphi|\iota\bar{\varphi}}(\Omega) = e^{\iota\varphi} \circ e^{\iota(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}} \circ e^{-\iota(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}} \circ e^{-\iota\varphi} \circ e^{\iota\varphi|\iota\bar{\varphi}}(\Omega). \tag{8}$$

Set

$$\tilde{\Omega} = e^{-\iota(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}} \circ e^{-\iota\varphi} \circ e^{\iota\varphi|\iota\bar{\varphi}}(\Omega), \tag{9}$$

where Ω and $\tilde{\Omega}$ apparently have a one-to-one correspondence. Here we follow the notation: $\bar{\varphi}\varphi = \varphi \lrcorner \bar{\varphi}$, $\mathbb{1}$ is the identity operator defined as

$$\mathbb{1} = \frac{1}{p+q} \left(\sum_i^n dz^i \otimes \frac{\partial}{\partial z^i} + \sum_i^n d\bar{z}^i \otimes \frac{\partial}{\partial \bar{z}^i} \right)$$

when it acts on (p, q) -forms of a complex manifold, and similarly for others. And it is easy to check that the operator

$$e^{-\iota(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}} \circ e^{-\iota\varphi} \circ e^{\iota\varphi|\iota\bar{\varphi}}$$

preserves the form types and thus $\tilde{\Omega}$ is still a (p, q) -form. In fact, for any (p, q) -form α on X_0 , we will find

$$\begin{aligned} & e^{-\iota(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}} \circ e^{-\iota\varphi} \circ e^{\iota\varphi|\iota\bar{\varphi}}(\alpha) \\ &= \alpha_{i_1 \dots i_p \bar{j}_1 \dots \bar{j}_q} dz^{i_1} \wedge \dots \wedge dz^{i_p} \wedge (\mathbb{1} - \bar{\varphi}\varphi) \lrcorner d\bar{z}^{j_1} \wedge \dots \wedge (\mathbb{1} - \bar{\varphi}\varphi) \lrcorner d\bar{z}^{j_q} \in A^{p,q}(X_0), \end{aligned}$$

where $\alpha = \alpha_{i_1 \dots i_p \bar{j}_1 \dots \bar{j}_q} dz^{i_1} \wedge \dots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \dots \wedge d\bar{z}^{j_q}$. Together with (8) and (9), Proposition 2.3 implies that

$$\begin{aligned} d(e^{\iota\varphi|\iota\bar{\varphi}}(\Omega)) &= d \circ e^{\iota\varphi} \circ e^{\iota(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}(\tilde{\Omega}) \\ &= e^{\iota\varphi} \circ (\bar{\partial} + [\partial, \iota\varphi] + \partial) \circ e^{\iota(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}(\tilde{\Omega}) \\ &= e^{\iota\varphi} (\bar{\partial}_\varphi + \partial) \sum_{k=0}^{+\infty} A_k \\ &= e^{\iota\varphi} \left(\bar{\partial}_\varphi A_0 + \sum_{k=0}^{+\infty} (\partial A_k + \bar{\partial}_\varphi A_{k+1}) \right), \end{aligned} \tag{10}$$

where

$$A_k := \frac{(\iota(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi})^k}{k!}(\tilde{\Omega})$$

is a $(p+k, q-k)$ -form and

$$\bar{\partial}_\varphi := \bar{\partial} + [\partial, \iota\varphi].$$

Thus, $d(e^{\iota\varphi|\iota\bar{\varphi}}(\Omega)) = 0$ amounts to

$$\bar{\partial}_\varphi A_0 = 0, \quad \partial A_k + \bar{\partial}_\varphi A_{k+1} = 0, \quad k = 0, 1, 2, \dots$$

PROPOSITION 4.2. For any given (p, q) -form Ω ,

$$d(e^{\iota_\varphi|_{\iota_{\bar{\varphi}}}}(\Omega)) = 0$$

is equivalent to

$$\sum_{k=0}^{\infty} \left(\bar{\partial} \circ \frac{\iota_\varphi^{k-1}}{(k-1)!} + \partial \circ \frac{\iota_\varphi^k}{k!} \right) A_{n-p-(l-k)} = 0, \tag{11}$$

where $\max\{1, n - p - q\} \leq l \leq \min\{2n - p - q, n + 1\}$, $\iota_\varphi^k = 0$ for $k < 0$ and $0! = 1$.

Proof. Recall that $\tilde{\Omega} = e^{-\iota_{(\mathbb{1}-\bar{\varphi}\varphi)}^{-1}\bar{\varphi}} \circ e^{-\iota_\varphi} \circ e^{\iota_\varphi|_{\iota_{\bar{\varphi}}}}(\Omega)$ and then

$$\begin{aligned} d(e^{\iota_\varphi|_{\iota_{\bar{\varphi}}}}(\Omega)) &= d \circ e^{\iota_\varphi} \circ e^{\iota_{(\mathbb{1}-\bar{\varphi}\varphi)}^{-1}\bar{\varphi}}(\tilde{\Omega}) \\ &= (\bar{\partial} \circ e^{\iota_\varphi} \circ e^{\iota_{(\mathbb{1}-\bar{\varphi}\varphi)}^{-1}\bar{\varphi}} + \partial \circ e^{\iota_\varphi} \circ e^{\iota_{(\mathbb{1}-\bar{\varphi}\varphi)}^{-1}\bar{\varphi}})(\tilde{\Omega}) \\ &= \sum_{k_1, k_2=0}^{+\infty} \left(\bar{\partial} \circ \frac{\iota_\varphi^{k_1}}{k_1!} + \partial \circ \frac{\iota_\varphi^{k_1}}{k_1!} \right) \circ \frac{\iota_{(\mathbb{1}-\bar{\varphi}\varphi)}^{-1}\bar{\varphi}^{k_2}}{k_2!}(\tilde{\Omega}). \end{aligned}$$

Note that the part of degree $(+(n - p - l + 1), -(n - p - l))$ in the operator $d \circ e^{\iota_\varphi} \circ e^{\iota_{(\mathbb{1}-\bar{\varphi}\varphi)}^{-1}\bar{\varphi}}$ is

$$\sum_{k=0}^{\infty} \left(\bar{\partial} \circ \frac{\iota_\varphi^{k-1}}{(k-1)!} + \partial \circ \frac{\iota_\varphi^k}{k!} \right) \circ \frac{\iota_{(\mathbb{1}-\bar{\varphi}\varphi)}^{-1}\bar{\varphi}^{n-p-l+k}}{(n-p-l+k)!}(\tilde{\Omega}) \tag{12}$$

since

$$\begin{aligned} (+ (n - p - l + 1), - (n - p - l)) &= (n - p - l + k)(1, -1) + (k - 1)(-1, 1) + (0, 1) \\ &= (n - p - l + k)(1, -1) + k(-1, 1) + (1, 0). \end{aligned}$$

This is exactly the left-hand side of (11). So $d(e^{\iota_\varphi|_{\iota_{\bar{\varphi}}}}(\Omega)) = 0$ is equivalent to the vanishing of (12) for each l such that

$$(p, q) + (+ (n - p - l + 1), - (n - p - l)) \in [0, n] \times [0, n],$$

i.e., $\max\{1, n - p - q\} \leq l \leq \min\{2n - p - q, n + 1\}$. □

Remark 4.3. We consider two special cases of Proposition 4.2.

(i) For $p = q = n - 1$, (11) is reduced to

$$\begin{cases} \partial A_0 + (\bar{\partial} + \partial \circ \iota_\varphi) A_1 = 0, \\ (\bar{\partial} + \partial \circ \iota_\varphi) A_0 + (\bar{\partial} \circ \iota_\varphi + \frac{1}{2} \partial \circ \iota_\varphi^2) A_1 = 0, \end{cases}$$

which is exactly the system of obstruction equations given in [RWZ16, (3.8)].

(ii) For $p = q = 1$, (11) is reduced to

$$\begin{cases} \partial \Omega - \partial \circ \iota_{\bar{\varphi}\varphi}(\Omega) + (\bar{\partial} + \partial \circ \iota_\varphi) \circ \iota_{\bar{\varphi}}(\Omega) = 0, \\ (\bar{\partial} + \partial \circ \iota_\varphi)(\Omega - \iota_{\bar{\varphi}\varphi}(\Omega)) + (\bar{\partial} \circ \iota_\varphi + \frac{1}{2} \partial \circ \iota_\varphi^2) \iota_{\bar{\varphi}}(\Omega) = 0, \\ \partial \circ \iota_{\bar{\varphi}}(\Omega) = 0. \end{cases}$$

Since $-\iota_{\bar{\varphi}}\varphi + \iota_{\varphi} \circ \iota_{\bar{\varphi}} = -\iota_{\varphi}\bar{\varphi} + \iota_{\bar{\varphi}} \circ \iota_{\varphi}$ and $2\iota_{\varphi} \circ \iota_{\bar{\varphi}}\varphi\alpha = \iota_{\varphi}^2 \circ \iota_{\bar{\varphi}}(\alpha)$ for $(1, 1)$ -form α as shown in [RWZ16, Proposition 2.6], we have

$$\begin{cases} \bar{\partial}\Omega = \bar{\partial} \circ (\iota_{\bar{\varphi}}\varphi - \iota_{\varphi} \circ \iota_{\bar{\varphi}})(\Omega) - \partial \circ \iota_{\varphi}(\Omega), \\ \partial\Omega = \partial \circ (\iota_{\varphi}\bar{\varphi} - \iota_{\bar{\varphi}} \circ \iota_{\varphi})(\Omega) - \bar{\partial} \circ \iota_{\bar{\varphi}}(\Omega), \\ \partial \circ \iota_{\bar{\varphi}}(\Omega) = 0, \end{cases}$$

which is exactly the system of obstruction equations given in [RWZ16, Proposition 2.7].

Unfortunately, the system (11) of obstruction equations consists of too many equations, and is difficult to solve. We try to reduce it to one with only two equations as in Proposition 4.5.

We will use the homogenous notation for a power series here and henceforth. Assuming that $\alpha(t)$ is a power series of (bundle-valued) (p, q) -forms, expanded as

$$\alpha(t) = \sum_{k=0}^{\infty} \sum_{i+j=k} \alpha_{i,j} t^i \bar{t}^j,$$

we use the notation

$$\begin{cases} \alpha(t) = \sum_{k=0}^{\infty} \alpha_k, \\ \alpha_k = \sum_{i+j=k} \alpha_{i,j} t^i \bar{t}^j, \end{cases}$$

where α_k is the k -order homogeneous part in the expansion of $\alpha(t)$ and all $\alpha_{i,j}$ are smooth (bundle-valued) (p, q) -forms on X_0 with $\alpha(0) = \alpha_{0,0}$.

LEMMA 4.4. *If $d(e^{\iota_{\varphi}\iota_{\bar{\varphi}}}\Omega)_{N_1} = 0$ for any $N_1 \leq N$, then*

$$(\bar{\partial}_{\varphi}A_0)_{N_1} = 0, \quad (\partial A_k + \bar{\partial}_{\varphi}A_{k+1})_{N_1} = 0, \quad k = 0, 1, 2, \dots$$

for any $N_1 \leq N$.

Proof. From (10), it follows that

$$e^{-\iota_{\varphi}}d(e^{\iota_{\varphi}\iota_{\bar{\varphi}}}\Omega) = \bar{\partial}_{\varphi}A_0 + \sum_{k=0}^{+\infty}(\partial A_k + \bar{\partial}_{\varphi}A_{k+1}).$$

For any $N_1 \leq N$,

$$0 = (e^{-\iota_{\varphi}}d(e^{\iota_{\varphi}\iota_{\bar{\varphi}}}\Omega))_{N_1} = (\bar{\partial}_{\varphi}A_0)_{N_1} + \sum_{k=0}^{+\infty}(\partial A_k + \bar{\partial}_{\varphi}A_{k+1})_{N_1}.$$

By comparing degrees, we complete the proof. □

As for (10), one can also have

$$d(e^{\iota_{\varphi}\iota_{\bar{\varphi}}}\alpha) = e^{\iota_{\varphi}\iota_{\bar{\varphi}}} \circ (e^{-\iota_{\varphi}}|^{-\iota_{\bar{\varphi}}} \circ e^{\iota_{\varphi}} \circ ([\partial, \iota_{\varphi}] + \bar{\partial} + \partial) \circ e^{-\iota_{\varphi}} \circ e^{\iota_{\varphi}\iota_{\bar{\varphi}}}\alpha).$$

A long local calculation shows that

$$d(e^{\iota_{\varphi}\iota_{\bar{\varphi}}}\alpha) = e^{\iota_{\varphi}\iota_{\bar{\varphi}}}(((\mathbf{1} - \bar{\varphi}\varphi)^{-1} - (\mathbf{1} - \bar{\varphi}\varphi)^{-1}\bar{\varphi})\lrcorner([\partial, \iota_{\varphi}] + \bar{\partial} + \partial)(\mathbf{1} - \bar{\varphi}\varphi + \bar{\varphi})\lrcorner\alpha). \tag{13}$$

Here we use the notation \lrcorner , first introduced in [RWZ16, § 2.1], to denote the *simultaneous contraction* on each component of a complex differential form. For example, $(\mathbf{1} - \bar{\varphi}\varphi + \bar{\varphi})\lrcorner\alpha$

means that the operator $(\mathbb{1} - \bar{\varphi}\varphi + \bar{\varphi})$ acts on α simultaneously as

$$\begin{aligned} &(\mathbb{1} - \bar{\varphi}\varphi + \bar{\varphi})\lrcorner(f_{i_1 \dots i_p \bar{j}_1 \dots \bar{j}_q} dz^{i_1} \wedge \dots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \dots \wedge d\bar{z}^{j_q}) \\ &= f_{i_1 \dots i_p \bar{j}_1 \dots \bar{j}_q} (\mathbb{1} - \bar{\varphi}\varphi + \bar{\varphi})\lrcorner dz^{i_1} \wedge \dots \wedge (\mathbb{1} - \bar{\varphi}\varphi + \bar{\varphi})\lrcorner dz^{i_p} \\ &\quad \wedge (\mathbb{1} - \bar{\varphi}\varphi + \bar{\varphi})\lrcorner d\bar{z}^{j_1} \wedge \dots \wedge (\mathbb{1} - \bar{\varphi}\varphi + \bar{\varphi})\lrcorner d\bar{z}^{j_q}, \end{aligned}$$

if α is locally expressed by

$$\alpha = f_{i_1 \dots i_p \bar{j}_1 \dots \bar{j}_q} dz^{i_1} \wedge \dots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \dots \wedge d\bar{z}^{j_q}.$$

This new simultaneous contraction is well defined since $\varphi(t)$ is a global $(1, 0)$ -vector valued $(0, 1)$ -form on X_0 (see [MK71, pp. 150–151]) as reasoned in [RZ18, Proof of Lemma 2.8]. Moreover, we know that

$$e^{-\iota_\varphi | -\iota_{\bar{\varphi}}} \circ e^{\iota_\varphi} = ((\mathbb{1} - \bar{\varphi}\varphi)^{-1} - (\mathbb{1} - \bar{\varphi}\varphi)^{-1} \bar{\varphi})\lrcorner : A^{p,q}(X_0) \rightarrow \bigoplus_{i=0}^{\min\{q, n-p\}} A^{p+i, q-i}(X_0).$$

Thus, by carefully comparing the types of forms in both sides of (13), we have

$$\bar{\partial}_t(e^{\iota_\varphi | \iota_{\bar{\varphi}}}(\alpha)) = e^{\iota_\varphi | \iota_{\bar{\varphi}}}((\mathbb{1} - \bar{\varphi}\varphi)^{-1} \lrcorner ([\partial, \iota_\varphi] + \bar{\partial})(\mathbb{1} - \bar{\varphi}\varphi)\lrcorner \alpha). \tag{14}$$

See [RZ18, Proposition 2.13] and [RWZ16, (2.14)] for more details of (14).

Here and henceforth we denote by $(\alpha)^{p,q}$ the (p, q) -type part of a $(p + q)$ -degree complex differential form α .

PROPOSITION 4.5. *The obstruction equation $d(e^{\iota_\varphi | \iota_{\bar{\varphi}}}(\Omega)) = 0$ is also equivalent to*

$$\begin{cases} \sum_{k=0}^{\infty} \left(\bar{\partial} \circ \frac{\iota_\varphi^{k-1}}{(k-1)!} + \partial \circ \frac{\iota_\varphi^k}{k!} \right) \circ \frac{\iota_{(\mathbb{1} - \bar{\varphi}\varphi)^{-1} \bar{\varphi}}^k}{k!}(\tilde{\Omega}) = 0, \\ \sum_{k=0}^{\infty} \left(\bar{\partial} \circ \frac{\iota_\varphi^{k-1}}{(k-1)!} + \partial \circ \frac{\iota_\varphi^k}{k!} \right) \circ \frac{\iota_{(\mathbb{1} - \bar{\varphi}\varphi)^{-1} \bar{\varphi}}^{k-1}}{(k-1)!}(\tilde{\Omega}) = 0. \end{cases} \tag{15}$$

Proof. From the proof of Proposition 4.2, it is easy to see that the left-hand side of the first equation in (15) is $(d(e^{\iota_\varphi | \iota_{\bar{\varphi}}}(\Omega)))^{p+1, q}$, while the other one is $(d(e^{\iota_\varphi | \iota_{\bar{\varphi}}}(\Omega)))^{p, q+1}$. Thus, (15) holds if $d(e^{\iota_\varphi | \iota_{\bar{\varphi}}}(\Omega)) = 0$.

Conversely, we assume that (15) holds. By (14) and (10), we compare types of forms to get

$$\begin{aligned} \bar{\partial}_t e^{\iota_\varphi | \iota_{\bar{\varphi}}}(\Omega) &= e^{\iota_\varphi | \iota_{\bar{\varphi}}} \circ (\mathbb{1} - \bar{\varphi}\varphi)^{-1} \lrcorner \bar{\partial}_\varphi A_0 \\ &= e^{\iota_\varphi | \iota_{\bar{\varphi}}} \circ (\mathbb{1} - \bar{\varphi}\varphi)^{-1} \lrcorner \left((d(e^{\iota_\varphi | \iota_{\bar{\varphi}}}(\Omega)))^{p, q+1} - \sum_{k=0}^{+\infty} \frac{\iota_\varphi^{k+1}}{(k+1)!} (\partial A_k + \bar{\partial}_\varphi A_{k+1}) \right). \end{aligned} \tag{16}$$

Similarly, we get

$$\partial_t e^{\iota_\varphi | \iota_{\bar{\varphi}}}(\Omega) = e^{\iota_\varphi | \iota_{\bar{\varphi}}} \circ (\mathbb{1} - \varphi\bar{\varphi})^{-1} \lrcorner \left((d(e^{\iota_\varphi | \iota_{\bar{\varphi}}}(\Omega)))^{p+1, q} - \sum_{k=0}^{+\infty} \frac{\iota_{\bar{\varphi}}^{k+1}}{(k+1)!} (\bar{\partial} A_k + \partial_\varphi A_{k+1}) \right),$$

where $\bar{\bullet}$ in the last term means that $\partial, \bar{\partial}, \varphi, \bar{\varphi}$ are replaced by $\bar{\partial}, \partial, \bar{\varphi}, \varphi$, respectively, while Ω takes no conjugation.

If (15) holds, i.e., $(d(e^{\iota_\varphi|_{\iota_{\bar{\varphi}}}}(\Omega)))^{p+1,q} = 0$ and $(d(e^{\iota_\varphi|_{\iota_{\bar{\varphi}}}}(\Omega)))^{p,q+1} = 0$, we will prove

$$d(e^{\iota_\varphi|_{\iota_{\bar{\varphi}}}}(\Omega)) = 0$$

by induction on orders. Obviously,

$$d(e^{\iota_\varphi|_{\iota_{\bar{\varphi}}}}(\Omega))_0 = d\Omega_0 = 0.$$

Now we assume that for any $N_1 \leq N$, $d(e^{\iota_\varphi|_{\iota_{\bar{\varphi}}}}(\Omega))_{N_1} = 0$. By Lemma 4.4, we have

$$(\bar{\partial}_\varphi A_0)_{N_1} = 0, \quad (\partial A_k + \bar{\partial}_\varphi A_{k+1})_{N_1} = 0, \quad k = 0, 1, 2, \dots$$

for any $N_1 \leq N$. For the $(N + 1)$ th order, (16) and the induction assumption for any $N_1 \leq N$ imply

$$\begin{aligned} (\bar{\partial}_t e^{\iota_\varphi|_{\iota_{\bar{\varphi}}}}(\Omega))_{N+1} &= ((e^{\iota_\varphi|_{\iota_{\bar{\varphi}}}} \circ (\mathbb{1} - \bar{\varphi}\varphi)^{-1} \lrcorner)^{-1} \circ \bar{\partial}_t e^{\iota_\varphi|_{\iota_{\bar{\varphi}}}}(\Omega))_{N+1} \\ &= (d(e^{\iota_\varphi|_{\iota_{\bar{\varphi}}}}(\Omega)))_{N+1}^{p,q+1} - \left(\sum_{k=0}^{+\infty} \frac{\iota_\varphi^{k+1}}{(k+1)!} (\partial A_k + \bar{\partial}_\varphi A_{k+1}) \right)_{N+1} \\ &= 0. \end{aligned}$$

Similarly, we have $(\partial_t e^{\iota_\varphi|_{\iota_{\bar{\varphi}}}}(\Omega))_{N+1} = 0$. So

$$d(e^{\iota_\varphi|_{\iota_{\bar{\varphi}}}}(\Omega))_{N+1} = (\bar{\partial}_t e^{\iota_\varphi|_{\iota_{\bar{\varphi}}}}(\Omega))_{N+1} + (\partial_t e^{\iota_\varphi|_{\iota_{\bar{\varphi}}}}(\Omega))_{N+1} = 0.$$

Thus, we complete the proof. □

Remark 4.6. For $p = q = 1$, if solving $de^{\iota_\varphi|_{\iota_{\bar{\varphi}}}}(\Omega) = 0$ for the orders $\leq N$, we proceed to the $(N + 1)$ th order. It is necessary to prove

$$\bar{\partial}(\varphi \lrcorner \Omega)_{N+1} = (\bar{\partial} e^{\iota_\varphi|_{\iota_{\bar{\varphi}}}}(\Omega))_{N+1}^{0,3} = (de^{\iota_\varphi|_{\iota_{\bar{\varphi}}}}(\Omega))_{N+1}^{0,3} = 0.$$

From (10), we have

$$(de^{\iota_\varphi|_{\iota_{\bar{\varphi}}}}(\Omega))_{N+1}^{0,3} = \left(\sum_{k=-1}^{+\infty} \frac{\iota_\varphi^{k+2}}{(k+2)!} (\partial A_k + \bar{\partial}_\varphi A_{k+1}) \right)_{N+1} = \left(\frac{\iota_\varphi^3}{3!} \circ \partial(\bar{\varphi} \lrcorner \Omega) \right)_{N+1} = 0,$$

where the last equality follows from Lemma 4.4.

Now we begin to solve (15) with two more lemmas. For the resolution of $\partial\bar{\partial}$ -equations, we need a lemma due to [Pop15, Theorem 4.1] (or [RWZ16, Lemma 3.14]).

LEMMA 4.7. *Let (X, ω) be a compact Hermitian complex manifold with the pure-type complex differential forms x and y . Assume that the $\partial\bar{\partial}$ -equation*

$$\partial\bar{\partial}x = y \tag{17}$$

admits a solution. Then an explicit solution of the $\partial\bar{\partial}$ -equation (17) can be chosen as

$$(\partial\bar{\partial})^* \mathbb{G}_{\text{BC}} y,$$

which uniquely minimizes the L^2 -norms of all the solutions with respect to ω . Besides, the equalities

$$\mathbb{G}_{\text{BC}}(\partial\bar{\partial}) = (\partial\bar{\partial})\mathbb{G}_{\text{A}} \quad \text{and} \quad (\partial\bar{\partial})^*\mathbb{G}_{\text{BC}} = \mathbb{G}_{\text{A}}(\partial\bar{\partial})^*$$

hold, where \mathbb{G}_{BC} and \mathbb{G}_{A} are the associated Green's operators of \square_{BC} and \square_{A} , respectively. Here \square_{BC} is defined in (7) and \square_{A} is the second Kodaira–Spencer operator (often also called Aepli Laplacian)

$$\square_{\text{A}} = \partial^* \bar{\partial}^* \bar{\partial} \partial + \bar{\partial} \partial \partial^* \bar{\partial}^* + \bar{\partial} \partial^* \bar{\partial} \bar{\partial}^* + \partial \bar{\partial}^* \bar{\partial} \partial^* + \bar{\partial} \bar{\partial}^* + \partial \partial^*.$$

Sketch of Proof for Lemma 4.7. We shall use the Hodge decomposition of \square_{BC} on X ,

$$A^{p,q}(X) = \ker \square_{BC} \oplus \text{Im}(\partial\bar{\partial}) \oplus (\text{Im } \partial^* + \text{Im } \bar{\partial}^*),$$

whose three parts are orthogonal to each other with respect to the L^2 -scalar product defined by ω , combined with the equality

$$\mathbf{1} = \mathbb{H}_{BC} + \square_{BC}\mathbb{G}_{BC},$$

where \mathbb{H}_{BC} is the harmonic projection operator. Then two observations follow:

- (1) $\square_{BC}\partial\bar{\partial}(\partial\bar{\partial})^* = \partial\bar{\partial}(\partial\bar{\partial})^*\square_{BC}$;
- (2) $\mathbb{G}_{BC}\partial\bar{\partial}(\partial\bar{\partial})^* = \partial\bar{\partial}(\partial\bar{\partial})^*\mathbb{G}_{BC}$.

It is clear that (1) implies (2), while the statement (1) is proved by a direct calculation,

$$\square_{BC}\partial\bar{\partial}(\partial\bar{\partial})^* = (\partial\bar{\partial})(\partial\bar{\partial})^*(\partial\bar{\partial})(\partial\bar{\partial})^* = \partial\bar{\partial}(\partial\bar{\partial})^*\square_{BC}.$$

Hence, we have

$$(\partial\bar{\partial})(\partial\bar{\partial})^*\mathbb{G}_{BC}y = \mathbb{G}_{BC}(\partial\bar{\partial})(\partial\bar{\partial})^*y = \mathbb{G}_{BC}\square_{BC}y = (\mathbf{1} - \mathbb{H}_{BC})y = y,$$

where $y \in \text{Im } \partial\bar{\partial}$ due to the solution-existence of the $\partial\bar{\partial}$ -equation.

To see that the solution $(\partial\bar{\partial})^*\mathbb{G}_{BC}y$ is the unique L^2 -norm minimum, we resort to the Hodge decomposition of the operator \square_A ,

$$A^{p,q}(X) = \ker \square_A \oplus (\text{Im } \partial + \text{Im } \bar{\partial}) \oplus \text{Im}(\partial\bar{\partial})^*, \tag{18}$$

where $\ker \square_A = \ker(\partial\bar{\partial}) \cap \ker \partial^* \cap \ker \bar{\partial}^*$. Let z be an arbitrary solution of the $\partial\bar{\partial}$ -equation (17), which decomposes into three components $z_1 + z_2 + z_3$ with respect to the Hodge decomposition (18) of \square_A . And we are able to obtain that

$$z_3 = \mathbb{G}_A(\partial\bar{\partial})^*y = (\partial\bar{\partial})^*\mathbb{G}_{BC}y.$$

Therefore,

$$\|z\|^2 = \|z_1\|^2 + \|z_2\|^2 + \|z_3\|^2 \geq \|z_3\|^2 = \|(\partial\bar{\partial})^*\mathbb{G}_{BC}y\|^2,$$

and the equality holds if and only if $z_1 = z_2 = 0$, i.e., $z = z_3 = (\partial\bar{\partial})^*\mathbb{G}_{BC}y$. □

LEMMA 4.8. *Let X be a complex manifold satisfying the $(p, q + 1)$ th and $(q, p + 1)$ th mild $\partial\bar{\partial}$ -lemmata. Consider the system of equations*

$$\begin{cases} \partial x = \bar{\partial}\zeta, \\ \bar{\partial} x = \partial\bar{\xi}, \end{cases} \tag{19}$$

where ζ, ξ are $(p + 1, q - 1)$ - and $(q + 1, p - 1)$ -forms on X , respectively. The system of equations (19) has a solution if and only if

$$\begin{cases} \partial\bar{\partial}\zeta = 0, \\ \bar{\partial}\partial\bar{\xi} = 0. \end{cases}$$

Proof. This lemma is inspired by [RWZ16, Observation 2.11]. The lemmata assumption will produce $\mu \in A^{p,q-1}$ and $\nu \in A^{p-1,q}$, satisfying the system of equations

$$\begin{cases} \partial\bar{\partial}\mu = \bar{\partial}\zeta, \\ \bar{\partial}\partial\nu = \partial\bar{\xi}. \end{cases}$$

The combined expression

$$\bar{\partial}\mu + \partial\nu$$

is our choice for the solution of the system (19). □

By Lemmata 4.7 and 4.8, we have the following proposition.

PROPOSITION 4.9. *With the same notation as in Lemmata 4.7 and 4.8, the system of equations (19) has a canonical solution*

$$x = \bar{\partial}(\partial\bar{\partial})^*\mathbb{G}_{\text{BC}}\bar{\partial}\zeta - \partial(\partial\bar{\partial})^*\mathbb{G}_{\text{BC}}\partial\bar{\xi}.$$

Now we assume that X_0 is a complex manifold that satisfies the $(q, p + 1)$ th and $(p, q + 1)$ th mild $\partial\bar{\partial}$ -lemmata. The obstruction (15) can be rewritten as

$$\begin{cases} \partial\tilde{\Omega} + \sum_{k=1}^{\infty} \left(\bar{\partial} \circ \frac{\iota_{\varphi}^{k-1}}{(k-1)!} + \partial \circ \frac{\iota_{\varphi}^k}{k!} \right) \circ \frac{\iota_{(\mathbb{1}-\bar{\varphi}\varphi)}^k}{k!} \bar{\varphi}(\tilde{\Omega}) = 0, \\ \bar{\partial}\tilde{\Omega} + \partial \circ \iota_{\varphi}(\tilde{\Omega}) + \sum_{k=2}^{\infty} \left(\bar{\partial} \circ \frac{\iota_{\varphi}^{k-1}}{(k-1)!} + \partial \circ \frac{\iota_{\varphi}^k}{k!} \right) \circ \frac{\iota_{(\mathbb{1}-\bar{\varphi}\varphi)}^{k-1}}{(k-1)!} \bar{\varphi}(\tilde{\Omega}) = 0. \end{cases} \tag{20}$$

Set

$$\tilde{\Omega}' := \tilde{\Omega} + \sum_{k=1}^{\infty} \frac{\iota_{\varphi}^k}{k!} \circ \frac{\iota_{(\mathbb{1}-\bar{\varphi}\varphi)}^k}{k!} \bar{\varphi}(\tilde{\Omega}).$$

Then (20) becomes

$$\begin{cases} \partial\tilde{\Omega}' = -\bar{\partial} \sum_{k=1}^{\infty} \frac{\iota_{\varphi}^{k-1}}{(k-1)!} \circ \frac{\iota_{(\mathbb{1}-\bar{\varphi}\varphi)}^k}{k!} \bar{\varphi}(\tilde{\Omega}), \\ \bar{\partial}\tilde{\Omega}' = -\partial \sum_{k=0}^{\infty} \frac{\iota_{\varphi}^{k+1}}{(k+1)!} \circ \frac{\iota_{(\mathbb{1}-\bar{\varphi}\varphi)}^k}{k!} \bar{\varphi}(\tilde{\Omega}). \end{cases}$$

In order to use the $(q, p + 1)$ th and $(p, q + 1)$ th mild $\partial\bar{\partial}$ -lemmata, we need to prove

$$\begin{cases} \partial\bar{\partial} \sum_{k=1}^{\infty} \frac{\iota_{\varphi}^{k-1}}{(k-1)!} \circ \frac{\iota_{(\mathbb{1}-\bar{\varphi}\varphi)}^k}{k!} \bar{\varphi}(\tilde{\Omega}) = 0, \\ \partial\bar{\partial} \sum_{k=0}^{\infty} \frac{\iota_{\varphi}^{k+1}}{(k+1)!} \circ \frac{\iota_{(\mathbb{1}-\bar{\varphi}\varphi)}^k}{k!} \bar{\varphi}(\tilde{\Omega}) = 0, \end{cases} \tag{21}$$

at the $(N + 1)$ th order if it has been solved for the orders $\leq N$.

Now we prove (21). Firstly, note that

$$\sum_{k=0}^{\infty} \frac{\iota_{\varphi}^{k+1}}{(k+1)!} \circ \frac{\iota_{(\mathbb{1}-\bar{\varphi}\varphi)}^k}{k!} \bar{\varphi}(\tilde{\Omega}) = (e^{\iota_{\varphi}} \circ e^{\iota_{(\mathbb{1}-\bar{\varphi}\varphi)}^{-1}\bar{\varphi}}(\tilde{\Omega}))^{p-1,q+1} = (e^{\iota_{\varphi}|\iota_{\bar{\varphi}}}(\tilde{\Omega}))^{p-1,q+1}.$$

Thus,

$$\begin{aligned} \left(\partial \bar{\partial} \sum_{k=0}^{\infty} \frac{\iota_{\varphi}^{k+1}}{(k+1)!} \circ \frac{\iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}^k}{k!}(\tilde{\Omega}) \right)_{N+1} &= \partial \bar{\partial} ((e^{\iota_{\varphi}|\iota_{\bar{\varphi}}}(\Omega))^{p-1,q+1})_{N+1} \\ &= (\partial \bar{\partial} e^{\iota_{\varphi}|\iota_{\bar{\varphi}}}(\Omega))_{N+1}^{p,q+2} \\ &= (\partial d e^{\iota_{\varphi}|\iota_{\bar{\varphi}}}(\Omega))_{N+1}^{p,q+2} \\ &= \partial (d e^{\iota_{\varphi}|\iota_{\bar{\varphi}}}(\Omega))_{N+1}^{p-1,q+2} \end{aligned}$$

since the obstruction equation is solved for the orders $\leq N$, i.e., $d(e^{\iota_{\varphi}|\iota_{\bar{\varphi}}}(\Omega))_{N_1} = 0$ for any $N_1 \leq N$. By Lemma 4.4, we have

$$(\bar{\partial}_{\varphi} A_0)_{N_1} = 0, \quad (\partial A_k + \bar{\partial}_{\varphi} A_{k+1})_{N_1} = 0, \quad k = 0, 1, 2, \dots$$

for any $N_1 \leq N$. It follows from (10) that

$$(d e^{\iota_{\varphi}|\iota_{\bar{\varphi}}}(\Omega))_{N+1}^{p-1,q+2} = \left(\sum_{k=-1}^{+\infty} \frac{\iota_{\varphi}^{k+2}}{(k+2)!} (\partial A_k + \bar{\partial}_{\varphi} A_{k+1}) \right)_{N+1} = 0.$$

So

$$\left(\partial \bar{\partial} \sum_{k=0}^{\infty} \frac{\iota_{\varphi}^{k+1}}{(k+1)!} \circ \frac{\iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}^k}{k!}(\tilde{\Omega}) \right)_{N+1} = \partial (d e^{\iota_{\varphi}|\iota_{\bar{\varphi}}}(\Omega))_{N+1}^{p-1,q+2} = 0.$$

Hence, we have proved the second equation of (21). Similarly, by the same argument (i.e., replace all φ (respectively, $\bar{\varphi}$) by $\bar{\varphi}$ (respectively, φ)), the first equation of (21) also holds.

By Proposition 4.9, one obtains a formal solution of (20) by induction

$$\begin{aligned} \tilde{\Omega}_l &= - \left(\sum_{k=1}^{\infty} \frac{\iota_{\varphi}^k}{k!} \circ \frac{\iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}^k}{k!}(\tilde{\Omega}) \right)_l - \bar{\partial}(\partial \bar{\partial})^* \mathbb{G}_{\text{BC}} \bar{\partial} \left(\sum_{k=1}^{\infty} \frac{\iota_{\varphi}^{k-1}}{(k-1)!} \circ \frac{\iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}^k}{k!}(\tilde{\Omega}) \right)_l \\ &\quad + \partial(\partial \bar{\partial})^* \mathbb{G}_{\text{BC}} \partial \left(\sum_{k=0}^{\infty} \frac{\iota_{\varphi}^{k+1}}{(k+1)!} \circ \frac{\iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}^k}{k!}(\tilde{\Omega}) \right)_l \\ &= - \left(\sum_{k=1}^{\min\{q,n-p\}} \frac{\iota_{\varphi}^k}{k!} \circ \frac{\iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}^k}{k!}(\tilde{\Omega}) \right)_l - \bar{\partial}(\partial \bar{\partial})^* \mathbb{G}_{\text{BC}} \bar{\partial} \left(\sum_{k=1}^{\min\{q,n-p\}} \frac{\iota_{\varphi}^{k-1}}{(k-1)!} \circ \frac{\iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}^k}{k!}(\tilde{\Omega}) \right)_l \\ &\quad + \partial(\partial \bar{\partial})^* \mathbb{G}_{\text{BC}} \partial \left(\sum_{k=0}^{\min\{q,n-p\}} \frac{\iota_{\varphi}^{k+1}}{(k+1)!} \circ \frac{\iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}^k}{k!}(\tilde{\Omega}) \right)_l. \end{aligned} \tag{22}$$

Remark 4.10. We are possibly able to obtain this (formal) solution (22) of (20) backwards by the invertibility of some operator in small t as shown in [LRW17, Remark 4.6], but it seems that to figure out this solution explicitly by the power series method is indispensable in this process.

4.3 Regularity argument

Here we adopt a strategy for a convergence argument [LZ18] suggested by Liu, which simplifies our argument involved in [RWZ16, RZ18].

From the induction expression (22), one obtains the formal expression of $\tilde{\Omega}$

$$\begin{aligned} \tilde{\Omega} = & - \sum_{i=1}^{\min\{q,n-p\}} \frac{\iota_{\varphi}^i}{i!} \circ \frac{\iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}^i}{i!} (\tilde{\Omega}) - \bar{\partial}(\partial\bar{\partial})^* \mathbb{G}_{\text{BC}} \bar{\partial} \sum_{i=1}^{\min\{q,n-p\}} \frac{\iota_{\varphi}^{i-1}}{(i-1)!} \circ \frac{\iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}^i}{i!} (\tilde{\Omega}) \\ & + \partial(\partial\bar{\partial})^* \mathbb{G}_{\text{BC}} \partial \sum_{i=0}^{\min\{q,n-p\}} \frac{\iota_{\varphi}^{i+1}}{(i+1)!} \circ \frac{\iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}^i}{i!} (\tilde{\Omega}) + \Omega_0. \end{aligned} \tag{23}$$

Set

$$\begin{aligned} F = & \partial(\partial\bar{\partial})^* \mathbb{G}_{\text{BC}} \partial \sum_{i=0}^{\min\{q,n-p\}} \frac{\iota_{\varphi}^{i+1}}{(i+1)!} \circ \frac{\iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}^i}{i!} \\ & - \sum_{i=1}^{\min\{q,n-p\}} \frac{\iota_{\varphi}^i}{i!} \circ \frac{\iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}^i}{i!} \\ & - \bar{\partial}(\partial\bar{\partial})^* \mathbb{G}_{\text{BC}} \bar{\partial} \sum_{i=1}^{\min\{q,n-p\}} \frac{\iota_{\varphi}^{i-1}}{(i-1)!} \circ \frac{\iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}^i}{i!} \end{aligned}$$

and write

$$\Omega_0 = (\mathbb{1} - F)\tilde{\Omega}. \tag{24}$$

We claim that $\tilde{\Omega}(t)$ converges in Hölder norm as $t \rightarrow 0$ by use of the following two *a priori* elliptic estimates: for any complex differential form ϕ ,

$$\|\bar{\partial}^* \phi\|_{k-1,\alpha} \leq C_1 \|\phi\|_{k,\alpha}$$

and

$$\|\mathbb{G}_{\text{BC}} \phi\|_{k,\alpha} \leq C_{k,\alpha} \|\phi\|_{k-4,\alpha},$$

where $k > 3$ and $C_{k,\alpha}$ depends only on k and α , not on ϕ (cf. [Kod86, Appendix.Theorem 7.4] for example). And we note that $\varphi(t)$ converges smoothly to zero as $t \rightarrow 0$. Thus, by (24), we estimate

$$\|\Omega_0\|_{k,\alpha} \geq (1 - \epsilon_{k,\alpha}) \|\tilde{\Omega}\|_{k,\alpha},$$

where $0 < \epsilon_{k,\alpha} \ll 1$ is some constant depending on k, α .

Finally, we proceed to the regularity of $\tilde{\Omega}(t)$ since there is possibly no uniform lower bound for the convergence radius obtained as above in the $C^{k,\alpha}$ -norm when k converges to $+\infty$. This argument lies heavily in the elliptic estimates [Kod86, Appendix 8], [DN55] and also [RWZ16, § 3.2].

Without loss of generality, we just consider the equation

$$\begin{aligned} \square \tilde{\Omega} = & -\square \sum_{k=1}^{\min\{q,n-p\}} \frac{\iota_{\varphi}^k}{k!} \circ \frac{\iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}^k}{k!} (\tilde{\Omega}) - \bar{\partial}\bar{\partial}^* \bar{\partial}(\partial\bar{\partial})^* \mathbb{G}_{\text{BC}} \bar{\partial} \sum_{k=1}^{\min\{q,n-p\}} \frac{\iota_{\varphi}^{k-1}}{(k-1)!} \circ \frac{\iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}^k}{k!} (\tilde{\Omega}) \\ & + \square \partial(\partial\bar{\partial})^* \mathbb{G}_{\text{BC}} \partial \sum_{k=0}^{\min\{q,n-p\}} \frac{\iota_{\varphi}^{k+1}}{(k+1)!} \circ \frac{\iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}^k}{k!} (\tilde{\Omega}) \end{aligned}$$

by applying the $\bar{\partial}$ -Laplacian $\square = \bar{\partial}^* \bar{\partial} + \bar{\partial} \bar{\partial}^*$ to the expression formula (23) and omitting the lower-order term $\square \Omega_0$ in this expression. By replacing the roles of

$$\iota_{\varphi} \circ \iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}, \quad \iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}, \quad \iota_{\varphi} + \frac{1}{2} \iota_{\varphi} \circ \iota_{\varphi} \circ \iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}$$

in the analogous strongly elliptic second-order pseudo-differential equation in the regularity argument of [RWZ16, §3.2]

$$\begin{aligned} \square \tilde{\Omega}(t) &= -\square(\iota_\varphi \circ \iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}(\tilde{\Omega}(t))) - \bar{\partial}\bar{\partial}^*\bar{\partial}(\partial\bar{\partial})^*\mathbb{G}_{\text{BC}}\bar{\partial}(\iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}(\tilde{\Omega}(t))) \\ &\quad + \square\partial(\partial\bar{\partial})^*\mathbb{G}_{\text{BC}}\partial((\iota_\varphi + \frac{1}{2}\iota_\varphi \circ \iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}})\tilde{\Omega}(t)), \end{aligned}$$

by

$$\sum_{i=1}^{\min\{q,n-p\}} \frac{\iota_\varphi^i}{i!} \circ \frac{\iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}^i}{i!}, \quad \sum_{i=1}^{\min\{q,n-p\}} \frac{\iota_\varphi^{i-1}}{(i-1)!} \circ \frac{\iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}^i}{i!}, \quad \sum_{i=0}^{\min\{q,n-p\}} \frac{\iota_\varphi^{i+1}}{(i+1)!} \circ \frac{\iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}^i}{i!},$$

respectively, we prove the following result. For each $l = 1, 2, \dots$, choose a smooth function $\eta^l(t)$ with values in $[0, 1]$

$$\eta^l(t) \equiv \begin{cases} 1 & \text{for } |t| \leq \left(\frac{1}{2} + \frac{1}{2^{l+1}}\right)r; \\ 0 & \text{for } |t| \geq \left(\frac{1}{2} + \frac{1}{2^l}\right)r, \end{cases}$$

where r is a positive constant to be determined. Inductively, by Douglis–Nirenberg’s interior estimates [Kod86, Appendix.Theorem 2.3], [DN55], for any $l = 1, 2, \dots$, $\eta^{2l+1}\tilde{\Omega}(t)$ is $C^{k+l,\alpha}$, where r can be chosen independent of l . Since $\eta^{2l+1}(t)$ is identically equal to 1 on $|t| < r/2$, which is independent of l , $\tilde{\Omega}(t)$ is C^∞ on X_0 with $|t| < r/2$. Then $\tilde{\Omega}(t)$ can be considered as a real analytic family of (p, q) -forms in t and thus is smooth on t .

5. Deformation invariance of Bott–Chern numbers

The main goal of this section is to study deformation invariance of Bott–Chern numbers on complex manifolds.

THEOREM 5.1. *If the reference fiber X_0 satisfies the $(p, q + 1)$ th and $(q, p + 1)$ th mild $\partial\bar{\partial}$ -lemmata and the deformation invariance of the $(p - 1, q - 1)$ -Aeppli number $h_{\mathbb{A}}^{p-1,q-1}(X_t)$ holds, then $h_{\text{BC}}^{p,q}(X_t)$ are deformation invariant.*

The following is a direct corollary.

COROLLARY 5.2. *If the reference fiber X_0 satisfies the $(p, 1)$ th mild $\partial\bar{\partial}$ -lemma, then $h_{\text{BC}}^{p,0}(X_t)$ and $h_{\text{BC}}^{0,p}(X_t)$ are deformation invariant.*

Resorting to the calculations for the Hodge and Bott–Chern numbers of manifolds in the Kuranishi family of the Iwasawa manifold (cf. [Ang13, Appendix]), we find the following example where neither the deformation invariance of the $(p, 0)$ - nor $(0, p)$ -Bott–Chern numbers is true when the condition of the $(p, 1)$ th mild $\partial\bar{\partial}$ -lemma does not hold on the reference fiber in Corollary 5.2. It indicates that the condition involved may not be omitted in order for the deformation invariance of $(p, 0)$ - and $(0, p)$ -Bott–Chern numbers.

Let \mathbb{I}_3 be the Iwasawa manifold of complex dimension 3 with η^1, η^2, η^3 denoted by the basis of the holomorphic one form $H^0(\mathbb{I}_3, \Omega^1)$ of \mathbb{I}_3 , satisfying the relation

$$d\eta^1 = 0, \quad d\eta^2 = 0, \quad d\eta^3 = -\eta^1 \wedge \eta^2.$$

And the convention $\eta^{12\bar{1}\bar{3}} := \eta^1 \wedge \eta^2 \wedge \bar{\eta}^1 \wedge \bar{\eta}^3$ will be used for simplicity.

Example 5.3 (The cases $(p, q) = (2, 0)$ and $(0, 2)$). The injectivity of $l_{\text{BC}, \partial}^{2,1}$ does not hold on \mathbb{I}_3 and in these cases $h_{\text{BC}}^{2,0}(X_t)$ and $h_{\text{BC}}^{0,2}(X_t)$ are deformation variant.

Proof. It is easy to check that the left-invariant $(2, 1)$ -form

$$\partial\eta^{3\bar{1}} = -\eta^{12\bar{1}}$$

stands for a non-trivial Bott–Chern class but a trivial class in $H_{\partial}^{2,1}(\mathbb{I}_3)$, which indicates non-injectivity of $l_{\text{BC}, \partial}^{2,1}$. The deformation variance of $h_{\text{BC}}^{2,0}(X_t)$ and $h_{\text{BC}}^{0,2}(X_t)$ can be got from [Ang13, Appendix]. \square

Now let us describe our basic philosophy to consider the deformation invariance of Bott–Chern numbers briefly. The Kodaira–Spencer’s upper semi-continuity theorem [KS60, Theorem 4] tells us that the function

$$t \mapsto h_{\text{BC}}^{p,q}(X_t) = \dim_{\mathbb{C}} H_{\text{BC}}^{p,q}(X_t, \mathbb{C})$$

is always upper semi-continuous for $t \in B$ and thus, to approach the deformational invariance of $h_{\text{BC}}^{p,q}(X_t)$, we only need to obtain the lower semi-continuity. Here our main strategy is a modified iteration procedure, originally from [LSY09] and developed in [Sun12, SY11, ZR13, LRY15], which is to look for an injective extension map from $H_{\text{BC}}^{p,q}(X_0)$ to $H_{\text{BC}}^{p,q}(X_t)$. More precisely, for the unique harmonic representative σ_0 of the initial Bott–Chern cohomology class in $H_{\text{BC}}^{p,q}(X_0)$, we try to construct a convergent power series

$$\sigma_t = \sigma_0 + \sum_{j+k=1}^{\infty} t^k \bar{t}^j \sigma_{k\bar{j}} \in A^{p,q}(X_0),$$

with σ_t varying smoothly on t such that for each small t :

- (i) $e^{\iota_{\varphi}|_{\iota_{\bar{\varphi}}}}(\sigma_t) \in A^{p,q}(X_t)$ is d -closed with respect to the differential structure on X_t with the induced family φ of Beltrami differentials;
- (ii) the extension map $H_{\text{BC}}^{p,q}(X_0) \rightarrow H_{\text{BC}}^{p,q}(X_t) : [\sigma_0]_d \mapsto [e^{\iota_{\varphi}|_{\iota_{\bar{\varphi}}}}(\sigma_t)]_d$ is injective.

Obviously, (i) amounts to Theorem 4.1. To guarantee (ii), it suffices to prove the following proposition.

PROPOSITION 5.4. *If the d -extension of $H_{\text{BC}}^{p,q}(X_0)$ as in Theorem 4.1 holds for a complex manifold X_0 , then the deformation invariance of $h_{\text{A}}^{p-1,q-1}(X_t)$ assures that the extension map*

$$H_{\text{BC}}^{p,q}(X_0) \rightarrow H_{\text{BC}}^{p,q}(X_t) : [\sigma_0]_d \mapsto [e^{\iota_{\varphi}|_{\iota_{\bar{\varphi}}}}(\sigma_t)]_d$$

is injective.

Proof. Here we follow an idea in [RZ18, Proposition 3.15]. Let us fix a family of smoothly varying Hermitian metrics $\{\omega_t\}_{t \in B}$ for the infinitesimal deformation $\pi : \mathcal{X} \rightarrow B$ of X_0 . Thus, if the Aeppli numbers $h_{\text{A}}^{p-1,q-1}(X_t)$ are deformation invariant, the Green’s operator $\mathbb{G}_{\text{A},t}$, acting on the $A^{p-1,q-1}(X_t)$, depends differentiably with respect to t from [KS60, Theorem 7] by Kodaira and Spencer. Using this, we ensure that this extension map cannot send a nonzero class in $H_{\text{BC}}^{p,q}(X_0)$ to a zero class in $H_{\text{BC}}^{p,q}(X_t)$.

If we suppose that

$$e^{\iota_{\varphi(t)}|_{\iota_{\bar{\varphi}(t)}}}(\sigma_t) = \partial_t \bar{\partial}_t \eta_t$$

for some $\eta_t \in A^{p-1, q-1}(X_t)$ when $t \in B \setminus \{0\}$, the Hodge decomposition of Bott–Chern Laplacian and the commutativity

$$\mathbb{G}_{\text{BC}}(\partial\bar{\partial}) = (\partial\bar{\partial})\mathbb{G}_{\text{A}}$$

in Lemma 4.7 yield that

$$\begin{aligned} e^{\iota_{\varphi(t)}|_{\iota_{\varphi(t)}^{\overline{}}}}(\sigma_t) &= \partial_t \bar{\partial}_t \eta_t = (\mathbb{H}_{\text{BC}, t} + \square_{\text{BC}, t} \mathbb{G}_{\text{BC}, t}) \partial_t \bar{\partial}_t (\eta_t) \\ &= \mathbb{G}_{\text{BC}, t} \square_{\text{BC}, t} \partial_t \bar{\partial}_t (\eta_t) \\ &= \mathbb{G}_{\text{BC}, t} \partial_t \bar{\partial}_t \bar{\partial}_t^* \partial_t^* \partial_t \bar{\partial}_t (\eta_t) \\ &= \partial_t \bar{\partial}_t \mathbb{G}_{\text{A}, t} \bar{\partial}_t^* \partial_t^* (e^{\iota_{\varphi(t)}|_{\iota_{\varphi(t)}^{\overline{}}}}(\sigma_t)), \end{aligned}$$

where $\mathbb{H}_{\text{BC}, t}$, $\square_{\text{BC}, t}$ are the harmonic projectors and the Bott–Chern Laplacian with respect to (X_t, ω_t) , respectively. Let t converge to 0 on both sides of the equality

$$e^{\iota_{\varphi(t)}|_{\iota_{\varphi(t)}^{\overline{}}}}(\sigma_t) = \partial_t \bar{\partial}_t \mathbb{G}_{\text{A}, t} \bar{\partial}_t^* \partial_t^* (e^{\iota_{\varphi(t)}|_{\iota_{\varphi(t)}^{\overline{}}}}(\sigma_t)),$$

which turns out that σ_0 is $\partial\bar{\partial}$ -exact on the reference fiber X_0 . Here we use the fact that the Green's operator $\mathbb{G}_{\text{A}, t}$ depends differentiably with respect to t . \square

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